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Transverse totally geodesic submanifolds of the tangent bundle

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Abstract. It is well-known that if ξ is a smooth vector field on a given Riemannian manifold M^n then ξ naturally defines a submanifold $\xi(M^n)$ transverse to the fibers of the tangent bundle TM^n with Sasaki metric. In this paper, we are interested in transverse totally geodesic submanifolds of the tangent bundle. We show that a transverse submanifold N^l of TM^n $(1 \le l \le n)$ can be realized locally as the image of a submanifold F^l of M^n under some vector field ξ defined along F^l. For such images $\xi(F^l)$, the conditions to be totally geodesic are presented. We show that these conditions are not so rigid as in the case of $l = n$, and we treat several special cases (ξ of constant length, ξ normal to F^l, M^n of constant curvature, M^n a Lie group and ξ a left invariant vector field).

Introduction

Let (M^n, g) be a Riemannian manifold and (TM^n, g_s) its tangent bundle equipped with the Sasaki metric [12]. Let ξ be a given smooth vector field on M^n. Then ξ naturally defines a mapping $\xi : M^n \to TM^n$ such that the submanifold $\xi(M^n) \subset TM^n$ is transverse to the fibers. This fact allows to ascribe to the vector field ξ some geometrical characteristics from the geometry of submanifolds. We say that the vector field ξ is *minimal,*

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totally umbilic or *totally geodesic* if $\xi(M^n)$ possesses the same property. In a similar way we can say about the *sectional, Ricci* or *scalar curvature* of a vector field. For the case of a *unit* vector field this approach has been proposed by H. Gluck and W. Ziller [6]. They proved that the Hopf vector field h on three-sphere S^3 is one with globally minimal volume, i.e. $h(S^3)$ is a globally minimal submanifold in the unit tangent bundle T_1S^3. Corresponding local consideration leads to the notion of the *mean curvature of a unit vector field* and a number of examples of locally minimal unit vector fields were found based on a preprint version of [5] (see [1], [2], [7] and references). In a different way, the second author found examples of unit vector fields of *constant mean curvature* [18] and completely described the *totally geodesic* unit vector fields on 2-dimensional manifolds of constant curvature [19]. The energy of a mapping $\xi : M^n \to T_1M^n$ can also be ascribed to the vector field ξ and we can say about the *energy* of a unit vector field (see [17], [4], [15] and references).

In contrast to unit vector fields, there are few results (both of local or global aspects) on the geometry of general vector fields treated as submanifolds in the *tangent bundle*. It is known [10] that if ξ is the zero vector field, then $\xi(M^n)$ is totally geodesic in TM^n. P. Walczak [14] treated the case when ξ is a non-zero vector field on M^n and proved that if ξ is a parallel vector field on M^n, then $\xi(M^n)$ is totally geodesic in TM^n. Moreover, if ξ is of constant length, then $\xi(M^n)$ is totally geodesic in TM^n if and only if ξ is a parallel vector field on M^n. The latter condition is rather burdensome. The basic manifold M^n should be a metrical product $M^{n-k} \times E^k$ $(k \geq 1)$, where E^k is a Euclidean (flat) factor.

Remark that $\xi(M^n)$ has maximal dimension among submanifolds in the tangent bundle, transverse to the fibers. In this paper, we study submanifolds N^l of TM^n with $l \leq n$ which are transverse to the fibers. We show in Section 2 that any transverse submanifold N^l of TM^n can be realized locally as the image of a submanifold F^l of M^n under some vector field ξ defined along F^l. We also investigate some cases when the image can be globally realized. Mainly, we are interested in submanifolds among this class which are totally geodesic. In this way, we get a chain of inclusions:

$$\xi(F^l) \subset \xi(M^n) \subset TM^n.$$

In comparison with the case when ξ is defined over the whole M^n or, at least, over a domain $D^n \subset M^n$ as in [14], the picture becomes different, because $\xi(F^l)$ can be totally geodesic in TM^n while $\xi(M^n)$ is not. Our considerations include also the case when the vector field is defined only on F^l, so that ξ defines a "direct" embedding $\xi : F^l \to TM^n$.

For $l = 1$ we get nothing else but a vector field along a curve in M^n which generates a geodesic in TM^n. S. Sasaki [12] described geodesic lines in TM^n in terms of vector fields along curves in M^n and found the differential equations on the curve and the corresponding vector field. Moreover, in the case when M^n is of constant curvature, K. Sato [13] explicitly described the curves and the vector fields.

Evidently, our approach takes an intermediate position between the above mentioned considerations for $l = 1$ and $l = n$. Necessary and sufficient conditions on $\xi(F^l)$ to be totally geodesic, that we make explicit in Section 3 (Proposition 3.1), have a clearer geometrical meaning if we suppose that ξ is of constant length along F^l (Theorem 3.2) or is a normal vector field along F^l (Theorem 3.3). Indeed, an application of Theorem 3.3 to the specific case of foliated Riemannian manifolds allows us to clarify the geometrical structure of $\xi(M^n)$ (Corollary 3.5).

The case of a base space M^n of constant curvature is discussed in detail in Section 4. An application to the case of a Riemannian manifold of constant curvature enlightens us as to the non rigidity of the totally geodesic property of $\xi(F^l)$, $l < n$, contrary to the case $l = n$.

Finally, an application of our results to Lie groups endowed with bi-invariant metrics gives a clear geometrical picture of our problem.

Remark. Throughout the paper

- M^n is a given Riemannian manifold with metric $\bar{g}$, F^l is a submanifold of M^n with the induced metric g, TM^n is the tangent bundle of M^n equipped with the Sasaki metric g_s;
- $\bar{\nabla}$, ∇, $\tilde{\nabla}$ are the Levi–Civita connections with respect to $\bar{g}$, g, g_s respectively;
- the indices range is fixed as $a, b, c = 1 \dots n$; $i, j, k = 1 \dots l$;
- all the vector fields are supposed sufficiently smooth, say of class C^∞.

1. Local geometry of $\xi(F^l)$

1.1. Tangent bundle of $\xi(F^l)$. Let $(M^n, \bar g)$ be an n-dimensional Riemannian manifold with metric $\bar g$. Denote by $\bar g(\cdot\,,\cdot)$ the scalar product with respect to $\bar g$. The *Sasaki metric* g_s on TM^n is defined by the following scalar product: if $\tilde X$, $\tilde Y$ are tangent vector fields on TM^n, then

$$g_s(\tilde X, \tilde Y) = \bar g(\pi_* \tilde X, \pi_* \tilde Y) + \bar g(K\tilde X, K\tilde Y) \tag{1}$$

where $\pi_* : TTM^n \to TM^n$ is the differential of the projection $\pi : TM^n \to M^n$ and $K : TTM^n \to TM^n$ is the *connection map* [3]. The local representations for π_* and K are the following ones. Let $(x^1, \dots, x^n)$ be a local coordinate system on M^n. Denote by $\partial/\partial x^a$ the natural tangent coordinate frame. Then, at each point $x \in M^n$, any tangent vector ξ can be decomposed as $\xi = \xi^a \frac{\partial}{\partial x^a}(x)$. The set of parameters $\{x^1, \dots, x^n; \xi^1, \dots, \xi^n\}$ forms the natural induced coordinate system in TM^n, i.e. for a point $z = (x, \xi) \in TM^n$, with $x \in M^n$, $\xi \in T_x M^n$, we have $x = (x^1, \dots, x^n)$, $\xi = \xi^a \frac{\partial}{\partial x^a}(x)$. The natural frame in $T_z TM^n$ is formed by $\left\{ \frac{\partial}{\partial x^a}(z), \frac{\partial}{\partial \xi^a}(z) \right\}$ and for any $\tilde X \in T_z TM^n$ we have the decomposition $\tilde X = \tilde X^a \frac{\partial}{\partial x^a}(z) + \tilde X^{n+a} \frac{\partial}{\partial \xi^a}(z)$. Now locally, the *horizontal* and *vertical* projections of $\tilde X$ are given by

$$\begin{aligned} \pi_* \tilde X &= \tilde X^a \frac{\partial}{\partial x^a}(\pi(z)), \\ K\tilde X &= (\tilde X^{n+a} + \bar\Gamma^a_{bc}(\pi(z))\xi^b \tilde X^c) \frac{\partial}{\partial x^a}(\pi(z)), \end{aligned} \tag{2}$$

where $\bar\Gamma^a_{bc}$ are the Christoffel symbols of the metric $\bar g$. The inverse operations are called *lifts*. If $\bar X = \bar X^a \partial/\partial x^a$ is a vector field on M^n then the vector fields on TM given by

$$\begin{aligned} \bar X^h &= \bar X^a \partial/\partial x^a - \bar\Gamma^a_{bc} \xi^b \bar X^c \partial/\partial \xi^a, \\ \bar X^v &= \bar X^a \partial/\partial \xi^a \end{aligned}$$

are called the *horizontal* and *vertical* lifts of X respectively. Remark that for any vector field $\bar X$ on M^n it holds

$$\begin{aligned} \pi_* \bar X^h &= \bar X, \qquad K\bar X^h = 0, \\ \pi_* \bar X^v &= 0, \qquad K\bar X^v = \bar X. \end{aligned} \tag{3}$$

Let F^l be an l-dimensional submanifold in M^n with a local representation given by

$$x^a = x^a(u^1, \dots, u^l).$$

Let ξ be a vector field on M^n defined in some neighborhood of (or only on) the submanifold F^l. Then the restriction of ξ to the submanifold F^l, called *a vector field on M^n along F^l*, generates a submanifold $\xi(F^l) \subset TM^n$ with a local representation of the form

$$\xi(F^l) : \begin{cases} x^a = x^a(u^1, \dots, u^l), \\ \xi^a = \xi^a(x^1(u^1, \dots, u^l), \dots, x^n(u^1, \dots, u^l)). \end{cases} \tag{4}$$

In what follows we will refer to the submanifold (4) as to one *generated by a vector field on M^n along F^l.*

The following proposition describes the tangent space of $\xi(F^l)$.

