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On stability of the Cauchy equation in normed spaces over fields with valuation

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Abstract. S. M. Ulam's problem was to give conditions for the existence of a linear mapping near an approximately linear mapping. D. H. Hyers, Th. M. Rassias and Z. Gajda have considered this problem in real Banach spaces. The purpose of this paper is to generalize their approach.

1. Introduction

In connection with a problem posed by Ulam, Th. M. Rassias [7] proved the following theorem on stability of linear mappings in Banach spaces.

Theorem 1. *Let E_1 and E_2 be two (real) Banach spaces and let $f : E_1 \to E_2$ be a mapping such that for each fixed $x \in E_1$ the transformation $t \mapsto f(tx)$ is continuous on $\mathbb{R}$. Moreover, assume that there exists $\varepsilon \in [0,\infty)$ and $\alpha \in [0,1)$ such that*

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon(\|x\|^\alpha + \|y\|^\alpha) \tag{1}$$

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for all $x, y \in E_1$. Then there exist a unique linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \delta \|x\|^\alpha \tag{2}$$

for all $x \in E_1$, where $\delta = \frac{2\varepsilon}{2-2^\alpha}$.

As it was mentioned by Rassias, the proof presented in [7] works for every α from the intervall $(-\infty, 1)$. Note that Theorem 1 involves the classical Hyers–Ulam stability of linear mappings when $\alpha = 0$, asymptotic stabililty in infinity when $\alpha < 0$, and local stability at the origin when $\alpha > 0$. Omitting the assumption that all the transformations $t \to f(tx)$ are continuous, one obtains the existence of a unique additive mapping $T : E_1 \to E_2$ with (2). Concerning the remaining cases, Z. Gajda [4] showed that the statement holds when $\alpha > 1$ with $\delta = \frac{2\varepsilon}{2^\alpha - 2}$, but it is not valid for $\alpha = 1$. Motivated by a problem of Z. Boros [2], we will prove a generalization of these results for mappings in Banach spaces over fields with arbitrary valuations. A particular case, when $\alpha = 0$ and E_2 is a Banach space over a p-adic field $\mathbb{Q}_p$, has been considered by W. A. Beyer [1] and J. Schwaiger [8] and some other cases follow from a theorem of G. L. Forti [3]. Our main result is established in Theorem 2.

2. Preliminaries

Let F be a field. We say that F is a field with the valuation v if $v : F \to \mathbb{R}$ is a positive definite, multiplicative and subadditive function with $v(0) = 0$. Moreover, if $v(x + y) \leq \max\{v(x), v(y)\}$ for all $x, y \in F$, then we say that v is a non-archimedean valuation, otherwise we say that v is an archimedean valuation. We say that $(X, \| \, \|)$ is a normed space over a field F with the valuation $| \, |_F$, if X is a linear space over F and the mapping $\| \, \| : X \to \mathbb{R}$ is positive definite, subadditive, and we have $\|\lambda x\| = |\lambda|_F \|x\|$ for all $\lambda \in F$, $x \in X$. A normed space is called non-archimedean if

$$\|x + y\| \leq \max\{\|x\| \, , \, \|y\|\} \text{ for all } x, y \in X.$$

If a (non-archimedean) normed space is complete with respect to the metric generated by the norm, it is called a (non-archimedean) Banach space.

If a field has zero characteristic, then it contains the rational numbers. If we have a valuation on a field of characteristic zero, then it involves a valuation on $\mathbb{Q}$. As it was proved by A. OSTROWSKI [6], if we have a valuation $|\ |_{\mathbb{Q}}$ on $\mathbb{Q}$, then one of the following statements holds:

1. There exists a $\beta \in (0,1]$ such that $|\ |_{\mathbb{Q}} = |\ |^{\beta}$, where $|\ |$ is the standard absolute value.
2. $|0|_{\mathbb{Q}} = 0$ and there exist a prime number p and $\varrho \in (0,1]$ such that if $x \in \mathbb{Q}\backslash\{0\}$ and $x = p^k \frac{n}{m}$ $(p \nmid n,\ p \nmid m)$, then $|x|_{\mathbb{Q}} = \varrho^k$.

