Publ. Math. Debrecen
64/1-2 (2004), 167–187

Distribution of additive and q-additive functions under some conditions

By I. KÁTAI (Budapest) and M. V. SUBBARAO (Edmonton)

Abstract. It is proved that an additive arithmetical function f under the fulfilment of the conditions of the Erdős–Wintner theorem has a limit distribution on the subset of the integers $\{n \le x \mid \omega(n) = k\}$, where $k = k(x) = (1 + o(1)) \log\log x$, and $\omega(n) =$ number of prime divisors of n. Similar theorems are proved for multiplicative and q-additive and q-multiplicative functions.

1. Introduction

1.1. Let $q \ge 2$ be an integer, the q-ary expansion of some $n \in \mathbb{N}_0$ let be defined as

$$n = \sum_{j=0}^{\infty} \varepsilon_j(n) q^j, \tag{1.1}$$

where the digits $\varepsilon_j(n)$ are taken from the set $\mathbb{A}_q = \{0, 1, \dots, q-1\}$. Let $\mathcal{A}_q$ be the set of q-additive functions, and $\overline{\mathcal{M}}_q$ be the set of q-multiplicative

Mathematics Subject Classification: 11K65, 11N60.
Key words and phrases: limit distribution, additive functions, q-additive functions, special conditions.
It was written while the author visited Alberta University for the kind invitation of the second named author funded by his NSERC Grant. Financially supported furthermore by OTKA T 031877, and the fund of Applied Number Theory Research Group of HAS.
Financially supported by an NSERC Grant of the author.

functions of modulus 1. $f : \mathbb{N}_0 \to \mathbb{R}$ belongs to $\mathcal{A}_q$, if $f(0) = 0$, and

$$f(n) = \sum_{j=0}^{\infty} f(\varepsilon_j(n) q^j).$$

We say that $g : \mathbb{N}_0 \to \mathbb{C}$ belongs to $\overline{\mathcal{M}}_q$, if $g(0) = 1$, $|g(bq^j)| = 1$ for every $b \in \mathbb{A}_q$, and

$$g(n) = \prod_{j=0}^{\infty} g(bq^j).$$

Let

$$\alpha(n) = \sum_{j=0}^{\infty} \varepsilon_j(n)$$

the so called "sum of digits" function and let

$$\beta_h(n) = \sum_{\varepsilon_j(n)=h} 1 \qquad (h = 1, \dots, q-1).$$

H. Delange [1] proved that for some $g \in \overline{\mathcal{M}}_q$, the limit

$$\lim \frac{1}{x} \sum_{n \le x} g(n) = M(g)$$

exists and $M(g) \neq 0$, if

$$m_j := \frac{1}{q} \sum_{c \in \mathbb{A}_q} g(cq^j) \neq 0 \qquad (j = 0, 1, 2, \dots) \tag{1.2}$$

and

$$\sum_{j=0}^{\infty} (1 - m_j) = \sum_{j=0}^{\infty} \frac{1}{q} \Big(\sum_{c \in \mathbb{A}_q} \big(1 - g(cq^j)\big) \Big) \tag{1.3}$$

is convergent. Furthermore,

$$M(g) = \prod_{j=0}^{\infty} m_j,$$

if (1.2) holds and (1.3) is convergent.

Hence he deduced that for $f \in \mathcal{A}_q$ the values $f(n)$ possess a limit distribution if and only if both of the next series are convergent:

$$\sum_j \sum_{b \in \mathbb{A}_q} f(bq^j), \tag{1.4}$$

$$\sum_j \sum_{b \in \mathbb{A}_q} f^2(bq^j). \tag{1.5}$$

Let

$$F(y) := \lim_{x \to \infty} \frac{1}{x} \# \{n < x \mid f(n) < y\}. \tag{1.6}$$

For some x and q let $N(x) = \left[\frac{\log x}{\log q}\right]$. Thus $N(q^N) = N$.

Let furthermore

$$M\left(N \mid r_1, r_2, \ldots, r_{q-1}\right) \tag{1.7}$$

be the number of integers $n < q^N$ for which $\beta_l(n) = r_l$ $(l = 1, \ldots, q-1)$. It is clear that (1.7) is equal to

$$\frac{N!}{r_0! r_1! r_2! \ldots r_{q-1}!},$$

where $r_0 := N - (r_1 + r_2 + \ldots + r_{q-1})$.

Let furthermore $S_N(\underline{r})$ be the set of the integers $n < q^N$ for which $r_j = \beta_j(n)$ $(j = 1, \ldots, q-1)$, $r_0 = N - (r_1 + \ldots + r_{q-1})$.

Let δ_N be a sequence tending to zero, and $\underline{r}$ be such a vector (for some N), for which

$$\left|\frac{qr_j}{N} - 1\right| < \delta_N \qquad (j = 0, 1, \ldots, q-1) \tag{1.8}$$

holds.

Theorem 1. *Assume that $f \in \mathcal{A}_q$ and that* (1.4) *and* (1.5) *are convergent. Let $r^{(N)} = \left(r_0^{(N)}, r_1^{(N)}, \ldots, r_{q-1}^{(N)}\right)$ be such a sequence of $\underline{r}$ for which* (1.8) *holds. Then*

$$\lim_{N \to \infty} \frac{1}{M\left(N \mid r_1^{(N)}, \ldots, r_{q-1}^{(N)}\right)} \#\left\{n < q^N \; n \in S_N\left(\underline{r}^{(N)}\right) \mid f(n) < y\right\} = F(y).$$

Theorem 2. *Let $g \in \overline{\mathcal{M}}_q$, such that* (1.2) *holds and* (1.3) *is convergent. Let $\underline{r}^{(N)}$ be a sequence of $\underline{r}$ satisfying the condition* (1.8)*. Then*

$$\frac{1}{M\left(N \mid r_1^{(N)}, \ldots, r_{q-1}^{(N)}\right)} \sum_{n \in S_N\left(\underline{r}^{(N)}\right)} g(n) = (1 + o_N(1))\, M(g).$$

Theorem 3. *Let $q = 2$, $f \in \mathcal{A}_2$, $f(2^j) = O(1)$ $(j \in \mathbb{N})$, $\eta_N = \frac{1}{N} \sum_{j=0}^{N-1} f(2^j)$,*

$$B_N^2 := \frac{1}{4} \sum_{j=0}^{N-1} \left(f(2^j) - \eta_N\right)^2.$$

Assume that $B_N \to \infty$.