Proposition 1.1. *A vector field $\tilde{X}$ on TM^n is tangent to $\xi(F^l)$ along $\xi(F^l)$ if and only if its horizontal-vertical decomposition is of the form*

$$\tilde{X} = X^h + (\bar{\nabla}_X \xi)^v,$$

where X is a tangent vector field on F^l, $\bar{\nabla}_X \xi$ is the covariant derivative of ξ in the direction of X with respect to the Levi–Civita connection of M^n and the lifts are considered as those on TM^n.

Proof. Let us denote by $\tilde{e}_i$ the vectors of the coordinate frame of $\xi(F^l)$. Then, evidently,

$$\tilde{e}_i = \left\{ \frac{\partial x^1}{\partial u^i}, \dots, \frac{\partial x^n}{\partial u^i};\ \ \frac{\partial \xi^1}{\partial u^i}, \dots, \frac{\partial \xi^n}{\partial u^i} \right\}.$$

Applying (2), we have

$$\pi_* \tilde{e}_i = \frac{\partial x^a}{\partial u^i} \frac{\partial}{\partial x^a} = \frac{\partial}{\partial u^i},$$

$$\begin{aligned} K \tilde{e}_i &= \left(\frac{\partial \xi^a}{\partial u^i} + \bar{\Gamma}^a_{bc} \xi^b \frac{\partial x^c}{\partial u^i} \right) \frac{\partial}{\partial x^a} = \left(\frac{\partial \xi^a}{\partial x^c} \frac{\partial x^c}{\partial u^i} + \bar{\Gamma}^a_{bc} \xi^b \frac{\partial x^c}{\partial u^i} \right) \frac{\partial}{\partial x^a} \\ &= \frac{\partial x^c}{\partial u^i} \left(\frac{\partial \xi^a}{\partial x^c} + \bar{\Gamma}^a_{bc} \xi^b \right) \frac{\partial}{\partial x^a} = \bar{\nabla}_i \xi, \end{aligned}$$

where $\bar\Gamma^a_{bc}$ are the Christoffel symbols of the metric $\bar g$ taken along F^l and $\bar\nabla_i$ means the covariant derivative of a vector field on M^n with respect to the Levi–Civita connection of $\bar g$ along the i-th coordinate curve of the submanifold $F^l \subset M^n$. Summing up, we have

$$\tilde e_i = \left(\frac{\partial}{\partial u^i}\right)^h + (\bar\nabla_i \xi)^v. \tag{5}$$

Let $\tilde X$ be a vector field on TM^n tangent to $\xi(F^l)$ along $\xi(F^l)$. Then the following decomposition holds $\tilde X = \tilde X^i \tilde e_i$. Set $X = \tilde X^i \partial/\partial u^i$. The vector field X is tangent to F^l and, taking into account (5), the decomposition of $\tilde X$ can be represented as $\tilde X = X^h + (\bar\nabla_X \xi)^v$, which completes the proof. □

Corollary 1.1. *Let (F^l, g) be a submanifold of a Riemannian manifold $(M^n, \bar g)$ with the induced metric. Let ξ be a vector field on M^n along F^l. Then the metric on $\xi(F^l)$, induced by the Sasaki metric of TM^n, is defined by the following scalar product*

$$g_s(\tilde X, \tilde Y) = g(X, Y) + \bar g(\bar\nabla_X \xi, \bar\nabla_Y \xi),$$

for all vector fields $\tilde X = X^h + (\bar\nabla_X \xi)^v$ and $\tilde Y = Y^h + (\bar\nabla_Y \xi)^v$ on $\xi(F^l)$, where X, Y are vector fields on F^l.

1.2. Normal bundle of $\xi(F^l)$. To describe the normal bundle of $\xi(F^l)$, we need one auxiliary notion. Let ξ be a given vector field on a submanifold $F^l \subset M^n$. Then $\bar\nabla$ enables us to define a point-wise linear mapping $\bar\nabla\xi : T_x F^l \to T_x M^n$, $X \to \bar\nabla_X \xi$, for all $x \in M^n$. Its dual mapping, with respect to the corresponding scalar products induced by g and $\bar g$, gives rise to the linear mapping $(\bar\nabla\xi)^* : T_x M^n \to T_x F^l$ defined by the formula

$$g((\bar\nabla\xi)^* W, X) = \bar g(\bar\nabla_X \xi, W) \text{ for all } W \in T_x M^n \text{ and } X \in T_x F^l. \tag{6}$$

We call the mapping $(\bar\nabla\xi)^* : T_x M^n \to T_x F^l$ the *conjugate derivative mapping*, or simply *conjugate derivative*. Remark, that if W is a vector field on M^n, then the application of $(\bar\nabla\xi)^*$ gives rise to a vector field $(\bar\nabla\xi)^* W$ on F^l by $[(\bar\nabla\xi)^* W]_x = (\bar\nabla\xi)^* W_x \in T_x F^l$ for all $x \in F^l$.

Now we can prove

Proposition 1.2. *Let η and Z be normal and tangent vector fields on F^l respectively. Then the lifts*

$$\eta^h,\ \eta^v - ((\bar{\nabla}\xi)^*\eta)^h,\ Z^v - ((\bar{\nabla}\xi)^*Z)^h$$

to the points of $\xi(F^l)$ span the normal bundle of $\xi(F^l)$ in TM^n.

Proof. Let $\tilde{X} = X^h + (\bar{\nabla}_X\xi)^v$ be a vector field on $\xi(F^l)$. Let η and Z be vector fields on F^l which are normal and tangent to F^l respectively. Taking into account (1), (3) and (6), we have

$$g_s(\tilde{X}, \eta^h) = \bar{g}(X, \eta) = 0$$

$$\begin{aligned} g_s(\tilde{X}, \eta^v - [(\bar{\nabla}\xi)^*\eta]^h) &= -\bar{g}(X, (\bar{\nabla}\xi)^*\eta) + \bar{g}(\bar{\nabla}_X\xi, \eta) \\ &= -\bar{g}(\bar{\nabla}_X\xi, \eta) + \bar{g}(\bar{\nabla}_X\xi, \eta) = 0 \end{aligned}$$

$$\begin{aligned} g_s(\tilde{X}, Z^v - [(\bar{\nabla}\xi)^*Z]^h) &= -\bar{g}(X, (\bar{\nabla}\xi)^*Z) + \bar{g}(\bar{\nabla}_X\xi, Z) \\ &= -\bar{g}(\bar{\nabla}_X\xi, Z) + \bar{g}(\bar{\nabla}_X\xi, Z) = 0 \end{aligned}$$

Let $\eta_1, \dots, \eta_p$ $(p = 1, \dots, n-l)$ be a normal frame of F^l while $f_1, \dots, f_l$ span T_xF^l at each point $x \in F^l$. Consider the vector fields

$$N_\alpha = \eta_\alpha^h, \quad P_\alpha = \eta_\alpha^v - ((\bar{\nabla}\xi)^*\eta_\alpha)^h, \quad F_i = f_i^v - ((\bar{\nabla}\xi)^*e_i)^h,$$

where $\alpha = 1, \dots, n-l$; $i = 1, \dots, n$. Let us show that these are linearly independent. Indeed, suppose that

$$\begin{aligned} &\lambda^\alpha N_\alpha + \mu^\alpha P_\alpha + \nu^i F_i \\ &\quad = \{\lambda^\alpha\eta_\alpha - \mu^\alpha(\bar{\nabla}\xi)^*\eta_\alpha - \nu^i(\bar{\nabla}\xi)^*e_i\}^h + \{\mu^\alpha\eta_\alpha + \nu^i f_i\}^v = 0. \end{aligned}$$