In the first case $|\ |_{\mathbb{Q}}$ is an archimedean valuation on $\mathbb{Q}$.

In the second case, it is easy to prove, that $|x+y|_{\mathbb{Q}} \le \max\{|x|_{\mathbb{Q}}\ ,\ |y|_{\mathbb{Q}}\}$, so $|\ |_{\mathbb{Q}}$ is a non-archimedean valuation on $\mathbb{Q}$. In case $\varrho = 1$ we have the trivial valuation. In case $\varrho = 1/p$ we say that $|\ |_{\mathbb{Q}}$ is the p-adic valuation and we denote it by $|\ |_p$. It is obvious that every $\varrho \in (0,1]$ can be produced as a non-negative power of $1/p$, so 2 goes to this form:

$\overline{2}$. There exists a prime number p and $\beta \ge 0$ such that $|\ |_{\mathbb{Q}} = |\ |_p^{\beta}$.

3. Results

At first we need some lemmas to prove our theorem.

Lemma 1. *Let $(X, \|\ \|_1)$ be a normed space over a field F of characteristic zero with a valuation $|\ |_F$, $(Y, \|\ \|_2)$ be a normed space over an arbitrary field, $f : X \to Y$ and α be a real number. If there exists a non-negative real number L such that*

$$\|f(x+y) - f(x) - f(y)\|_2 \le L \max\{\|x\|_1^{\alpha},\ \|y\|_1^{\alpha}\} \quad \text{for every } x, y \in X, \tag{3}$$

then

$$\|f(nx) - nf(x)\|_2 \le L\sigma(n,\alpha)\|x\|_1^{\alpha}$$

for all $n \in \mathbb{N}$, where

$$\sigma(n,\alpha) = (n-1) + \big(|n-1|_F^{\alpha} + \cdots + |2|_F^{\alpha} + |1|_F^{\alpha}\big). \tag{4}$$

(In this paper $0^{\alpha} = 0$ for $\alpha \ne 0$ and $0^0 = 1$.)

Proof. We prove the lemma by induction on n. For $n=1$ the statement is trivial. Let us suppose, that $n>1$ and $\|f\big[(n-1)x\big]-(n-1)f(x)\|_2\leq L\sigma(n-1,\alpha)\|x\|_1^\alpha$. Then

$$\begin{aligned}
&\|f(nx)-nf(x)\|_2\\
&\qquad=\|f(nx)-f\big[(n-1)x\big]-f(x)+f\big[(n-1)x\big]-(n-1)f(x)\|_2\\
&\qquad\leq\|f(nx)-f\big[(n-1)x\big]-f(x)\|_2+\|f\big[(n-1)x\big]-(n-1)f(x)\|_2\\
&\qquad\leq L\max\big\{\|(n-1)x\|_1^\alpha,\ \|x\|_1^\alpha\big\}+L\sigma(n-1,\alpha)\|x\|_1^\alpha\\
&\qquad\leq L\big(|n-1|_F^\alpha+1+\sigma(n-1,\alpha)\big)\|x\|_1^\alpha=L\sigma(n,\alpha)\|x\|_1^\alpha. \qquad\square
\end{aligned}$$

Lemma 2. *Let $(X,\|\ \|_1)$ be a normed space over a field F of characteristic zero with a valuation $|\ |_F$, $(Y,\|\ \|_2)$ be a normed space over a field K of characteristic zero with a valuation $|\ |_K$, $f:X\to Y$, α be a real number and s be a positive integer. Let us consider the functions $f_n:X\to Y$,*

$$f_n(x)=\frac{1}{s^n}f(s^nx)\quad(x\in X,\ n\in\mathbb{N}).\tag{5}$$

If the function f satisfies (3) and $|s|_F^\alpha<|s|_K$, then

$$\|f(x)-f_n(x)\|_2\leq\frac{L\sigma(s,\alpha)}{|s|_K-|s|_F^\alpha}\|x\|_1^\alpha$$

for all $x\in X$ and $n\in\mathbb{N}$.