Let $\rho_N \to 0$. Then

$$\lim_{N \to \infty} \frac{1}{\binom{N}{k}} \left\{ n < 2^N \;\middle|\; \frac{f(n) - k\eta_N}{B_N} < y,\ \alpha(n) = k \right\} = \phi(y), \tag{1.9}$$

uniformly as $N \to \infty$, and $k = k^{(N)}$ satisfies

$$|k/N - 1/2| < \rho_N. \tag{1.10}$$

1.2. Let $\mathcal{A}$ be the set of real valued additive and $\mathcal{M}$ be the complex valued multiplicative functions. We say that $f \in \mathcal{A}$, if $f(mn) = f(m) + f(n)$ holds for all coprime pairs of m, n. We say that $g \in \mathcal{M}$ if $g(mn) = g(m) \cdot g(n)$ whenever $(m, n) = 1$, and $g(1) = 1$.

Let $\overline{\mathcal{M}} \subseteq \mathcal{M}$ be the set of g for which additionally $|g(n)| = 1$ $(n \in \mathbb{N})$ holds.

Let $\omega(n)$ be the number of prime factors, $\Omega(n)$ be the number of prime power divisors of n. Then $\omega, \Omega \in \mathcal{A}$.

Let $U_k = \{n \mid \omega(n) = k\}$, $V_k = \{n \mid \Omega(n) = k\}$, furthermore $\pi_k(x) = \#\{n \leq x \mid n \in U_k\}$, $\prod_k(x) = \#\{n \leq x \mid n \in V_k\}$. For the sake of simplicity let $x_1 := \log x$, $x_2 = \log x_1$.

By using a theorem of J. Kubilius [2], one can prove that

$$\pi_k(x) = \frac{x}{x_1} \frac{x_2^{k-1}}{(k-1)!} \left(1 + O\left(\frac{1}{\sqrt{x_2}}\right)\right) \tag{1.11}$$

$$\prod\nolimits_k(x) = \frac{x}{x_1} \frac{x_2^{k-1}}{(k-1)!} \left(1 + O\left(\frac{1}{\sqrt{x_2}}\right)\right) \tag{1.12}$$

whenever $x \to \infty$ and $k/x_2 \to 1$ as $(x \to \infty)$. These formulas follow directly from Theorem 21.4 in Elliott [4].

A classical theorem of Erdős and Wintner (see in [4], Chapter 5) says. An additive function f has a limit distribution if and only if each of the next three series are convergent:

$$\sum_{|f(p)|>1} 1/p, \quad \sum_{|f(p)|\le 1} \frac{f(p)}{p}, \quad \sum_{|f(p)|\le 1} \frac{f^2(p)}{p}. \tag{1.13}$$

Assume that $\delta_x \downarrow 0$, and that $k = k(x)$ is such a sequence of integers for which

$$\left|\frac{k}{x_2} - 1\right| < \delta_x. \tag{1.14}$$

We shall prove the following assertions.

Theorem 4. *Assume that for $f \in \mathcal{A}$ the series' in* (1.13) *are convergent and that* (1.14) *holds.*

Then

$$\lim_{x\to\infty} \frac{1}{\prod_k(x)} \# \{n \le x,\ n \in V_k,\ f(n) < y\} = F(y),$$

where F is a distribution function.

Theorem 5. *Let $g \in \overline{\mathcal{M}}$ and assume that*

$$\sum_p \frac{1-g(p)}{p}$$

is convergent. Then

$$\frac{1}{\prod_k(x)} \sum_{\substack{n\le x\\ n\in V_k}} g(n) = (1+o_x(1))M(g) \qquad (x \to \infty)$$

uniformly as $x \to \infty$ and (1.14) *is satisfied.*

Here

$$M(g) = \prod_p e_p, \quad e_p = \left(1 - \frac{1}{p}\right)\left(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \dots\right).$$

Remark 1. The Theorems 4 and 5 remain valid if we change V_k to U_k, and $\prod_k$ to π_k. The proofs became somewhat more complicated.

2. Theorem 1 can be interpreted as a result on the joint distribution of the functions f, $\beta_1, \dots, \beta_{q-1}$. Theorem 3 can be stated as a joint distribution law of the functions f and α. A similar formulation can be made for Theorem 4.

3. The referee of the paper suggested us to mention the possibility of the reformulation of our theorems written above. We appreciate his or her kind remarks.

2. Proof of Theorem 1

Let

$$\sum\nolimits_1 := \sum_{n \in S_N(\underline{r})} f(n), \quad \sum\nolimits_2 = \sum_{n \in S_N(\underline{r})} f^2(n).$$

Then

$$\sum\nolimits_1 = \sum_{b=1}^{q-1} \sum_{j=0}^{N-1} f(bq^j) \cdot M\left(N-1 \mid r_b - 1, \dots\right)$$

and the components in $M(N-1 \mid \dots)$ are $r_1, \dots, r_{b-1}; r_b - 1, r_{b+1}, \dots, r_{q-1}$. Thus

$$\frac{M(N-1 \mid \dots, r_b - 1, \dots)}{M(N \mid \dots, r_b, \dots)} = \frac{r_b}{N},$$

consequently

$$\frac{1}{M(N \mid r_1, \dots, r_{q-1})} \sum\nolimits_1 = \sum_{b=1}^{q-1} \frac{r_b}{N} \sum_{j=0}^{N-1} f(bq^j).$$