Because of the fact that the horizontal and vertical components are linearly independent, we see that $\mu^\alpha\eta_\alpha + \nu^i f_i = 0$ which is possible iff $\mu^\alpha = 0$, $\nu^i = 0$. Then, from the horizontal part of the decomposition above we see that $\lambda^\alpha = 0$. So, N_α, P_α and F_i are linearly independent, which completes the proof. □

Remark. In the case when ξ is a normal vector field, the images $(\bar{\nabla}\xi)^*\eta$ and $(\bar{\nabla}\xi)^*Z$ have a simple and natural meaning, namely

$$(\bar{\nabla}\xi)^*\eta = g^{ik}\bar{g}(\nabla_k^\perp\xi, \eta)\frac{\partial}{\partial u^i}, \quad (\bar{\nabla}\xi)^*Z = -A_\xi Z,$$

where $\nabla^{\perp}$ is the normal bundle connection of F^l and A_ξ is the shape operator of F^l with respect to the normal vector field ξ. In fact, $(\bar{\nabla}\xi)^*\eta$ is the vector field on F^l dual to the 1-form $\bar{g}(\nabla^{\perp}_k\xi, \eta)du^k$.

2. Characterization of submanifolds of TM^n transverse to fibers

It is clear that all totally geodesic vector fields along submanifolds of M^n generate submanifolds in TM^n which are transverse to the fibers of TM^n. We study in this section the converse question. We start with the local case.

Proposition 2.1. *Let N^l be an embedded submanifold in the tangent bundle of a Riemannian manifold M^n, which is transverse to the fiber at a point $z \in N^l$, then there is a submanifold F^l of M^n containing $x = \pi(z)$, a neighborhood U of x in M^n, a neighborhood V of z in TM^n and a vector field ξ on M^n along $F^l \cap U$ such that $N^l \cap V = \xi(F^l \cap U)$.*

Proof. Since T_zN^l is transverse to the vertical subspace V_zTM^n of TTM^n at z, $\pi_* \restriction T_zN^l : T_zN^l \to T_xM^n$ is injective, and so there is an open neighborhood W of z in TM^n such that $\pi_* \restriction T_{z'}N^l : T_{z'}N^l \to T_{\pi(z')}M^n$ is injective for all $z' \in W \cap N^l$. Hence $\pi \restriction W \cap N^l : W \cap N^l \to M^n$ is an immersion, and thus there exist a cubic centered coordinate system (U, φ) about $x = \pi(z)$ and a neighborhood V of z in W such that $\pi \restriction V \cap N^l$ is 1:1 and $\pi(V \cap N^l)$ is a part of a slice F^l of (U, φ) [16, p. 28]. The slice F^l is a submanifold of M^n and we have $\pi \restriction V \cap N^l : V \cap N^l \to U \cap F^l$ is an imbedding onto, and so there is a C^∞-mapping $\xi : F^l \cap U \to N^l \cap V$ such that $\pi \circ \xi = Id_{F^l \cap U}$. In other words, ξ is a vector field on M^n along $F^l \cap U$ such that $N^l \cap V = \xi(F^l \cap U)$. □

The global version of the last result requires further conditions.

Theorem 2.1. *Let N^n be a connected compact n-dimensional submanifold of the tangent bundle of a connected simply connected Riemannian manifold M^n, which is everywhere transverse to the fibers of TM^n. Then M^n is also compact, and there is a vector field ξ on M^n such that $\xi(M^n) = N^n$.*

PROOF. The fact that N^n is everywhere transverse to the fibers of TM^n implies that $\pi \restriction N^n : N^n \to M^n$ is an immersion. Since M^n and N^n are connected of the same dimension and N^n is compact, then M^n is compact and $\pi \restriction N^n$ is a covering projection (cf. [8, Vol. 1, p. 178]). Now, M^n is simply connected and so $\pi \restriction N^n$ is a diffeomorphism. Let $\xi : M^n \to N^n$ be the inverse of $\pi \restriction N^n$. Then ξ is a vector field on M^n and $\xi(M^n) = N^n$. □

In a similar way, we can show the following:

Theorem 2.2. *Let N^l be a connected compact submanifold of the tangent bundle of a connected simply connected manifold M^n, which is transverse to the fibers it meets and projects onto a simply connected submanifold F^l of M^n. Then F^l is compact and there is a vector field ξ on M^n along F^l such that $\xi(F^l) = N^l$.*

In the particular case of horizontal totally geodesic submanifolds of TM^n, i.e. whose tangent space at any point is horizontal, we can state the following:

Theorem 2.3. *Let N^l be a connected complete totally geodesic horizontal submanifold of the tangent bundle of a connected Riemannian manifold M^n which projects into a simply connected Riemannian submanifold F^l of M^n. Then F^l is also complete and totally geodesic in M^n and there is a parallel vector field ξ on M^n along F^l such that $\xi(F^l) = N^l$.*

PROOF. By hypothesis, for all $z \in N^l$, T_zN^l is a horizontal subspace of T_zTM^n with respect to the Levi–Civita connection of $\bar{g}$. Hence $\pi \restriction N^l : N^l \to F^l$ is an isometric submersion of N^l into F^l, with N^l and F^l connected and of the same dimension. Since N^l is complete, also F^l is complete and N^l is a covering space of F^l (cf. [8, Vol. 1, p. 176]). The fact that F^l is simply connected implies that $\pi \restriction N^l : N^l \to F^l$ is an isometry, and there is an isometry $\xi : F^l \to N^l$ such that $\pi \restriction N^l \circ \xi = Id_{F^l}$, i.e. ξ is a vector field on M^n along F^l.

Now, F^l is totally geodesic. Indeed, let X and Y be vector fields on F^l, and denote by the same letters some of their extensions to M^n. If we denote by X^h and Y^h their horizontal lifts to TM^n, then $X^h \restriction N^l$ and $Y^h \restriction N^l$ are vector fields on TM^n along N^l. For all $z \in N^l$, T_zN^l being horizontal, $\pi_* \restriction T_zN^l : T_zN^l \to T_xM^n$ is bijective. Since $\pi_*(X^h(z)) =$

$X(\pi(z))$ and $\pi_*(Y^h(z)) = Y(\pi(z))$, we have that $X^h(z)$ and $Y^h(z)$ are tangent to N^l. Thus $(\tilde{\nabla}_{X^h}Y^h) \restriction N^l$ is tangent to N^l and hence horizontal. Consequently $(\tilde{\nabla}_{X^h}Y^h) \restriction N^l = (\bar{\nabla}_X Y)^h \restriction N^l$ and is tangent to N^l. Hence $\bar{\nabla}_X Y = \pi_* \circ (\bar{\nabla}_X Y)^h$ is tangent to F^l and so F^l is totally geodesic. It remains to prove that ξ is parallel along F^l. In fact, for all $x \in F^l$ and $X \in T_x F^l$, the vector $X^h + (\bar{\nabla}_X \xi)^v$ is tangent to $\xi(F^l) = N^l$ at $\xi(x)$ and is mapped onto X. Since $T_{\xi(x)}N^l$ is a horizontal space, $\bar{\nabla}_X \xi = 0$. Therefore, ξ is parallel along F^l. □

Corollary 2.1. *Let N^n be a connected complete totally geodesic horizontal n-dimensional submanifold of the tangent bundle of a connected simply connected Riemannian manifold M^n. Then M^n is also complete and there is a parallel vector field ξ on M^n such that $\xi(M^n) = N^n$.*

3. The conditions on $\xi(F^l)$ to be totally geodesic

Evidently, geometrical properties of the submanifold $\xi(F^l)$ depend on the submanifold F^l and the vector field ξ. If one does not pose any restrictions on them, the geometry of $\xi(F^l)$ becomes rather intricate. Nevertheless, it is possible to formulate the conditions on $\xi(F^l)$ to be totally geodesic in more or less geometrical terms.

To do this, we introduce the notion of a ξ-connection on the Riemannian manifold M^n.

Definition 3.1. Let M^n be a Riemannian manifold with Riemannian connection $\bar{\nabla}$ and curvature tensor $\bar{R}$. Let ξ be a fixed smooth vector field on M^n. Denote by $\mathfrak{X}(M^n)$ the set of all smooth vector fields on M^n. The mapping $\overset{*}{\nabla}\colon \mathfrak{X}(M^n) \times \mathfrak{X}(M^n) \to \mathfrak{X}(M^n)$ defined by

$$\overset{*}{\nabla}_{\bar{X}}\, \bar{Y} = \bar{\nabla}_{\bar{X}}\bar{Y} + \frac{1}{2}\Big[\bar{R}(\xi, \bar{\nabla}_{\bar{X}}\xi)\bar{Y} + \bar{R}(\xi, \bar{\nabla}_{\bar{Y}}\xi)\bar{X}\Big] \tag{7}$$

is a torsion-free affine connection on M^n. It is called the ξ-connection.