Proof. Applying Lemma 1 we have:

$$\begin{aligned}
\|f(x)-f_n(x)\|_2&=\left\|\sum_{k=0}^{n-1}\left(\frac{1}{s^k}f(s^kx)-\frac{1}{s^{k+1}}f(s^{k+1}x)\right)\right\|_2\\
&\leq\sum_{k=0}^{n-1}\frac{1}{|s|_K^{k+1}}\left\|sf(s^kx)-f(s^{k+1}x)\right\|_2\leq\frac{L\sigma(s,\alpha)}{|s|_K}\sum_{k=0}^{n-1}\left(\frac{|s|_F^\alpha}{|s|_K}\right)^k\|x\|_1^\alpha\\
&\leq\frac{L\sigma(s,\alpha)}{|s|_K}\sum_{k=0}^{\infty}\left(\frac{|s|_F^\alpha}{|s|_K}\right)^k\|x\|_1^\alpha=\frac{L\sigma(s,\alpha)}{|s|_K-|s|_F^\alpha}\|x\|_1^\alpha
\end{aligned}$$

for every $x\in X$ and $n\in\mathbb{N}$, which is our statement. $\square$

Lemma 3. *Let $(X, \| \ \|_1)$ be a normed space over a field F of characteristic zero with a valuation $| \ |_F$, $(Y, \| \ \|_2)$ be a normed space over a field K of characteristic zero with a valuation $| \ |_K$, $f : X \to Y$, α be a real number and s be a positive integer. Let us consider the functions $g_n : X \to Y$,*

$$g_n(x) = s^n f\left(\frac{1}{s^n}x\right) \quad (x \in X,\ n \in \mathbb{N}). \tag{6}$$

If the function f satisfies (3) and $|s|_F^\alpha > |s|_K$, then

$$\|f(x) - g_n(x)\|_2 \le \frac{L\sigma(s,\alpha)}{|s|_F^\alpha - |s|_K}\|x\|_1^\alpha$$

for all $x \in X$ and $n \in \mathbb{N}$.

Proof. Using Lemma 1 with $n = s$ and x/s in place of x, we get that

$$\left\| f(x) - sf\left(\frac{1}{s}x\right)\right\|_2 \le \frac{L\sigma(s,\alpha)}{|s|_F^\alpha}\|x\|_1^\alpha.$$

Therefore, for every $m \in \mathbb{N}$ and $x \in X$:

$$\|f(x) - g_n(x)\|_2 = \left\|\sum_{k=0}^{n-1}\left(s^k f\left(\frac{1}{s^k}x\right) - s^{k+1} f\left(\frac{1}{s^{k+1}}x\right)\right)\right\|_2$$

$$\le \sum_{k=0}^{n-1}|s|_K^k \left\|f\left(\frac{1}{s^k}x\right) - sf\left(\frac{1}{s^{k+1}}x\right)\right\|_2 \le \frac{L\sigma(s,\alpha)}{|s|_F^\alpha}\sum_{k=0}^{n-1}\left(\frac{|s|_K}{|s|_F^\alpha}\right)^k \|x\|_1^\alpha$$

$$\le \frac{L\sigma(s,\alpha)}{|s|_F^\alpha}\sum_{k=0}^{\infty}\left(\frac{|s|_K}{|s|_F^\alpha}\right)^k \|x\|_1^\alpha = \frac{L\sigma(s,\alpha)}{|s|_F^\alpha - |s|_K}\|x\|_1^\alpha. \qquad \square$$

Lemma 4. *Let $(X, \| \ \|_1)$ be a normed space over a field F of characteristic zero with a valuation $| \ |_F$, $(Y, \| \ \|_2)$ be a normed space over a field K of characteristic zero with a valuation $| \ |_K$, $f : X \to Y$ and α be a real number. If $f : X \to Y$ is additive, $\|f(x)\|_2 \le M\|x\|_1^\alpha$ for some fixed $M \in \mathbb{R}$ and for all $x \in X$, and there exists a positive integer s such that $|s|_F^\alpha \ne |s|_K$, then $f(x) = 0$ for all $x \in X$.*