Similarly, for $\sum_2$, we have

$$\begin{aligned} \frac{1}{M(N \mid r_1, \dots, r_{q-1})} \sum\nolimits_2 = \sum_{b_1 \neq b_2} \frac{r_{b_1} r_{b_2}}{N(N-1)} \sum_{j_1 \neq j_2} \sum f(b_1 q^{j_1}) f(b_2 q^{j_2}) \\ + \sum_b \frac{r_b(r_b - 1)}{N(N-1)} \sum_{j_1 \neq j_2} f(bq^{j_1}) f(bq^{j_2}) \end{aligned}$$

$$+\sum_{b=1}^{q-1}\frac{r_b}{N}\sum_j f^2(bq^j).$$

Let

$$E_N := \sum_{b=1}^{q-1}\frac{r_b}{N}\sum_{j=0}^{N-1} f(bq^j)$$

and consider the sum

$$\Delta_N = \frac{1}{M(N \mid r_1,\dots,r_b)}\sum_{n\in S_N(\underline{r})}\left(f(n)-E_N\right)^2.$$

Then

$$\Delta_N = \frac{1}{M(N \mid r_1,\dots,r_b)}\sum\nolimits_2 - E_N^2,$$

which by the notation

$$A(b) = \sum_{j=0}^{N-1} f(bq^j), \quad D(b) = \sum_{j=0}^{N-1} f^2(bq^j)$$

$$C(b_1,b_2) = \sum_{j=0}^{N-1} f(b_1q^j)f(b_2q^j)$$

can be written as

$$\begin{aligned}\Delta_N = \frac{1}{(N-1)}\Big(\sum_b \frac{r_b}{N}A(b)\Big)^2 - \sum_2 \frac{r_b}{N(N-1)}A^2(b)\\ - \sum_{b_1\neq b_2}\frac{r_{b_1}r_{b_2}}{N(N-1)}C(b_1,b_2) + \sum_b \frac{r_b(N-r_b)}{N}D(b).\end{aligned} \tag{2.1}$$

Hence we can deduce that

$$\Delta_N < c\sum_{j=0}^{N-1}\sum_{b=0}^{q-1} f^2(bq^j), \tag{2.2}$$

where c is a constant which may depend on q.

Indeed,

$$A^2(b) \le \Big(\sum_j 1\Big)\sum_{j=0}^{N-1} f^2(bq^j) = ND(b),$$

thus the first summand on the right hand side of (2.1) is less than $c\sum_b D(b)$.

For the third summand we observe first that

$$|C(b_1,b_2)| \le \sqrt{D(b_1)}\,\sqrt{D(b_2)},$$

whence

$$\left|\sum_{b_1\ne b_2} \frac{r_{b_1}\cdot r_{b_2}}{N(N-1)} C(b_1,b_2)\right| \le 2\left(\sum \sqrt{D(b)}\right)^2 \le 2q\sum D(b).$$

Let $\varepsilon>0$ be fixed, M be a suitable large integer.

The integers $n<q^{N+M}$ can be written as $n=t+mq^M$, where $t\in[0,q^M-1]$, $m\in[0,q^N-1]$.

Let $\beta_j(t)=\eta_j$ $(j=1,\dots,q-1)$.

For fixed t, there exist exactly

$$M(N\mid r_1-\eta_1,\dots,r_{q-1}-\eta_{q-1}) \tag{2.3}$$

integers m, for which $\beta_j(n)=r_j$ $(j=1,\dots,q-1)$ satisfy. For fixed M and $N\to\infty$, the quantity (2.3) is

$$\left(\frac{1}{q}\right)^M (1+o_N(1))M(N+M\mid r_1,\dots,r_{q-1}). \tag{2.4}$$

Let $f_M(u)=f(u\cdot q^M)$ $(\in\mathcal{A}_q)$.

Let $R>0$ be a fixed number, and

$$\kappa_N^{(R)}(r_1-\eta_1,\dots,r_{q-1}-\eta_{q-1}) = \#\left\{|f_M(n)|\ge R,\ n\in S_N(\underline{\tilde r})\right\}$$
$$\underline{\tilde r}=(r_1-\eta_1,\dots,r_{q-1}-\eta_{q-1}).$$

By using the convergence of (1.4), (1.5) and applying (2.2), we obtain that

$$\begin{aligned}\kappa_N^{(R)}&(r_1-\eta_1,\dots,r_{q-1}-\eta_{q-1})\\ &< 2\frac{\tau_R(M)}{q^M} M\left(N+M\mid r_1,\dots,r_{q-1}\right),\end{aligned} \tag{2.5}$$

where $\tau_R(M)\to 0$ as $M\to\infty$.

Let $y \in \mathbb{R}$ be fixed. Let $t^{(1)}, t^{(2)}, \dots, t^{(P)}$ be those integers in $[0, q^M - 1]$ for which $f(t^{(j)}) < y - R$. For a fixed $t^{(j)}$, the number of those $m < q^M$ for which $n = t^{(j)} + m \cdot q^M \in S_{N+M}(\underline{r})$, and $f(n) \geq y$, is less than

$$\frac{2\tau_R(M)}{q^M} M\left(N + M \mid r_1, \dots, r_{q-1}\right). \tag{2.6}$$

Let $s^{(1)}$, $s^{(2)}, \dots, s^{(\pi)}$ be those integers in $[0, q^M - 1]$ for which $f(s^{(l)}) < y + R$. Similarly as above, the number of those $m < q^N$ for which $n = s^{(l)} + m \cdot q^M \in S_{N+M}(\underline{r})$ and $f(n) \leq -y$ is less than (2.6). Thus the number of those $n \in S_{N+M}(\underline{r})$ for which $f(n) < y$ holds, is no less than

$$\sum_{t^j} \frac{1}{q^M} \left(1 - 2\tau_R(M)\right) M(N + M \mid r_1, \dots, r_{q-1})$$
$$\geq \left(\frac{1 - 2\tau_R(M)}{q^M}\right) \Big(\sum_{t^{(j)}} 1\Big) M(N + M \mid r_1, \dots, r_{q-1}).$$

Consequently

$$\lim_{N\to\infty} \frac{1}{M\left(N \mid r_1^{(N)}, \dots, r_{q-1}^{(N)}\right)}$$
$$\times \# \left\{n < q^N \mid n \in S_N(\underline{r}),\ f(n) < y\right\} \geq F(y - 0)$$

and similarly, arguing with $s^{(\nu)}$ instead of $t^{(\mu)}$ we deduce that

$$\limsup_{N\to\infty} \frac{1}{M\left(N \mid r_1^{(N)}, \dots, r_{q-1}^{(N)}\right)}$$
$$\times \left\{n < q^N \mid n \in S_N(\underline{r}),\ f(n) < y\right\} \leq F(y + 0).$$

Here F is defined by (1.6).