Remark that if ξ is a parallel vector field or the manifold M^n is flat, then the ξ-connection is the same as the Levi–Civita connection of M^n.

It is easy to check that (7) indeed defines a torsion-free affine connection. Now we can state the main technical tool for the further considerations.

Proposition 3.1. *Let F^l be a submanifold in a Riemannian manifold M^n. Let ξ be a vector field on M^n along F^l. Then $\xi(F^l)$ is totally geodesic in TM^n if and only if*

(a) *F^l is totally geodesic with respect to the ξ-connection* (7)*;*

(b) *for any vector fields X, Y on F^l*

$$\bar{\nabla}_X \bar{\nabla}_Y \xi = \bar{\nabla}_{\overset{*}{\nabla}_X Y} \xi + \frac{1}{2}\bar{R}(X,Y)\xi.$$

Proof. By definition, the submanifold $\xi(F^l)$ is totally geodesic in TM^n if and only if $g_s(\tilde{\nabla}_{\tilde{X}}\tilde{Y}, \tilde{N}) = 0$ for any vector fields $\tilde{X}$, $\tilde{Y}$ tangent to $\xi(F^l)$ along $\xi(F^l)$ and $\tilde{N}$ normal to $\xi(F^l)$. To calculate $\tilde{\nabla}_{\tilde{X}}\tilde{Y}$, we use the Kowalski formulas [9].

For any vector fields $\bar{X}$, $\bar{Y}$ on M^n, the covariant derivatives of various combinations of lifts to the point $(x,\xi) \in TM^n$ can be found as follows

$$\begin{aligned} &\tilde{\nabla}_{\bar{X}^h}\bar{Y}^h = (\bar{\nabla}_{\bar{X}}\bar{Y})^h - \frac{1}{2}(\bar{R}(\bar{X},\bar{Y})\xi)^v, \quad \tilde{\nabla}_{\bar{X}^v}\bar{Y}^h = \frac{1}{2}(\bar{R}(\xi,\bar{X})\bar{Y})^h, \\ &\tilde{\nabla}_{\bar{X}^h}\bar{Y}^v = (\bar{\nabla}_{\bar{X}}\bar{Y})^v + \frac{1}{2}(\bar{R}(\xi,\bar{Y})\bar{X})^h, \quad \tilde{\nabla}_{\bar{X}^v}\bar{Y}^v = 0. \end{aligned} \tag{8}$$

where $\bar{\nabla}$ and $\bar{R}$ are the Levi–Civita connection and the curvature tensor of M^n respectively.

Let $\tilde{X} = X^h + (\bar{\nabla}_X\xi)^v$ and $\tilde{Y} = (Y)^h + (\bar{\nabla}_Y\xi)^v$ be vector fields tangent to $\xi(F^l)$. Then, applying (8), we easily find

$$\begin{aligned} \tilde{\nabla}_{\tilde{X}}\tilde{Y} = &\left(\bar{\nabla}_X Y + \frac{1}{2}\bar{R}(\xi,\bar{\nabla}_X\xi)Y + \frac{1}{2}\bar{R}(\xi,\bar{\nabla}_Y\xi)X\right)^h \\ &+ \left(\bar{\nabla}_X\bar{\nabla}_Y\xi - \frac{1}{2}\bar{R}(X,Y)\xi\right)^v \end{aligned}$$

or

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = (\overset{*}{\nabla}_X Y)^h + \left(\bar{\nabla}_X\bar{\nabla}_Y\xi - \frac{1}{2}\bar{R}(X,Y)\xi\right)^v.$$

Using Proposition 1.2, we see that the totally geodesic property of $\xi(F^l)$ is equivalent to

$$\begin{cases} \bar g(\overset{*}{\nabla}_X Y, \eta) = 0, \\ \bar g(\overset{*}{\nabla}_X Y, (\nabla\xi)^*\eta) = \bar g(\bar\nabla_X \bar\nabla_Y \xi - \frac{1}{2}\bar R(X,Y)\xi, \eta), \\ \bar g(\overset{*}{\nabla}_X Y, (\nabla\xi)^* Z) = \bar g(\bar\nabla_X \bar\nabla_Y \xi - \frac{1}{2}\bar R(X,Y)\xi, Z), \end{cases} \tag{9}$$

for any vector fields X, Y, Z tangent to F^l and any vector field η orthogonal to F^l.

From $(9)_1$ we see that F^l must be autoparallel with respect to $\overset{*}{\nabla}$ and hence totally geodesic [8]. Thus, $\overset{*}{\nabla}_X Y$ is tangent to F^l and it is possible to apply (6). Therefore, we can rewrite the equations $(9)_2$ and $(9)_3$ as

$$\begin{cases} \bar g(\bar\nabla_{\overset{*}{\nabla}_X Y}\xi - \bar\nabla_X \bar\nabla_Y \xi + \frac{1}{2}\bar R(X,Y)\xi, \eta) = 0, \\ \bar g(\bar\nabla_{\overset{*}{\nabla}_X Y}\xi - \bar\nabla_X \bar\nabla_Y \xi + \frac{1}{2}\bar R(X,Y)\xi, Z) = 0 \end{cases}$$

for any vector fields η normal and Z tangent to F^l along F^l. Thus, we conclude

$$\bar\nabla_X \bar\nabla_Y \xi = \bar\nabla_{\overset{*}{\nabla}_X Y}\xi + \frac{1}{2}\bar R(X,Y)\xi,$$

which completes the proof. □

For the cases when $l = 1$ and $l = n$, we get the known conditions for the totally geodesic property of $\xi(F^l)$.

Corollary 3.1. *If $l = 1$ and $\xi(F^l)$ is a curve Γ in TM^n then this curve is a geodesic if and only if*

$$\begin{cases} x'' + \bar R(\xi, \xi')x' = 0, \\ \xi'' = 0, \end{cases}$$

where $(\,')$ means the covariant derivative with respect to the natural parameter of Γ and $x(\sigma) = (\pi \circ \Gamma)(\sigma)$ (cf. [12]);

Proof. Indeed, in this case $\tilde X = \tilde Y = \Gamma' = (x')^h + (\xi')^v$, $\bar X = \bar Y = x'$ and $\overset{*}{\nabla}_{\bar X} \bar Y = x'' + \bar R(\xi, \xi')x'$. Thus, $x(\sigma)$ is geodesic with respect to the ξ-connection iff $x'' + \bar R(\xi, \xi')x' = 0$ and the rest of the proof is evident. □

Corollary 3.2. *If $l = n$ and $F^l = M^n$, then $\xi(M^n)$ is totally geodesic in TM^n if and only if for any vector fields $\bar{X}, \bar{Y}$ on M^n (cf. [14])*

$$\bar{\nabla}_{\bar{X}}\bar{\nabla}_{\bar{Y}}\xi = \bar{\nabla}_{\overset{*}{\nabla}_{\bar{X}}\bar{Y}}\xi + \frac{1}{2}\bar{R}(\bar{X},\bar{Y})\xi.$$

Proof. In this case, only (b) of Proposition 3.1 should be checked, which completes the proof. □

The result of Corollary 3.2 can be expressed in more geometrical terms. To do this, introduce a symmetric bilinear mapping $h_\xi : \mathfrak{X}(M^n)\times\mathfrak{X}(M^n) \to \mathfrak{X}(M^n)$ by

$$h_\xi(\bar{X},\bar{Y}) = \frac{1}{2}\Big[\bar{R}(\xi,\nabla_{\bar{X}}\xi)\bar{Y} + \bar{R}(\xi,\nabla_{\bar{Y}}\xi)\bar{X}\Big], \tag{10}$$

for all $\bar{X}$, $\bar{Y} \in \mathfrak{X}(M^n)$. Then the definition of the ξ-connection takes as similar form as the Gauss decomposition

$$\overset{*}{\nabla}_{\bar{X}}\ \bar{Y} = \bar{\nabla}_{\bar{X}}\bar{Y} + h_\xi(\bar{X},\bar{Y}). \tag{11}$$

Define a *"shape operator"* A_ξ for the field ξ by

$$A_\xi\bar{Y} = -\bar{\nabla}_{\bar{Y}}\xi, \quad \text{for all } \bar{Y} \in \mathfrak{X}(M^n). \tag{12}$$

Then the covariant derivative of the $(1,1)$-tensor field A_ξ is given by

$$(\bar{\nabla}_{\bar{X}}A_\xi)\bar{Y} = -\bar{\nabla}_{\bar{X}}\bar{\nabla}_{\bar{Y}}\xi + \bar{\nabla}_{\bar{\nabla}_{\bar{X}}\bar{Y}}\xi.$$