Proof. Since f is additive, it is $\mathbb{Q}$-linear. Thus the existence of some $s \in \mathbb{N}$ with $|s|_F^\alpha \ne |s|_K$ implies that there is some rational number r

$\left(r \in \{s, \frac{1}{s}\}\right)$ such that $|r|_F^\alpha < |r|_K$. But then we have for any $x \in X$, $x \neq 0$, and any $n \in \mathbb{N}$ that

$$|r|_K^n \|f(x)\|_2 = \|f(r^n x)\|_2 \leq M|r|_F^{n\alpha} \|x\|_1^\alpha$$

therefore with $q := |r|_F^\alpha / |r|_K (\in]0,1[)$ we have

$$\|f(x)\|_2 \leq Mq^n \|x\|_1^\alpha,$$

which for $n \to \infty$ gives $\|f(x)\|_2 = 0$. □

Now we can formulate our main theorem.

THEOREM 2. *Let $(X, \| \ \|_1)$ be a normed space over a field F of characteristic zero with a valuation $| \ |_F$, $(Y, \| \ \|_2)$ be a Banach space over a field K of characteristic zero with a valuation $| \ |_K$, $f : X \to Y$ and α be a real number. If the function f satisfies* (3) *and there exists a positive integer s such that $|s|_F^\alpha \neq |s|_K$, then there exists a unique additive function $g : X \to Y$ for which*

$$\|f(x) - g(x)\|_2 \leq C \|x\|_1^\alpha \quad (x \in X) \tag{7}$$

with some $C \in \mathbb{R}$. Moreover g satisfies (7) *with $C = \frac{L\sigma(s,\alpha)}{\left||s|_K - |s|_F^\alpha\right|}$.*

PROOF. I. At first we prove the existence part of the theorem. We consider two cases.

1. $|s|_F^\alpha < |s|_K$. Let us consider the functions $f_1, f_2, f_3, \dots$ defined by (5). We prove that $(f_n(x))$ is a Cauchy sequence for each fixed $x \in X$. Let $m, n \in \mathbb{N}$ such that $n > m$. Using Lemma 2, from

$$\begin{aligned}
\|f_m(x) - f_n(x)\|_2 &= \left\| \frac{1}{s^m} f\left(s^m x\right) - \frac{1}{s^n} f\left(s^n x\right) \right\|_2 \\
&= \left| \frac{1}{s^m} \right|_K \left\| f\left(s^m x\right) - \frac{1}{s^{n-m}} f\left(s^{n-m} s^m x\right) \right\|_2 \\
&= \frac{1}{|s|_K^m} \left\| f\left(s^m x\right) - f_{n-m}\left(s^m x\right) \right\|_2 \\
&\leq \frac{1}{|s|_K^m} \frac{L\sigma(s,\alpha)}{|s|_K - |s|_F^\alpha} \left\| s^m x \right\|_1^\alpha = \left(\frac{|s|_F^\alpha}{|s|_K} \right)^m \frac{L\sigma(s,\alpha)}{|s|_K - |s|_F^\alpha} \|x\|_1^\alpha
\end{aligned}$$

we conclude that $(f_n(x))$ is a Cauchy sequence for each $x \in X$. Since Y is complete, the definition $g(x) = \lim\limits_{n\to\infty} f_n(x)$ $(x \in X)$ is correct. Furthermore, for every $x, y \in X$