Thus, if y is a continuity point of F, then

$$\lim_{N\to\infty} \frac{1}{M\left(N \mid r_1^{(N)}, \dots, r_{q-1}^{(N)}\right)} \# \left\{n < q^N \mid n \in S_N(r),\ f(n) < y\right\} = F(y).$$

The proof of Theorem 1 is completed.

3. Proof of Theorem 2

Let $g(n) \in \overline{\mathcal{M}}_q$, and assume that $\sum(1-m_j)$ is convergent. It implies that $g(bq^j) \to 1$ $(j \to \infty,\ b \neq 0)$. Let $f \in \mathcal{A}_q$ be a real valued function, defined for bq^j by $g(bq^j) = e^{if(bq^j)}$. We may assume that $f(bq^j) \in [-\pi, \pi]$, furthermore, we obtain that the series

$$\sum f(bq^j), \quad \sum f^2(bq^j)$$

are convergent.

Let $f_l(n) = f(n \cdot q^l)$, $g(n \cdot q^l) = e^{if_l(n)}$. Let M be a large integer and consider the sum,

$$\frac{1}{M\left(N+M \mid r_1, \ldots, r_{q-1}\right)} \sum_{n<q^{N+M}} g(n). \tag{3.1}$$

Since $n = t + q^M m$ implies that $g(n) = g(t)e^{if_M}(m)$, and

$$\sum_{\substack{m<q^N \\ m \in S_N(r_1-\eta_1,\ldots,r_{q-1}-\eta_{q-1})}} \left(1 - e^{if_M(m)}\right) = o(1) M\left(N \mid r_1-\eta_1, \ldots, r_{q-1}-\eta_{q-1}\right)$$

due to the fact that the convergence of (1.3) implies that $\sum_j \sum_{b=0}^{q-1} f(bq^j)$ and $\sum_j \sum_{b=0}^{q-1} f^2(bq^j)$ are convergent.

Hence, by (2.1), applying for f_M instead of f, and by the remark that (2.3) is equal to (2.4), we obtain that (3.1) equals to

$$\left(\frac{1}{q^M} \sum_{t<qM} g(t)\right)\left(1 + o_M(1)\right).$$

Hence the Theorem 2 readily follows.

4. Proof of Theorem 3

Let $\mu_r = \int_{-\infty}^{\infty} x^r d\phi(x)$ be the r'th moment of the normal-law.

Let

$$f_N^*(2^j) = \frac{f(2^j) - \eta_N}{B_N} \qquad (j = 0, \ldots, N-1)$$

and $f_N^*(n) = \sum_{j=0}^{N-1} \varepsilon_j(n) f_N^*(2^j)$ for $n < 2^N$. Then $f_N^*(n)$ can be interpreted as a random variable Θ_N which is a sum of the independent random variables $\xi_0, \dots, \xi_{N-1}$, such that

$$P(\xi_j = 0) = 1/2 = P\left(\xi_j = \frac{f(2^j) - \eta_N}{B_N}\right).$$

One can calculate that $E\Theta_N = 0$, and that

$$E\Theta_N^2 = \sum_{u \neq v} E(\xi_u \xi_v) + \sum E\xi_u^2 = 1.$$

Since $\max_{0 \le j < N} |f_N^*(2^j)| < \frac{C}{B_N} \to 0$ $(N \to \infty)$, from known theorem of probability theory we obtain easily that the moments $E\eta_N^r$ converge to μ_r.

Furthermore

$$E\eta_N^r = \sum_{t=1}^{r} \frac{1}{2^r} \sum_{h_1, \dots, h_t} \Delta_N(h_1, \dots, h_t)$$

where

$$\Delta_N(h_1, \dots, h_t) = \sum_{l_1, \dots, l_t} f_N^{*h_1}(2^{l_1}) \dots f_N^{*h_t}(2^{l_t}),$$

$h_1, \dots, h_t$ are positive integers such that $h_1 + \dots + h_t = r$, and $l_1, \dots, l_t$ are running over the sequences of mutually distinct numbers from the set $\{0, 1, \dots, N-1\}$.

Since

$$\left(T_{N,k}^{(r)} := \right) \frac{1}{\binom{N}{k}} \sum_{\substack{n < 2^N \\ \alpha(n) = k}} f_N^{*r}(n) = \sum_{t=1}^{r} \frac{\binom{N-t}{k-t}}{\binom{N}{k}} \sum_{h_1, \dots, h_t} \Delta_N(h_1, \dots, h_t),$$

and

$$\binom{N-t}{k-t} = \frac{1}{2^t}\,(1 + o_n(1)) \binom{N}{k},$$

furthermore that $\Delta_N(h_1, \dots, h_t)$ are bounded in N, we obtain that $T_{N,k}^{(r)} \to \mu_r$, if $N \to \infty$, and $k = k_N$ satisfies the condition stated in the theorem.

Now by the Frechet–Shohat theorem we get immediately the assertion.