Hence we see that the Codazzi-type equation $\bar{R}(\bar{X},\bar{Y})\xi = (\bar{\nabla}_{\bar{Y}}A_\xi)\bar{X} - (\bar{\nabla}_{\bar{X}}A_\xi)\bar{Y}$ holds. In these notations

$$\bar{\nabla}_{\overset{*}{\nabla}_{\bar{X}}\bar{Y}}\xi + \frac{1}{2}\bar{R}(\bar{X},\bar{Y})\xi - \bar{\nabla}_{\bar{X}}\bar{\nabla}_{\bar{Y}}\xi = \bar{\nabla}_{h_\xi(\bar{X},\bar{Y})}\xi + \frac{1}{2}\Big[(\bar{\nabla}_{\bar{X}}A_\xi)\bar{Y} + (\bar{\nabla}_{\bar{Y}}A_\xi)\bar{X}\Big].$$

If we introduce a symmetric bilinear mapping $\Omega_\xi : \mathfrak{X}(M^n) \times \mathfrak{X}(M^n) \to \mathfrak{X}(M^n)$ defined by

$$\Omega_\xi(\bar{X},\bar{Y}) = \bar{\nabla}_{h_\xi(\bar{X},\bar{Y})}\xi + \frac{1}{2}\Big[(\bar{\nabla}_{\bar{X}}A_\xi)\bar{Y} + (\bar{\nabla}_{\bar{Y}}A_\xi)\bar{X}\Big],$$

then Corollary 3.2 can be reformulated as

Corollary 3.3. *If ξ is a smooth vector field on a Riemannian manifold M^n then $\xi(M^n)$ is totally geodesic in TM^n if and only if for any vector fields $\bar{X}, \bar{Y}$ on M^n*

$$\Omega_\xi(\bar{X}, \bar{Y}) = \bar{\nabla}_{h_\xi(\bar{X},\bar{Y})}\xi + \frac{1}{2}\Big[(\bar{\nabla}_{\bar{X}} A_\xi)\bar{Y} + (\bar{\nabla}_{\bar{Y}} A_\xi)\bar{X}\Big] \equiv 0, \qquad (13)$$

where h_ξ and A_ξ are defined by (10) *and* (12) *respectively.*

Remark. The statement of Proposition 3.1 can also be reformulated in these terms, namely, *let F^l be a submanifold in a Riemannian manifold M^n and ξ be a vector field on M^n along F^l. Then $\xi(F^l)$ is totally geodesic in TM^n if and only if F^l is totally geodesic with respect to the ξ-connection* (7) *and Ω_ξ vanishes on the tangent bundle of F^l*

Now, combining Theorem 2.1 with Proposition 3.1, we obtain

Corollary 3.4. *On a connected simply connected compact n-dimensional Riemannian manifold, vector fields satisfying* (b) *of Proposition 3.1 generate the only connected compact totally geodesic n-dimensional submanifolds of the tangent bundle which are transverse to fibers.*

As has been shown in [20], for the case of the unit tangent bundle, the Hopf vector fields on odd dimensional spheres generate totally geodesic submanifolds in T_1S^n. For the tangent bundle the situation is different.

Theorem 3.1. *A non-zero Killing vector field on a space of non-zero constant curvature (M^n, c) never generates a totally geodesic submanifold in TM^n. Moreover, a manifold with positive sectional curvature does not admit a non-zero Killing vector field with totally geodesic property.*

Proof. Let ξ be a Killing vector field on a space M^n of constant curvature c. Then A_ξ is a skew-symmetric linear operator, i.e.

$$\bar{g}(A_\xi \bar{X}, \bar{Y}) + \bar{g}(\bar{X}, A_\xi \bar{Y}) = 0, \qquad (14)$$

and moreover,

$$(\bar{\nabla}_{\bar{X}} A_\xi)\bar{Y} = \bar{R}(\xi, \bar{X})\bar{Y} \qquad (15)$$

for all vector fields $\bar{X}$, $\bar{Y}$ on M^n (cf. [8]). Since M^n is of non-zero constant curvature, the equation (13) can be simplified in the following way:

$$(\bar{\nabla}_{\bar{X}} A_\xi)\bar{Y} + (\bar{\nabla}_{\bar{Y}} A_\xi)\bar{X} = \bar{R}(\xi, \bar{X})\bar{Y} + \bar{R}(\xi, \bar{Y})\bar{X}$$

$$= c\Big[2\bar g(\bar X,\bar Y)\xi - \bar g(\xi,\bar X)\bar Y - \bar g(\xi,\bar Y)\bar X\Big]$$

$$\begin{aligned}\bar R(\xi,\bar\nabla_{\bar X}\xi)\bar Y + \bar R(\xi,\bar\nabla_{\bar Y}\xi)\bar X &= c\Big[\bar g(\bar\nabla_{\bar X}\xi,\bar Y) + \bar g(\bar X,\bar\nabla_{\bar Y}\xi)\bar X)\Big]\xi\\ &\quad - c\Big[(\bar g(\xi,\bar X)\bar\nabla_{\bar Y}\xi + \bar g(\xi,\bar Y)\bar\nabla_{\bar X}\xi)\Big]\\ &= c\Big[\bar g(\xi,\bar X)A_\xi\bar Y + \bar g(\xi,\bar Y)A_\xi\bar X\Big].\end{aligned}$$

So, ξ is totally geodesic if

$$\bar g(\xi,\bar X)\bar Y + \bar g(\xi,\bar Y)\bar X - \bar\nabla_{\bar g(\xi,\bar X)A_\xi\bar Y + \bar g(\xi,\bar Y)A_\xi\bar X}\xi = 2\bar g(\bar X,\bar Y)\xi,$$

or

$$\bar g(\xi,\bar X)\Big[\bar Y + A_\xi(A_\xi\bar Y)\Big] + \bar g(\xi,\bar Y)\Big[\bar X + A_\xi(A_\xi\bar X)\Big] = 2\bar g(\bar X,\bar Y)\xi,$$

for all vector fields $\bar X$, $\bar Y$ on M^n. Choosing $\bar X$, $\bar Y$ such that $\bar X_x \neq 0$ and $\bar X_x = \bar Y_x \perp \xi_x$, we get $2|\bar X_x|^2\xi_x = 0$. Therefore, $\xi = 0$ for all $x \in M^n$.

Let ξ be a non-zero Killing vector field on a manifold with *positive* (non-constant) sectional curvature. From (14) it follows that $A_\xi\xi \perp \xi$. If $A_\xi\xi = 0$, then, after setting $Y = \xi$ in (14), we conclude that ξ has a constant length and therefore can be totally geodesic if it is a parallel vector field [14]. In this case, $M^n = M^{n-1}\times E^1$ and we come to a contradiction. Suppose that $A_\xi\xi \neq 0$. Then $\xi \wedge A_\xi\xi$ is a non-zero bivector field. Setting $\bar Y = \bar X$ in (13) and using (15), we have

$$A_\xi\Big[\bar R(\xi, A_\xi\bar X)\bar X\Big] + \bar R(\xi,\bar X)\bar X = 0.$$

Taking a scalar product in both sides with ξ and applying (14), we get

$$-\bar g(\bar R(\xi, A_\xi\bar X)\bar X, A_\xi\xi) + K_{\xi\wedge\bar X}|\xi\wedge\bar X|^2 = 0.$$

Finally, setting $\bar X = A_\xi\xi$, we have $K_{\xi\wedge\bar X} = 0$ and come to a contradiction. □

The next theorem is analogous to the one proved by P. Walczak [14], but does not have similar rigid consequences for the structure of M^n.

Theorem 3.2. *Let ξ be a vector field of constant length along a submanifold $F^l \subset M^n$. Then $\xi(F^l)$ is a totally geodesic submanifold in TM^n if and only if F^l is totally geodesic in M^n and ξ is a parallel vector field on M^n along F^l.*

Proof. The condition $|\xi| = \text{const}$ implies $\bar{g}(\bar{\nabla}_X\xi, \xi) = 0$ for any vector field X tangent to F^l. As $\xi(F^l)$ is supposed to be totally geodesic, it follows from the second condition of Proposition 3.1 that $\bar{g}(\bar{\nabla}_X\bar{\nabla}_Y\xi, \xi) = 0$. Hence $\bar{g}(\bar{\nabla}_X\xi, \bar{\nabla}_Y\xi) = 0$ for any $X, Y \in T_xF^l$, $x \in F^l$. Supposing $X = Y$, we see that $\bar{\nabla}_X\xi = 0$, i.e. ξ is parallel along F^l in the ambient space and the second condition of Proposition 3.1 is fulfilled. Moreover, the condition $\bar{\nabla}_X\xi = 0$ means that the ξ-connection (7) coincides with the Levi–Civita connection of M^n, so that by Proposition 3.1 F^l is totally geodesic in M^n.