$$\begin{aligned}
0 \le \|g(x+y) - g(x) - g(y)\|_2 &= \lim_{n\to\infty} \|f_n(x+y) - f_n(x) - f_n(y)\|_2 \\
&= \lim_{n\to\infty} \left\| \frac{1}{s^n} f\left(s^n x + s^n y\right) - \frac{1}{s^n} f\left(s^n x\right) - \frac{1}{s^n} f\left(s^n y\right) \right\|_2 \\
&= \lim_{n\to\infty} \frac{1}{|s|_K^n} \left\| f\left(s^n x + s^n y\right) - f\left(s^n x\right) - f\left(s^n y\right) \right\|_2 \\
&\le \lim_{n\to\infty} \left(\frac{|s|_F^\alpha}{|s|_K} \right)^n L \max\{\|x\|_1^\alpha\ , |y|_1^\alpha\} = 0,
\end{aligned}$$

thus, g is an additive function. Finally, by Lemma 2 we have

$$\|f(x) - g(x)\|_2 = \lim_{n\to\infty} \|f(x) - f_n(x)\|_2 \le \frac{L\sigma(s,\alpha)}{|s|_K - |s|_F^\alpha} \|x\|_1^\alpha. \tag{8}$$

2. $|s|_F^\alpha > |s|_K$. Let us consider the functions $g_1, g_2, g_3, \dots$ defined by (6). We prove that $(g_n(x))$ is a Cauchy sequence for each fixed $x \in X$. Let $m, n \in \mathbb{N}$ such that $n > m$. Using Lemma 3, from

$$\begin{aligned}
\|g_m(x) - g_n(x)\|_2 &= \left\| s^m f\left(\frac{1}{s^m} x\right) - s^n f\left(\frac{1}{s^n} x\right) \right\|_2 \\
&= |s^m|_K \left\| f\left(\frac{1}{s^m} x\right) - s^{n-m} f\left(\frac{1}{s^{n-m}} \frac{1}{s^m} x\right) \right\|_2 \\
&= |s|_K^m \left\| f\left(\frac{1}{s^m} x\right) - g_{n-m}\left(\frac{1}{s^m} x\right) \right\|_2 \\
&\le |s|_K^m \frac{L\sigma(s,\alpha)}{|s|_F^\alpha - |s|_K} \left\| \frac{1}{s^m} x \right\|_1^\alpha = \left(\frac{|s|_K}{|s|_F^\alpha} \right)^m \frac{L\sigma(s,\alpha)}{|s|_F^\alpha - |s|_K} \|x\|_1^\alpha
\end{aligned}$$

we conclude that $(g_n(x))$ is a Cauchy sequence for each $x \in X$. Since Y is complete, the definition $g(x) = \lim\limits_{n\to\infty} g_n(x) \quad (x \in X)$ is correct. Furthermore, for every $x, y \in X$

$$\begin{aligned}
0 \le \|g(x+y) - g(x) - g(y)\|_2 &= \lim_{n\to\infty} \|g_n(x+y) - g_n(x) - g_n(y)\|_2 \\
&= \lim_{n\to\infty} \left\| s^n f\left(\frac{1}{s^n} x + \frac{1}{s^n} y\right) - s^n f\left(\frac{1}{s^n} x\right) - s^n f\left(\frac{1}{s^n} y\right) \right\|_2
\end{aligned}$$

$$= \lim_{n\to\infty} |s|_K^n \left\| f\left(\frac{1}{s^n}x + \frac{1}{s^n}y\right) - f\left(\frac{1}{s^n}x\right) - f\left(\frac{1}{s^n}y\right)\right\|_2$$
$$\le \lim_{n\to\infty} \left(\frac{|s|_K}{|s|_F^\alpha}\right)^n L \max\{\|x\|_1^\alpha\ , |y|_1^\alpha\} = 0,$$

thus, g is an additive function. Finally, by Lemma 3 we have

$$\|f(x) - g(x)\|_2 = \lim_{n\to\infty} \|f(x) - g_n(x)\|_2 \le \frac{L\sigma(s,\alpha)}{|s|_F^\alpha - |s|_K}\|x\|_1^\alpha. \tag{9}$$

Now, applying (8) and (9) we get (7) with $C = \frac{L\sigma(s,\alpha)}{\left||s|_K - |s|_F^\alpha\right|}$.