5. Two lemmas

Let $\prod_k(x \mid A)$ be the number of those integers $n \le x$, for which $n \in V_k$ and $(n, A) = 1$. Let

$$F(s,z) = \sum \frac{z^{\Omega(n)}}{n^s} = \prod_P \frac{1}{1 - z/p^s},$$

$$F_A(s,z) = \sum_{(n,A)=1} \frac{z^{\Omega(n)}}{n^s} = \prod_{p|A} \left(1 - \frac{z}{p^s}\right) \cdot F(s,z).$$

Hence we obtain that

$$\prod\nolimits_k(x \mid A) = \sum_{\delta|A} \mu(\delta) \prod\nolimits_{k-\omega(\delta)} \left(\frac{x}{\delta}\right). \tag{5.1}$$

Let $\eta := \frac{k}{x_2}$.
If $A \le x_2$, then $\omega(\delta) \le \omega(A) \le \frac{c \log A}{\log\log} \le \frac{c x_3}{x_4}$ whenever $\delta \mid A$. We have

$$\log(\log x - \log\delta) = \log\log x + \log\left(1 - \frac{\log\delta}{\log x}\right)$$

$$\big(x_2 + O\left(\varepsilon(x)\right)\big)^{k-1} = x_2^{k-1} \exp\left(\varepsilon(x) \frac{k}{x_2}\right).$$

From (1.12) we have that

$$\prod\nolimits_{k-\omega(\delta)} \left(\frac{x}{\delta}\right) = \frac{1}{\delta} \prod\nolimits_{k-w(\delta)}(x) \left(1 + O\left(\frac{1}{\sqrt{x_2}}\right)\right).$$

Furthermore

$$\prod\nolimits_{k-\omega(\delta)}(x) = \prod\nolimits_k(x) \prod_{j=0}^{w(\delta)-1} \frac{k-j}{x_2 - j} \left(1 + O\left(\frac{1}{\sqrt{x_2}}\right)\right)^{\omega(\delta)},$$

and

$$\prod_{j=0}^{\omega(\delta)-1} \frac{k-j}{x_2 - j} = \eta^{\omega(\delta)} \left(1 + O\left(\frac{1}{\sqrt{x_2}}\right)\right).$$

Thus, by (5.1) we deduce that

$$\frac{\prod_k(x,A)}{\prod_k(x)} = \left(\sum_{\delta\mid A} \frac{\mu(\delta)}{\delta}\eta^{\omega(\delta)}\right) + O\left(\sum_{\delta\mid A} \frac{\eta^{\omega(\delta)}}{\delta}\left(e^{c\frac{\omega(\delta)}{\sqrt{x_2}}} - 1\right)\right). \tag{5.2}$$

Let $\psi_\eta(A) := \prod_{p\mid A}(1-\frac{\eta}{P})$.
The error term is less than

$$\ll \frac{1}{\sqrt{x_2}} \sum_{\delta\mid A} \frac{\eta^{\omega(\delta)}\omega(\delta)}{\delta} = \frac{\eta}{\sqrt{x_2}}\left(\sum_{p\mid A}\frac{1}{p}\right)\prod_{p\mid A}(1+\eta/p)\,.$$

We proved.

Lemma 6. *If $A \le x_2$, then*

$$\frac{\prod_k(x \mid A)}{\prod_k(x)} = \psi_\eta(A) + O\left(\frac{1}{\sqrt{x_2}}\left(\sum_{p\mid A} 1/p\right)\prod_{p\mid A}\left(1+\frac{\eta}{p}\right)\right).$$

Lemma 7. *Let $r \ge 1$ be fixed, $\varepsilon(x) \to 0$, $p_1 < p_2 < \ldots < p_r \left(< x^{\varepsilon(x)}\right)$ be primes, $\eta = \frac{k}{x_2} \to 1$ as $x \to \infty$. Then*

$$\lim \frac{\prod_k(x \mid p_1 \ldots p_r)}{\prod_k(x)} = \prod_{j=1}^{r}\left(1-\frac{1}{p_j}\right).$$

Proof. It is enough to observe that $\log x = (1+o(1))\log\frac{x}{\delta}$ $(\log\log x/\delta)^{k-1} = \left(x_2 + \log\left(1-\frac{\log\delta}{\log x}\right)\right)^{k-1} = (x_2 + O\left(\varepsilon(x)\right))^{k-1} = x_2^{k-1}\exp(\varepsilon(x)\eta) = (1+o_x(1))x_2^{k-1}$. $\square$

6. Proof of Theorem 4

Let Y be a large constant, P_Y be the product of primes up to Y. Let $\mathcal{E}_Y$ be the set of integers m, for which $P(m) \le Y$ and $\mathcal{T}_Y$ be those integers ν for which $(\nu, P_Y) = 1$. Let c_Y be a constant depending on Y, such that $c_Y \to \infty$ as $Y \to \infty$. Let $E(n) = \prod_{\substack{p^\alpha \| n\\ p\le Y}} p^\alpha$, and

$$F(n) := \frac{n}{E(n)}.$$

It is known that

$$\limsup_{x\to\infty} \frac{1}{x} \# \left\{ n < x \mid E(n) > Y^{c_Y} \right\} \le \Delta(c_Y), \tag{6.1}$$

where $\Delta(c_Y) \to 0$ as $Y \to \infty$.

From the convergence of the series' under (1.13), there exists a sequence $\varepsilon_p \downarrow 0$ such that $\sum_{|f(p)|>\varepsilon_p} 1/p < \infty$. Let $\mathcal{P}^* = \{p \mid |f(p)| > \varepsilon_p\}$.

The density of the integers $n \le x$ for which $p^2 \mid n$ for at least one prime $p > Y$ is less than c/Y. By using Turán's method we obtain that

$$\sum_{\substack{\nu \le X \\ (\nu, P_Y)=1}} \left(\sum_{\substack{p \mid \nu \\ p \notin \mathcal{P}^*}} f(p) - A_{x,y} \right)^2 \le cX \prod_{p \le Y} \left(1 - \frac{1}{p}\right) \sum_{\substack{p \notin \mathcal{P}^* \\ Y < p \le X}} \frac{f^2(p)}{P},$$

where

$$A_{x,Y} = \sum_{\substack{p \notin \mathcal{P}^* \\ Y < p \le x}} \frac{f(p)}{p}.$$

Let $\Delta > 0$.