On the other hand, if F^l is totally geodesic in M^n and $\bar{\nabla}_X\xi = 0$ for any tangent vector field X on F^l, then both conditions from Proposition 3.1 are satisfied evidently. □

Giving more restrictions on the vector field, we can a more geometrical result.

Theorem 3.3. *Let ξ be a normal vector field on a submanifold $F^l \subset M^n$, which is parallel in the normal bundle. Then $\xi(F^l)$ is totally geodesic in TM^n if and only if F^l is totally geodesic in M^n.*

Proof. If ξ is a normal vector field to F^l and parallel in the normal bundle, then $\bar{\nabla}_X\xi = -A_\xi X$ for each vector field X on F^l, where A_ξ is the shape operator of F^l with respect to ξ, and hence $\bar{g}(\bar{\nabla}_X\xi, \xi) = 0$. This means that $|\xi| = \text{const}$ along F^l.

Let $\xi(F^l)$ be totally geodesic in TM^n. Then from (b) of Proposition 3.1 we see that $\bar{g}(\bar{\nabla}_X\bar{\nabla}_Y\xi, \xi) = 0$, which implies $|\bar{\nabla}_X\xi| = 0$ for each X tangent to F^l. In this case, along F^l the ξ-connection (7) coincides with the Levi–Civita connection of M^n and (a) of Proposition 3.1 implies the totally geodesic property of F^l.

Conversely, if ξ is a normal vector field which is parallel in the normal bundle of F^l and F^l is totally geodesic, then $\bar{\nabla}_X\xi = 0$ for any vector field X tangent to F^l. Evidently, both conditions of Proposition 3.1 are fulfilled. □

The application of Theorem 3.3 to the specific case of a foliated Riemannian manifold allows to clarify the geometrical structure of $\xi(M^n)$. The manifold M^n is said to be *ν-foliated* if it admits a family $\mathcal{F}$ of connected ν-dimensional submanifolds $\{\mathcal{F}_\alpha; \alpha \in A\}$ called *leaves* such that (i) $M^n = \bigcup_{\alpha\in A}\mathcal{F}_\alpha$; (ii) $\mathcal{F}_\alpha \cap \mathcal{F}_\beta = \emptyset$ for $\alpha \neq \beta$; (iii) there exists a coordinate covering $\mathcal{U}$ of M^n such that in each local chart $U \in \mathcal{U}$ the leaves can be expressed locally as level submanifolds, i.e. $u^{\nu+1} = c_{\nu+1}, \dots, u^n = c_n$.

The family $\mathcal{F}$ is called a *ν-foliation* and *hyperfoliation* for $\nu = n-1$. The hyperfoliation is said to be *transversally orientable* if M^n admits a vector field ξ transversal to the leaves. Moreover, with respect to the Riemannian metric on M^n, this vector field can be chosen as a field of unit normals for each leaf.

A submanifold $F^{k+\nu} \subset M^n$ is called *ν-ruled* if $F^{k+\nu}$ admits a ν-foliation $\{\mathcal{F}_\alpha; \alpha \in A\}$ such that each leaf $\mathcal{F}_\alpha$ is totally geodesic in M^n. The leaves $\mathcal{F}_\alpha$ are called *elements* or *generators* [11].

Corollary 3.5. *Let M^n be a Riemannian manifold admitting a totally geodesic transversally orientable hyperfoliation $\mathcal{F}$. Let ξ be a field of normals of the foliation having constant length. Then $\xi(M^n)$ is an $(n-1)$-ruled submanifold in TM^n with the elements $\xi(\mathcal{F}_\alpha)$.*

Proof. Indeed, let $\mathcal{F}_\alpha$ be a leaf of the hyperfoliation and ξ be a vector field of constant length on M^n which is a field of normals along each leaf. Applying Theorem 3.3, we get that $\xi(\mathcal{F}_\alpha)$ is totally geodesic in TM^n for each α. Since $\xi : M^n \to \xi(M^n)$ is a homeomorphism, $\xi(\mathcal{F}_\alpha) \cap \xi(\mathcal{F}_\beta) = \emptyset$ for $\alpha \neq \beta$ and $\xi(M^n) = \bigcup_{\alpha \in A} \xi(\mathcal{F}_\alpha)$. Finally, if $\mathcal{F}_\alpha$ is given by $u^n = c_n$ within a local chart U then from (4) we see that $\xi(\mathcal{F}_\alpha)$ is given by the same equalities within the local chart $\xi(U)$. So, $\xi(\mathcal{F}) = \{\xi(\mathcal{F}_\alpha); \alpha \in A\}$ form a hyperfoliation on $\xi(M^n)$ with totally geodesic leaves in TM^n. $\square$

4. The case of a base space of constant curvature

If the ambient space is of constant curvature $c \neq 0$ and ξ is a normal vector field on a submanifold $F^l \subset M^n$, then the necessary and sufficient condition on ξ to generate a totally geodesic submanifold in TM^n takes a rather simple form.

Theorem 4.1. *Let F^l be a submanifold of a space $M^n(c)$ of constant curvature $c \neq 0$. Let ξ be a normal vector field on F^l. Then $\xi(F^l)$ is totally geodesic in TM^n if and only if F^l is totally geodesic in $M^n(c)$ and ξ is parallel in the normal bundle.*

Proof. The curvature tensor of $M^n(c)$ is of the form

$$\bar R(\bar X,\bar Y)\bar Z = c(\bar g(\bar Y,\bar Z)\bar X - \bar g(\bar X,\bar Z)\bar Y). \tag{16}$$

If ξ is a normal vector field on F^l then $\bar\nabla_X\xi = -A_\xi X + \nabla^\perp_X\xi$. As $A_\xi X$ is tangent and $\nabla^\perp_X\xi$ is normal to F^l, from (16) we find

$$\bar R(\xi,\bar\nabla_X\xi)Y = -cg(A_\xi X,Y)\xi$$

for any vector fields X, Y on F^l. Thus, the conditions from Proposition 3.1 mean that

$$\begin{cases} \bar\nabla_X Y - cg(A_\xi X,Y)\xi \text{ is tangent to } F^l, \\ \bar\nabla_{\bar\nabla_X Y - cg(A_\xi X,Y)\xi}\xi = \bar\nabla_X\bar\nabla_Y\xi. \end{cases} \tag{17}$$

Multiplying $(17)_1$ by ξ and by normal vector field η orthogonal to ξ, we have

$$\begin{cases} g(A_\xi X,Y)(1-c|\xi|^2)=0, \\ g(A_\eta X,Y)=0. \end{cases}$$

If ξ is of constant length $|\xi|^2=\frac{1}{c}$ $(c>0)$ then by Theorem 3.2, F^l is totally geodesic in M^n, otherwise F^l is totally geodesic immediately.

So, F^l is totally geodesic and therefore $\bar\nabla_X\xi = \nabla^\perp_X\xi$, $\bar\nabla_X Y = \nabla_X Y$. The condition $(17)_2$ now takes the form

$$\nabla^\perp_{\nabla_X Y}\xi = \nabla^\perp_X\nabla^\perp_Y\xi. \tag{18}$$

Set $Y=\nabla_V Z$, where V and Z are arbitrary vector fields tangent to F^l. Then from (18), we get

$$\nabla^\perp_{\nabla_X\nabla_V Z}\xi = \nabla^\perp_X\nabla^\perp_{\nabla_V Z}\xi.$$

Applying (18) to $\nabla^\perp_{\nabla_V Z}\xi$ in the right-hand side of the above equation, we see that $\nabla^\perp_{\nabla_V Z}\xi = \nabla^\perp_V\nabla^\perp_Z\xi$ and therefore,

$$\nabla^\perp_{\nabla_X\nabla_V Z}\xi = \nabla^\perp_X\nabla^\perp_V\nabla^\perp_Z\xi. \tag{19}$$

Interchanging the roles of X and V, we get

$$\nabla^\perp_{\nabla_V\nabla_X Z}\xi = \nabla^\perp_V\nabla^\perp_X\nabla^\perp_Z\xi. \tag{20}$$

Finally, applying again (18) to the bracket $[X,V]$ and Z, we get

$$\nabla^{\perp}_{\nabla_{[X,V]}Z}\xi = \nabla^{\perp}_{[X,V]}\nabla^{\perp}_{Z}\xi. \tag{21}$$

Combining (19), (20) and (21), we obtain

$$\nabla^{\perp}_{R(X,V)Z}\xi = R^{\perp}(X,V)\nabla^{\perp}_{Z}\xi$$

where R is the curvature tensor of F^l and $R^{\perp}$ is the normal curvature tensor. Since F^l is totally geodesic and $M^n(c)$ is of constant curvature, $R^{\perp}(X,Y)\eta \equiv 0$ for any normal vector field η and, moreover,

$$R(X,Y)Z = c(g(Y,Z)X - g(X,Z)Y).$$

So, we have

$$c\nabla^{\perp}_{g(Y,Z)X-g(X,Z)Y}\xi = 0.$$

Setting X orthogonal to Y and $Y = Z$ we get $\nabla^{\perp}_{X}\xi = 0$ for any vector field X on F^l, which completes the necessary part of the proof. The sufficient part is trivial. □

The application of Theorem 4.1 to the case of a space of constant curvature shows the difference between our considerations and WALCZAK's [14]. Let S^n be the unit sphere and S^{n-1} be the unit totally geodesic great sphere in S^n. Denote by D^n an open equatorial zone around S^{n-1} where the unit geodesic vector field orthogonal to S^{n-1} is regularly defined. Then D^n is a Riemannian manifold of constant positive curvature and S^{n-1} is a totally geodesic submanifold in D^n.