II. To prove the uniqueness of g we suppose that $g_1,\ g_2 : X \to Y$ are additive functions such that

$$\|f(x) - g_1(x)\|_2 \le LC_1\|x\|_1^\alpha$$

and

$$\|f(x) - g_2(x)\|_2 \le LC_2\|x\|_1^\alpha$$

for all $x \in X$ and some $C_1, C_2 \in \mathbb{R}$. Then

$$\begin{aligned}\|g_1(x) - g_2(x)\|_2 &\le \|f(x) - g_1(x)\|_2 + \|f(x) - g_2(x)\|_2 \\ &\le L(C_1 + C_2)\|x\|_1^\alpha \quad (x \in X).\end{aligned}$$

Clearly, $g_1 - g_2$ is additive. Applying Lemma 4, we obtain that $g_1 - g_2 = 0$, so $g_1 = g_2$. □

Now we are going to investigate the linearity of the additive approximating function.

Definition. Let F be a field with a valuation $|\ |_F$, $t_0 \in F$ and $0 < \delta \in \mathbb{R}$. The open ball of radius δ and center t_0 is the set

$$B_\delta(t_0) = \{t \in F : |t - t_0|_F < \delta\}.$$

Lemma 5. *Let F be a field of characteristic zero with some non-trivial valuation $|\ |_F$ such that $\mathbb{Q}$ is dense in F with respect to this valuation. Let Y be a normed space over F. Moreover assume that $g : F \to Y$ is additive. If g is bounded on some open ball then g is of the form $g(t) = ct$.*

Proof. If g is bounded on some open ball then by the additivity, g is also bounded on some neighborhood $B_\delta(0)$ of 0 : $\|g(t)\| \leq M$ for all $t \in B_\delta(0)$. Let $q \in \mathbb{Q} \setminus \{0\}$ such that $|q|_F \neq 1$. Since $\left|\frac{1}{q}\right|_F = \frac{1}{|q|_F}$, we may assume that $|q|_F > 1$. Let $k \in \mathbb{N}$ such that $|q|_F \leq k$. Let $t \neq 0$ be arbitrary. There is some $m \in \mathbb{Z}$ such that $\delta/k \leq |q^{-m}t|_F < \delta$, implying that $|q^m|_F \leq (k/\delta)|t|_F$. Thus $|q^{-m}|_F\|g(t)\| = \|g(q^{-m}t)\| \leq M$ which implies $\|g(t)\| \leq M|q^m|_F \leq \frac{kM}{\delta}|t|_F = M'|t|_F$. Consequently g is continuous. Choose a sequence s_n of rationals converging to t. Then $g(s_n) = s_n g(1)$ tends to $g(t)$. Thus $g(t) = tg(1)$. □

Theorem 3. *Let us suppose that F and Y satisfy the assumptions of Lemma 5, $(X, \|\ \|_1)$ is a normed space over F, $f : X \to Y$ and α is a real number. Moreover let $(Y, \|\ \|_2)$ be a Banach space. If the function f satisfies (3), for every $x \in X$ the mapping $f_x : t \mapsto f(tx)$ $(t \in F)$ is bounded on an open ball $B_{\delta_x}(t_x)$ of non-zero center $t_x \in F$ and radius $\delta_x > 0$, and there exists a positive integer s such that $|s|_F^\alpha \neq |s|_K$, then there exists a unique linear function $g : X \to Y$ for which*

$$\|f(x) - g(x)\|_2 \leq C\|x\|_1^\alpha \quad (x \in X) \tag{10}$$

with some $C \in \mathbb{R}$. Moreover g satisfies (10) with $C = \frac{L\sigma(s,\alpha)}{\left||s|_K - |s|_F^\alpha\right|}$.

Proof. The existence and the uniqueness of g is proved in Theorem 2.