From the convergence of (1.13) we deduce that

$$\frac{1}{x} \# \{\nu < x,\ (\nu, P_Y) = 1 \,\big|\, |f(\nu)| > \Delta\} \le c(\Delta, Y) \prod_{p<Y} \left(1 - \frac{1}{p}\right), \tag{6.2}$$

where $c(\Delta, Y) \to 0$ as $Y \to \infty$.

Let

$$G_x(\xi) = \frac{1}{x} \# \left\{ n \le x \mid f(n) < \xi \right\}.$$

Let $J^{(\lambda)}$ be the set of those $m \in \mathcal{E}_Y$ for which $f(m) < \lambda$, and $m \le Y^{c_Y}$.

From (6.2) we obtain that

$$G_X(\xi) \le \frac{\varphi(P_Y)}{P_Y} \left(\sum_{\substack{m \le Y^{c_Y} \\ m \in J^{\xi+\Delta}}} \frac{1}{m} \right) + c(\Delta, Y) + O(\Delta(c_Y)) \tag{6.3}$$

and

$$G_X(\xi) \ge \frac{\varphi(P_Y)}{P_Y} \sum_{\substack{m \le Y^{c_Y} \\ m \in J^{\xi-\Delta}}} \frac{1}{m} - c(\Delta, Y) + O(\Delta(c_Y)). \tag{6.4}$$

Let

$$\mathcal{T}_{k,m} = \left\{ \nu \le \frac{x}{m} \;\middle|\; (\nu, P_Y) = 1,\ \Omega(\nu) = k - \Omega(m) \right\}.$$

By using Lemma 5.2 in Prachar [3], according to

$$\#\{n \le x \mid P(n) \le Y\} < \exp\left(-\frac{\log\log\log Y}{\log Y} x_1 + 2\log\log Y \right)$$

for $Y > Y_0$, we deduce that

$$\limsup_{x\to\infty} \frac{1}{\prod_k(x)} \sum_{m > Y^{c_Y}} \# \{\mathcal{T}_{k,m}\} \le \Delta_1(c_Y),$$

where $\Delta_1(c_Y) \to 0$ as $Y \to \infty$, uniformly for k under the condition (1.14).

Observe furthermore that the number of those integers $n \in V_k$ for which there exists a prime $p > Y$ such that $p^2 \mid n$ is at most $o_Y(1) \prod_k(x)$.

From Lemma 1,

$$\frac{\prod_k(x \mid P_Y)}{\prod_k(x)} = \psi_\eta(P_Y) + O\left(\frac{\log\log Y}{\sqrt{x_2}} \cdot (\log Y)^c \right), \tag{6.5}$$

furthermore

$$\prod\nolimits_k(x \mid pP_Y) = \prod\nolimits_k(x \mid P_Y) - \prod\nolimits_k\left(\frac{x}{p} \;\middle|\; P_Y \right) \quad \text{if} \quad p > P_Y. \tag{6.6}$$

It is clear that $\prod_k(x \mid pP_Y) = \prod_k(x \mid P_Y)$ if $p \le Y$.

Let S_x be the set of those integers $n \le x$, $n \in V_k$, for which there exists at least one prime divisor $p \in \mathcal{P}^*$, $p > Y$. Then

$$\#S_x \le \sum_{\substack{p\in\mathcal{P}^* \\ p > Y}} \prod\nolimits_{k-1}\left(\frac{x}{p}\right) = \sum\nolimits_1 + \sum\nolimits_2$$

where in $\sum_1$ we sum over the primes $p < x^{1-\rho(Y)}$ and in $\sum_2$ for the others. We have

$$\sum\nolimits_2 \le \sum_{\substack{m < x^{\rho(x)} \\ \Omega(m) = k-1}} \sum_{p < \frac{x}{m}} 1 \le \frac{cx}{x_1} \cdot \frac{x_2^{k-1}}{(k-1)!} \left(1 + \frac{\log\rho(Y)}{x_2} \right)^{k-1}$$

i.e.

$$\frac{1}{\prod_k(x)}\sum\nolimits_2 < c\left(1+\frac{\log\rho(Y)}{x_2}\right)^{k-1}.$$

Let now $\rho(Y)$ be defined so that

$$\rho^2(Y)=\sum_{\substack{Y<p<\infty\\ p\in\mathcal{P}^*}} 1/p.$$

Then

$$\limsup_{x\to\infty}\frac{1}{\prod_k(x)}\#S_x\le c\rho(Y)^{1/2}. \tag{6.7}$$

Let $f_x\in\mathcal{A}$ be defined for prime powers p^α as follows:

$$f_x(p^\alpha)=\begin{cases}0 & \text{if } \alpha\ge 2,\quad p>Y,\\ 0 & \text{if } \alpha=1,\quad p\in\mathcal{P}^*,\ p>Y,\\ f(p^\alpha) & \text{otherwise.}\end{cases}$$

Let $h(n)=f(n)-f_x(n)$. Then, for each fixed $\Delta>0$,

$$\limsup_{x\to\infty}\frac{1}{\prod_k(x)}\#\{n\le x\mid n\in V_k, |\,h(n)|>\Delta\}\le c(\Delta,Y), \tag{6.8}$$

where $c(\Delta,Y)\to 0$ as $Y\to\infty$, for every fixed $\Delta>0$.

This assertion is obvious from (6.7).