Let ξ be a unit (or of constant length) geodesic vector field on $D^n \subset S^n$ which is normal to the totally geodesic great sphere S^{n-1}. Then $\xi(D^n)$ is not totally geodesic in TD^n while the restriction of ξ to S^{n-1} generates the totally geodesic submanifold $\xi(S^{n-1})$ in TD^n.

Indeed, ξ is of constant length and by Walczak's result, $\xi(D^n)$ can be totally geodesic in TD^n only if ξ is a parallel vector field on D^n [14], which is impossible due to positive curvature of D^n. On the other hand, ξ is parallel in the normal bundle of $S^{n-1} \subset D^n$ and we can apply Theorem 4.1 to see that $\xi(S^{n-1})$ is totally geodesic in TD^n.

As concerns flat Riemannian manifolds, WALCZAK has shown that every totally geodesic vector field on a flat Riemannian manifold is harmonic (cf. [14]) and that, consequently, on a compact flat Riemannian manifold, a vector field is totally geodesic if and only if it is parallel. We shall give a similar result for vector fields along submanifolds.

Theorem 4.2. *Let F^l be a compact oriented submanifold in a flat Riemannian manifold M^n. Let ξ be a vector field on F^l. Then $\xi(F^l)$ is totally geodesic in TM^n if and only if F^l is totally geodesic in M^n and ξ is parallel along F^l.*

Proof. Since M^n is flat, the ξ-connection is the same as the Levi–Civita connection on M^n. So, by Proposition 3.1, $\xi(F^l)$ is totally geodesic if and only if F^l is totally geodesic and

$$\bar{\nabla}_X \bar{\nabla}_Y \xi = \bar{\nabla}_{\bar{\nabla}_X Y} \xi \tag{22}$$

for all vector fields X and Y on F^l.

Suppose now that $\xi(F^l)$ is totally geodesic. Then F^l is totally geodesic and is thus flat. Hence locally we can choose vector fields $X_1, X_2, \dots, X_l$ tangent to F^l such that $\bar{\nabla}_{X_i} X_j = \nabla_{X_i} X_j = 0$, and $\bar{g}(X_i, X_j) = g(X_i, X_j) = \delta_{ij}$, for all $i, j = 1, \dots, l$. Putting $X = Y = X_i$ in the identity (22), we obtain $\bar{\nabla}_{X_i} \bar{\nabla}_{X_i} \xi = 0$. Hence, $\sum_{i=1}^l \bar{g}(\bar{\nabla}_{X_i} \bar{\nabla}_{X_i} \xi, \xi) = 0$, i.e.

$$\sum_{i=1}^{l} X_i.\bar{g}(\bar{\nabla}_{X_i}\xi, \xi) = \sum_{i=1}^{l} |\bar{\nabla}_{X_i}\xi|^2. \tag{23}$$

If we consider the function f defined by $f(x) = \frac{1}{2}\bar{g}_x(\xi, \xi)$, for all $x \in F^l$, then we can define a global vector field X_f on F^l by the local formula $X_f = g(\bar{\nabla}_{X_i}\xi, \xi)X_i$. Formula (23) can thus be written locally as $\operatorname{div} X_f = \sum_{i=1}^l |\bar{\nabla}_{X_i}\xi|^2$.

Integrating both sides of the last equality and applying Green's theorem, we obtain $\sum_{i=1}^l \int_{F^l} |\bar{\nabla}_{X_i}\xi|^2 dv = 0$, and hence $\bar{\nabla}_{X_i}\xi = 0$, for all $i = 1, \dots, l$. Therefore ξ is parallel along F^l.

The sufficient part of the theorem is trivial. □

Remarks. 1. If in Theorem 4.2 the field ξ is a normal vector field along F^l, then $\bar{\nabla}_X \xi$ is also normal for each vector field X on F^l. Indeed, for the X_i's constructed in the proof of the theorem, we have $\bar{g}(\bar{\nabla}_{X_i}\xi, X_j) = X_i.\bar{g}(\xi, X_j) = 0$, and so $\bar{\nabla}_{X_i}\xi$ is normal to F^l. Hence the identity (22) can be written as

$$\bar{\nabla}^{\perp}_{X}\bar{\nabla}^{\perp}_{Y}\xi = \bar{\nabla}_{\nabla_X Y}\xi. \tag{24}$$

Also, ξ is parallel if and only if it is parallel in the normal bundle. Hence $\xi(F^l)$ is totally geodesic if and only if F^l is totally geodesic and ξ is parallel in the normal bundle.

2. The condition of compactness is necessary. Indeed, if we consider $\mathbb{R}^n$ with its canonical coordinates $(x_1, x_2, \dots, x_n)$ and its canonical Euclidean metric, and the hypersurface $\mathbb{R}^{n-1}$ which is identified with the subspace given by: $x_n = 0$, then $\mathbb{R}^{n-1}$ is an oriented totally geodesic submanifold of $\mathbb{R}^n$. We have $\bar{\nabla}_{\partial/\partial x_i}\partial/\partial x_j = 0$ for all $i, j = 1, \dots, n$. We consider the vector field ξ on $\mathbb{R}^n$ along $\mathbb{R}^{n-1}$ defined by $\xi(x) = x_1\partial/\partial x_n(x)$, where x_1 is the first component of x. Now, to show that $\xi(\mathbb{R}^{n-1})$ is totally geodesic in $T\mathbb{R}^n$, it suffices to check that (22) is verified. In fact, $\bar{\nabla}_{\partial/\partial x_i}\bar{\nabla}_{\partial/\partial x_j}\xi = \bar{\nabla}_{\partial/\partial x_i}\delta_{1j}\partial/\partial x_n = 0$. But $\bar{\nabla}_{\partial/\partial x_1}\xi = \partial/\partial x_n$, and so ξ is not parallel.

5. The case of Lie groups with bi-invariant metrics

Let us consider a connected Lie group G^n equipped with a bi-invariant metric $\bar{g}$, i.e. invariant by both left and right translations. We shall generalize the results of P. Walczak [14] on totally geodesic left invariant vector fields on G^n to left invariant vector fields along Lie subgroups.

Let H^l be a Lie subgroup of G^n. The metric g induced from $\bar{g}$ on H^l is a bi-invariant metric. If we denote by $\bar{\nabla}$ and ∇ the Levi–Civita connections on G^n and H^l respectively, then we have $\bar{\nabla}_X Y = \frac{1}{2}[X, Y]$, for all X, Y of $\mathfrak{g}$, the Lie algebra of G^n, and $\nabla_X Y = \frac{1}{2}[X, Y]$, for all X, Y of $\mathfrak{h}$, the Lie algebra of H^l.

Lemma 5.1. *A connected complete submanifold F^l of G^n containing the identity element e of G^n, such that T_eF^l is a subalgebra of $\mathfrak{g}$, is totally geodesic if and only if F^l is a Lie subgroup H^l of G^n.*

Proof. If we denote by exp the exponential mapping $\exp : \mathfrak{g} \to G^n$ of the Lie group G^n, and by $\exp_x : T_xG^n \to G^n$ the exponential map at a point x of G^n with respect to the Levi–Civita connection of the metric g, then for all $x \in G^n$, $\exp_x = \exp \circ (L_{x^{-1}})_*$, where L_x is the left translation of G^n by x. Indeed, we show firstly that $\exp_e = \exp$. Let $X \in \mathfrak{g} \equiv T_eG^n$ and $\gamma(t) = \exp tX$. It suffices to check that γ is a geodesic. We have $\dot\gamma(t) = (L_{\gamma(t)})_*(\dot\gamma(0)) = (L_{\gamma(t)})_*(X)$, and thus $\bar\nabla_{\dot\gamma(t)}\dot\gamma(t) = \bar\nabla_{X(\gamma(t))}X(\gamma(t))$, where X denotes also the left invariant vector field on G^n corresponding to X. Hence $\bar\nabla_{\dot\gamma(t)}\dot\gamma(t) = \frac{1}{2}[X,X](\gamma(t)) = 0$, and so $\exp_e = \exp$. Now, our assertion follows from the fact that left translations are isometries.