Let us consider an arbitrary $x \in X$ and the function $g_x : F \to Y$, $g_x(t) = g(tx)$. Then

$$\|g_x(t)\|_2 \leq \|f(tx) - g(tx)\|_2 + \|f_x(t)\|_2 \leq M'|t|_F^\alpha + \|f_x(t)\|_2,$$

for all $t \in B_\delta(t_x)$ with $t_x \neq 0$, where $M' = C\|x\|_1^\alpha$ is a constant. We may choose $\delta_x' = \min\left\{\frac{1}{2}|t_x|_F,\ \delta_x\right\}$ implying that for $t \in B_{\delta_x'}(t_x)$ we have $0 < \frac{1}{2}|t_x|_F \leq |t|_F \leq \frac{3}{2}|t_x|_F$. Thus $|t|_F^\alpha \leq M = \max\{(\frac{1}{2})^\alpha,\ (\frac{3}{2})^\alpha\}|t_x|_F^\alpha$ for all $t \in B_{\delta_x'}(t_x)$. Therefore the additive function g_x is bounded on $B_{\delta_x'}(t_x)$. Consequently, by Lemma 5, g_x is linear, so $g(tx) = g_x(t) = tg_x(1) = tg(x)$ for all $t \in F$. □

4. Concluding remarks

Let us notice, that

$$\max\left\{\|x\|_1^\alpha, \|y\|_1^\alpha\right\} \le \|x\|_1^\alpha + \|y\|_1^\alpha \le 2\max\left\{\|x\|_1^\alpha, \|y\|_1^\alpha\right\},$$

so using $2L$ instead of L in Theorem 2 we can write the following:

Theorem 4. *Let $(X, \|\ \|_1)$ be a normed space over a field F of characteristic zero with a valuation $|\ |_F$, $(Y, \|\ \|_2)$ be a Banach space over a field K of characteristic zero with a valuation $|\ |_K$, $f : X \to Y$ and α be a real number. If there exists a positive integer s such that $|s|_F^\alpha \ne |s|_K$ and the function $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\|_2 \le L(\|x\|_1^\alpha + \|y\|_1^\alpha) \quad \text{for every } x, y \in X, \tag{11}$$

then there exists a unique additive function $g : X \to Y$ for which

$$\|f(x) - g(x)\|_2 \le \frac{2L\sigma(s,\alpha)}{\left||s|_K - |s|_F^\alpha\right|}\|x\|_1^\alpha \tag{12}$$

for all $x \in X$.

This theorem gives the theorem of Rassias (see [7]) and the theorem of Gajda (see [4]) if the valuations on F and K are usual absolute values and $s = 2$. It is to be noted that the bound in their result is half as much as here, but if we prove Lemma 1 distinguishing four cases according to the valuation on the field F and the sign of α, we get the same bound.

If $|2|_F^\alpha < |2|_K$, then Theorem 2 follows from G. L. Forti's stability theorem [3] for a class of functional equations

$$g[F(x,y)] = H[g(x), g(y)].$$

Now let us consider the exeptional case when $|s|_F^\alpha = |s|_K$ for every $s \in \mathbb{N}$. Then $|r|_F^\alpha = |r|_K$ for all $r \in \mathbb{Q}$, therefore it is sufficient to verify the valuations on $\mathbb{Q}$. There are three cases.

1. $\alpha = 0$ and $|\ |_K$ is the trivial valuation on $\mathbb{Q}$.
2. $\alpha \ne 0$, $|\ |_F = |\ |^{\beta_1}$ and $|\ |_K = |\ |^{\beta_2}$ for some $\beta_1, \beta_2 \in (0,1]$, where $\alpha\beta_1 = \beta_2$.

3. $\alpha \neq 0$ and there exists a prime number p such that $|\ |_F = |\ |_p^{\beta_1}$ and $|\ |_K = |\ |_p^{\beta_2}$ for some $\beta_1, \beta_2 \geq 0$, where $\alpha\beta_1 = \beta_2$.

Gajda [4] has constructed a counterexample for the case when $\alpha = 1$ and the restriction of $|\ |_F$ and $|\ |_K$ to $\mathbb{Q}$ is the usual absolute value. Similar example can be given for case 2 in the general setting. However, this example does not work for the remaining cases. Therefore, it is not yet decided whether the Cauchy-equation is stabil in order α in Cases 1 and 3.