Let $X\in[\sqrt{x},x]$, $k_1=k+O(x_3)$,

$$\sum\nolimits_1(X,k_1)=\sum_{\substack{\nu\le X\\ (\nu,P_Y)=1\\ \Omega(\nu)=k_1}} f_x(\nu), \tag{6.9}$$

$$\sum\nolimits_2(X,k_1)=\sum_{\substack{\nu\le X\\ (\nu,P_Y)=1\\ \Omega(\nu)=k_1}} f_x^2(\nu). \tag{6.10}$$

We have

$$\sum\nolimits_1(X,k_1)=\sum_{p^\alpha<x^\varepsilon} f_x(p^\alpha)\prod\nolimits_{k_1-\alpha}\left(\frac{x}{p^\alpha}\,\middle|\,pP_Y\right), \tag{6.11}$$

$$\sum\nolimits_2(X,k_1) = \sum_{p^\alpha<x^\varepsilon} f_x^2(p^\alpha)\prod\nolimits_{k_1-\alpha}\left(\frac{x}{p^\alpha}\;\Big|\; pP_Y\right)$$
$$+\sum_{\substack{p^\alpha q^\beta\le x\\ p^\alpha,\,q^\beta\le x^\varepsilon\\ p\ne q}} f_x(p^\alpha)f_x(q^\beta)\prod\nolimits_{k_1-\alpha-\beta}\left(\frac{x}{p^\alpha q^\beta}\;\Big|\; pqP_Y\right). \quad (6.12)$$

From (6.1), (6.2) we obtain that

$$\frac{1}{\prod_{k_1}(X)}\sum\nolimits_1(X,k_1) = \sum_{Y\le p<X^\varepsilon}\frac{f_x(p)}{p}\left(\psi_\eta(P_Y)+O\left(\frac{(\log Y)^C}{\sqrt{x_2}}\right)\right)$$
$$+O\Bigg(\sum_{\substack{p^\alpha<x^\varepsilon\\ p>Y,\ \alpha\ge 2}}\frac{1}{p^\alpha}\Bigg)$$
$$=\psi_\eta(P_Y)E_x+O\left(\frac{1}{Y\log Y}\right),$$
$$E_x=\sum_{Y\le p<x^\varepsilon}\frac{f_x(p)}{P}.$$

Furthermore,

$$\frac{1}{\prod_{k_1}(X)}\sum\nolimits_2 = \psi_\eta(P_Y)\sum\frac{f_x^2(p)}{p}+O\left(\frac{1}{Y\log Y}\right)$$
$$+\psi_\eta(P_Y)\sum_{p\ne q}\frac{f_x(p)f_x(q)}{pq}$$
$$=\psi_\eta(P_Y)\left(E_x^2+\sum\frac{f_x^2(p)}{p}\right)+O\left(\frac{1}{Y\log Y}\right).$$

Hence we obtain that

$$\frac{1}{\prod_{k_1}(X)}\sum_{\substack{\nu\le X\\(\nu,P_Y)=1\\ \Omega(\nu)=k_1}}\left(f_x(\nu)-E_x^2\right)\le c\psi_\eta(P_Y)\sum_{Y<p<x^\varepsilon}\frac{f_x^2(p)}{p}+O\left(\frac{1}{Y\log Y}\right)$$

and so

$$\begin{aligned} &\frac{1}{\prod_{k_1}(X \mid P_Y)} \sum_{\substack{\nu \le x \\ (\nu, P_Y)=1 \\ \Omega(\nu)=k_1}} (f_x(\nu) - E_x)^2 \\ &\qquad \le c_1 \sum_{Y<p<x^{\varepsilon}} \frac{f_x^2(p)}{p} + O\left(\frac{\log Y}{Y}\right). \end{aligned} \tag{6.13}$$

Let $\Delta > 0$ be a small constant as above. Then, by a suitable large Y, from (6.13) we obtain that

$$\frac{1}{\prod_{k_1}(X \mid P_Y)} \# \{\nu \le X,\ \nu \in V_{k_1},\ (\nu, P_Y) = |f_x(\nu)| \ge \Delta\} < \Delta \tag{6.14}$$

for each k_1, and for every large x.

Let

$$F_x^{(k)}(\xi) := \frac{1}{\prod_k(x)} \# \{f_x(n) < \xi,\ n \in V_k,\ n < x\}.$$

Let $J^{(\lambda)}$ as earlier be the collection of elements from $\mathcal{E}_Y$ for which $f(m) < \lambda$, $m < Y^{c_Y}$.

Let $\lambda = \xi - \Delta$, $m \in J^{(\xi-\Delta)}$. Then, from (6.14) with $X = \frac{x}{m}$, $k_1 = k - \Omega(m)$, we have that for all but $\Delta \prod_{k_1}(\frac{x}{m} \mid P_Y)$ of integers ν, $f_x(m\nu) < \xi$.

Hence we obtain that

$$F_x^{(k)}(\xi) \ge (1-\Delta) \sum_{m \in J^{(\xi-\Delta)}} \frac{\prod_{k-\Omega(m)}\left(\frac{x}{m} \mid P_Y\right)}{\prod_k(x)} + o_x(1) \tag{6.15}$$

and similarly,

$$F_x^{(k)}(\xi) \le (1+\Delta) \sum_{m \in J^{(\xi+\Delta)}} \frac{\prod_{k+\Omega(m)}\left(\frac{x}{m} \mid P_Y\right)}{\prod_k(x)} + o_x(1). \tag{6.16}$$

Since

$$\frac{\prod_{k-\Omega(m)}\left(\frac{x}{m} \mid P_Y\right)}{\prod_k(x)} = \frac{\eta^{\Omega(m)}}{m} \psi_\eta(P_Y)\ (+o_x(1))$$

therefore

$$F_x^{(k)}(\xi) \ge (1-\Delta) \sum_{m \in J^{(\xi-\Delta)}} \frac{\eta^{\Omega(m)}}{m} \psi_\eta(P_Y) + o_x(1), \tag{6.17}$$

$$F_x^{(k)}(\xi) \le (1+\Delta) \sum_{m \in J^{(\xi+\Delta)}} \frac{\eta^{\Omega(m)}}{m} \psi_\eta(P_Y) + o_x(1). \tag{6.18}$$

From the Erdős–Wintner theorem we know that

$$\lim_{x\to\infty} G_x(\xi) = G(\xi)$$

exists for each continuity point ξ of G.