We consider a Lie subgroup H^l of G^n and $\mathfrak{h} = T_eH^l$ its Lie algebra. If $X \in \mathfrak{h}$, then $\exp_e tX = \exp tX \in H^l$, for all t in a neighborhood of 0, i.e. H^l contains the geodesic starting from e and with initial condition X, and by the left translations, H^l contains all geodesics starting from points of H^l with initial vectors tangent to H^l at these points. Thus H^l is totally geodesic.

Conversely, suppose that F^l is a connected complete submanifold of G^n such that $e \in F^l$ and $T_eF^l =: \mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Let H^l be the connected subgroup of G^n with Lie algebra $\mathfrak{h}$. H^l is then a connected totally geodesic submanifold of G^n with $T_eH^l = T_eF^l$. Therefore $H^l = F^l$. □

Proposition 5.1. *A left invariant vector field on G^n along a submanifold F^l generates a totally geodesic submanifold of TG^n if and only if it is parallel along F^l and F^l is totally geodesic.*

Proof. A left invariant vector field on G^n is necessarily of constant length, and we apply Theorem 3.2. □

Corollary 5.1. *A left invariant vector field ξ on G^n along a Lie subgroup H^l is totally geodesic if and only if it is an element of the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$.*

Proof. By Lemma 5.1, H^l is a totally geodesic submanifold in G^n. Thus, by virtue of Proposition 5.1, ξ is totally geodesic if and only if ξ is parallel along H^l.

Suppose that ξ is totally geodesic. Then $\bar\nabla_X\xi = 0$, for all $X \in \mathfrak{h}$; i.e. ξ is in the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$.

Conversely, if ξ is in the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$, then $\bar\nabla_X\xi = 0$, for all $X \in \mathfrak{h}$. Let $x \in H^l$ and $z \in T_xH^l$. It suffices to prove that $\bar\nabla_z\xi = 0$. But $X := (L_{x^{-1}})_*(z) \in T_eH^l \equiv \mathfrak{h}$, and consequently $\bar\nabla_z\xi = (\bar\nabla_X\xi)(x) = 0$. □

Corollary 5.2. (a) *There are no non-zero left invariant totally geodesic vector fields on a semi-simple Lie subgroup of a Lie group with a bi-invariant Riemannian metric.*

(b) *Every left invariant vector field along a subgroup of an abelian Lie group with a bi-invariant Riemannian metric generates a totally geodesic submanifold of the tangent bundle.*

Theorem 5.1. *Let N^l be a connected complete totally geodesic embedded submanifold of the tangent bundle of a connected Lie group G^n equipped with a bi-invariant Riemannian metric such that $H^l = \pi(N^l)$ is a Lie subgroup of G^n. Suppose that N^l is horizontal at a point z of T_eG^n.*

(a) *If $z \in T_eH^l$, then N^l is the image of H^l by a left invariant vector field on H^l which belongs to the center of $\mathfrak{h}$. In particular, if H^l is semi-simple, then H^l is the only connected totally geodesic embedded submanifold of TG^n which is tangent to H^l at e and orthogonal to the fiber at a point of T_eG^n.*

(b) *If H^l is simple, then N^l is the image of H^l by a left invariant vector field on G^n along H^l which belongs to the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$.*

Proof. Using Proposition 2.1, there is a neighborhood U of e in G^n, a neighborhood V of z in TG^n and a vector field Y on M^n along $H^l \cap U$ such that $N^l \cap V = Y(H^l \cap U), Y(e) = z$. We have $T_zN^l = T_z(N^l \cap V) = T_zY(H^l \cap U)$. Then each vector of T_zN^l can be written as $X^h + (\bar{\nabla}_X Y)^v$, for some $X \in \mathfrak{h}$. But T_zN^l is a subset of the horizontal subspace of TTG^n at z, so at e we have $\bar{\nabla}_X Y = 0$ for all $X \in \mathfrak{h}$. On the other hand, since $N^l \cap V = Y(H^l \cap U)$ is totally geodesic, the second assertion of Proposition 3.1 reduces at e to the identity

$$\bar{\nabla}_{X_1}\bar{\nabla}_{X_2}Y = \frac{1}{2}\bar{R}(X_1, X_2)Y, \text{ for all vector fields } X_1, X_2 \text{ on } H^l.$$

Then for all $W \in \mathfrak{g} = T_eG^n$, we have

$$\bar{g}(\bar{\nabla}_{X_1(e)}\bar{\nabla}_{X_2}Y, W) = \frac{1}{2}\bar{g}(\bar{R}(X_1(e), X_2(e))Y(e), W).$$

If we extend W to a vector field X_3 along H^l, which is orthogonal to $\bar{\nabla}_{X_2}Y$ in a neighborhood of e in H^l, then we can write

$$\bar{g}(\bar{\nabla}_{X_1(e)}\bar{\nabla}_{X_2}Y, W) = -\bar{g}(\bar{\nabla}_{X_2(e)}Y, \bar{\nabla}_{X_1(e)}X_3) = 0,$$

and consequently, $\bar{g}(\bar{R}(X_1(e), X_2(e))Y(e), W) = 0$, for all $X_1(e), X_2(e) \in \mathfrak{h} = T_eH^l$ and $W \in \mathfrak{g} = T_eG^n$. Therefore we have

$$R(\cdot,\cdot)Y(e) = 0, \text{ when applied to vectors in } T_eH^l.$$

Let us denote by ξ the left invariant vector field on G^n along H^l such that $Y(e) = \xi(e)$. Then $\bar{R}(\cdot,\cdot)\xi(e) = 0$ when applied to vectors in T_eH^l, and hence

$$\bar{R}(\cdot,\cdot)\xi = 0, \text{ when applied to elements of } \mathfrak{h}. \tag{25}$$

Consider now two cases.

(a) If $\xi(e) = z \in T_eH^l$, then $\xi \in \mathfrak{h}$, and we have, by virtue of (25), $\bar{R}(X,\xi)\xi = 0$, for all $X \in \mathfrak{h}$. Thus $|\,[\xi, X]\,|^2 = 4\bar{g}(\bar{R}(\xi, X)X, \xi) = 0$ for all $X \in \mathfrak{h}$. It follows that ξ belongs to the center of $\mathfrak{h}$.

(b) If H^l is simple, then $[\mathfrak{h},\mathfrak{h}] = \mathfrak{h}$. But $\bar{\nabla}_{[X_1,X_2]}\xi = \frac{1}{2}[[X_1, X_2], \xi] = -2R(X_1, X_2)\xi = 0$, for all X_1, $X_2 \in \mathfrak{h}$, by virtue of (25). Since $[\mathfrak{h},\mathfrak{h}] = \mathfrak{h}$, we deduce easily that $\bar{\nabla}_X\xi = 0$, for all $X \in \mathfrak{h}$, or equivalently $[X, \xi] = 0$, for all $X \in \mathfrak{h}$. It follows that ξ belongs to the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$.

In both cases, ξ belongs to the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$. Hence, by Lemma 5.1, H^l is totally geodesic in G^n, and Proposition 5.1 implies then that $\xi(H^l)$ is a complete totally geodesic submanifold of TG^n. Therefore $\xi(H^l) = N^l$, because $\xi_*(T_eH^l) = T_zN^l$ and N^l and H^l are connected. □

Corollary 5.3. *Let N^l be a connected complete horizontal totally geodesic submanifold of the tangent bundle of a connected Lie group G^n equipped with a bi-invariant Riemannian metric such that $H^l = \pi(N^l)$ is a simply connected submanifold of G^n containing the identity element. Suppose that $\mathfrak{h} := \pi_*(T_zN^l)$ is a Lie subalgebra of $\mathfrak{g}$ for a point z of $T_eG^n \cap N^l$. If $Z \in T_eH^l$ (resp. $\mathfrak{h}$ is simple), then H^l is a Lie subgroup of G^n and N^l is the image of H^l by a left invariant vector field on H^l (resp. on G^n along H^l) which belongs to the center of $\mathfrak{h}$ (resp. centralizer of $\mathfrak{h}$ in $\mathfrak{g}$).*

Proof. By Theorem 2.3, H^l is complete and totally geodesic. It follows from Lemma 5.1 that H^l is a Lie subgroup of G^n. Now, our corollary follows from Theorem 5.1. □

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