If we have an additive function $g : X \to Y$, $\alpha \geq 0$ and the function $\phi : X \to Y$ satisfies the inequality $\|\phi(x)\|_2 \leq C\|x\|_1^\alpha$ for some $C \in \mathbb{R}$, then the function $f : X \to Y$, $f(x) = g(x)+\phi(x)$ satisfies (3) with $L = (2^\alpha+2)C$, since

$$\begin{aligned}
&\|f(x+y) - f(x) - f(y)\|_2 = \|\phi(x+y) - \phi(x) - \phi(y)\|_2 \\
&\quad \leq \|\phi(x+y)\|_2 + \|\phi(x)\|_2 + \|\phi(y)\|_2 \leq C\|x+y\|_1^\alpha + C\|x\|_1^\alpha + C\|x\|_1^\alpha \\
&\quad \leq C(\|x\|_1 + \|y\|_1)^\alpha + C\|x\|_1^\alpha + C\|x\|_1^\alpha \\
&\quad \leq 2^\alpha C \max\{\|x\|_1^\alpha, \|y\|_1^\alpha\} + C\|x\|_1^\alpha + C\|x\|_1^\alpha \\
&\quad \leq (2^\alpha + 2)C \max\{\|x\|_1^\alpha, \|y\|_1^\alpha\}.
\end{aligned}$$

Now let us see some examples for the function ϕ with different domains and ranges.

Example. Let p be a prime, $\mathbb{Q}_p$ denote the p-adic field that is the completion for the p-adic valuation of $\mathbb{Q}$, $(Y, \|\ \|_2)$ be a real normed space, $\alpha \geq 0$ and $y \in Y$. Let $\phi : \mathbb{Q}_p \to Y$, $\phi(x) = |x|_p^\alpha y$ for all $x \in \mathbb{Q}_p$. Then $\|\phi(x)\|_2 = |x|_p^\alpha \|y\|_2 = \|y\|_2\|x\|_1^\alpha$.

Example. Let p be a prime, $(Y, \|\ \|_2)$ be a normed space over $\mathbb{Q}_p$, $\alpha \geq 0$ and $y \in Y$. Then let $\phi : \mathbb{R} \to Y$, $\phi(0) = 0$ and $\phi(x) = \frac{1}{\Theta_p(|x|)^\alpha} y$ for all $x \in \mathbb{R} \setminus \{0\}$, where $\Theta_p(x)$ denotes the biggest power of p which is not greater than x ($x \in \mathbb{R}^+$). We can use that $\left|\frac{1}{\Theta_p(|x|)^\alpha}\right|_p = \Theta_p(|x|)^\alpha \leq |x|^\alpha$ ($x \neq 0$), so the function ϕ is an appropriate function.

References

[1] W. A. Beyer, Approximately Lorentz transformations and p-adic fields, *preprint, Los Alamos National Laboratory* (1986), LA-DC-9486.

[2] Z. Boros, 1. Problem and Remarks, Report of Meeting: The First Katowice–Debrecen Winter Seminar on Functional Equations, *Annales Math. Silesianae* **15** (2001), 98–99.

[3] G. L. Forti, An existence and stability theorem for a class of functional equations, *Stochastica* **4** (1980), 23–30.

[4] Z. Gajda, On Stability of additive mappings, *Internat. J. Math. Math. Sci.* **14** (1991), 431–434.

[5] D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U.S.A.* **27** (1941), 222–224.

[6] A. Ostrowski, Über einige Lösungen der Funktionalgleichung $\phi(x)\phi(y) = \phi(xy)$, *Acta Math.* **41** (1918), 271–284.

[7] Th. M. Rassias, On the stability of linear mapping in Banach spaces, *Proc. Amer. Math. Soc.* **72** (1978), 297–300.

[8] J. Schwaiger, Functional equations for homogeneous polynomials arising from multilinear mappings and their stability, *Ann. Math. Silesianae* **8** (1994), 157–171.

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