From (6.17), (6.18) we obtain that

$$\liminf F_x^{(k)}(\xi) \ge (1-\Delta) \left(\sum_{m \in J^{(\xi-\Delta)}} 1/m \right) \frac{\varphi(P_Y)}{P_Y}, \tag{6.19}$$

$$\limsup F_x^{(k)}(\xi) \le (1+\Delta) \left(\sum_{m \in J^{(\xi+\Delta)}} 1/m \right) \frac{\varphi(P_Y)}{P_Y}. \tag{6.20}$$

Comparing with (6.3) we deduce that

$$\begin{aligned} \liminf F_x^{(k)}(\xi) &\ge (1-\Delta)^2 \limsup G_x(\xi - 2\Delta) \\ &\quad + O(c(\Delta, Y)) + O(\Delta(cy)) + O(\Delta) \end{aligned} \tag{6.21}$$

and similarly that

$$\limsup F_x^{(k)}(\xi) \le (1+\Delta)^2 \limsup G_x(\xi+2\Delta) + O(c(\Delta,Y)) + O(\Delta). \tag{6.22}$$

Hence, by $Y \to \infty$, then $\Delta \to 0$ we obtain that

$$\lim_{x\to\infty} F_x^{(k)}(\xi) = G(\xi) \quad (= F(\xi))$$

holds.

Thus Theorem 4 is true.

7. Proof of Theorem 5

The proof is very similar to that of Theorem 4. Let Y be a large constant, $C_Y \to \infty$, as $Y \to \infty$. Then

$$\frac{1}{\prod_k(x)} \sum_{\substack{n \le x \\ n \in V_k}} g(n) = \frac{1}{\prod_k(x)} \sum_{\substack{m \in \mathcal{E}_Y \\ m \le Y^{c_Y}}} g(m) \sum_{\substack{\nu < \frac{x}{m} \\ (\nu, P_Y)=1 \\ \Omega(\nu) = k-(\Omega)(m)}} g(\nu) + O(\Delta(c_Y)), \tag{7.1}$$

where $\Delta(c_Y) \to 0$ as $Y \to \infty$.

Let $f(p^\alpha) = \arg g(p^\alpha)$ for prime power p^α, and let the domain of f be extended to $n \in \mathcal{F}_Y$. The convergence of

$$\sum_p \frac{1-g(p)}{p}$$

implies the convergence of the series

$$\sum_{|f(p)|\geq 1} \frac{1}{p}, \quad \sum_{|f(p)|<1} \frac{f(p)}{p}, \quad \sum_{|f(p)|<1} \frac{f^2(p)}{p}. \tag{7.2}$$

Let us observe that

$$\sum_{\substack{\nu < \frac{x}{m} \\ (\nu, P_Y)=1 \\ \Omega(\nu)=k-\Omega(m)}} (g(\nu)-1) = i \sum_{\substack{\nu < \frac{x}{m} \\ (\nu, P_Y)=1 \\ \Omega(\nu)=k-\Omega(m)}} f(\nu) + O\Bigg(\sum_{\substack{\nu < \frac{x}{m} \\ (\nu, P_Y)=1 \\ \Omega(\nu)=k-\Omega(m)}} f^2(\nu) \Bigg). \tag{7.3}$$

Arguing as earlier, we can prove that the right hand side of (7.3) is less than $c(Y) \prod_{k-\Omega(m)} \left(\frac{x}{m} \mid P_Y\right)$, where $c(Y) \to 0$ as $Y \to \infty$.

Thus, from (7.1),

$$\frac{1}{\prod_k(x)} \sum_{\substack{n \leq x \\ n \in V_k}} g(n) = (1 + o_x(1)) \sum_{\substack{m \in \mathcal{E}_Y \\ m \leq Y^{c_Y}}} \frac{g(m)}{m} \eta^{\Omega(m)} \prod_{p<Y} \left(1 - \frac{1}{p}\right) + O(c(Y)),$$

whence

$$\limsup_{x \to \infty} \left| \frac{1}{\prod_k(x)} \sum_{\substack{n \leq x \\ n \in V_k}} g(n) - T(Y) \right| \leq O(c(Y))$$

where

$$T(Y) = \Bigg(\sum_{\substack{m \in \mathcal{E}_Y \\ m \leq Y^{c_4}}} \frac{g(m)}{m} \Bigg) \prod_{p<Y} (1 - 1/p).$$

Furthermore,

$$T(Y) = \prod_{p \leq Y} \left(1 - \frac{1}{p}\right) \left(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \dots\right) + o_Y(1)$$

and so

$$\lim_{Y\to\infty} T(Y) = \prod_p \left(1-\frac{1}{p}\right)\left(1+\frac{g(p)}{p}+\frac{g(p^2)}{p^2}+\dots\right)$$

consequently Theorem 5 it follows immediately.

References

[1] H. Delange, Sur les fonctions q-additives ou q-multiplicatives, *Acta Arithmetica* **21** (1972), 285–298.

[2] J. Kubilius, Translations of Math. Monographs 11, *Amer. Math. Soc.*, *Providence, RI.*

[3] K. Prachar, Primzahlverteilung, *Springer Verlag*, *Berlin*, 1957.

[4] P. D. T. A. Elliott, Probabilistic number theory, *Springer Verlag*, *New York*, 1960.

I. KÁTAI
DEPARTMENT OF COMPUTER ALGEBRA
EÖTVÖS LORÁND UNIVERSITY
H-1117 BUDAPEST
PÁZMÁNY PÉTER SÉTÁNY 1/C
HUNGARY

E-mail: katai@compalg.inf.elte.hu

M. V. SUBBARAO
UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA, T6G 2G1
CANADA

E-mail: m.v.subbarao@ualberta.ca

(Received February 6, 2003; revised May 12, 2003)