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A characterization of radicals in finite groups

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Abstract. We give a characterization of an $\mathfrak{F}$-radical of a finite group using Shemetkov's concept of generalized centralizers of chief factors.

1. Introduction

All groups considered in this paper are finite. The reader is assumed to be familiar with the theory of formations. We shall adhere to the notations used in [1].

It is well-known that the Fitting subgroup $F(G)$ of a group G coincides with the intersection of centralizers of all chief factors of G. An analogous result hold for the p-nilpotent radical $F_p(G)$ of G (see [1], p. 45). In [2] we extended these results using Shemetkov's concept of an f-centralizer (see [3]). In this paper we improve results in [2] by considering an intersection of f-normalizers of some family of non-Frattini chief factors.

2. Preliminary results

We remind that a formation is a class of groups closed under taking homomorphic images and subdirect products. A formation $\mathfrak{F}$ is called: 1) P-saturated if $G/O_p(G) \cap \Phi(G) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$; 2) $\mathfrak{N}_p$-saturated if $G/\Phi(O_p(G)) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$; 3) p-solubly saturated

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if from $G/N \in \mathfrak{F}$, where N is a p-subgroup contained in the Frattini subgroup of a p-soluble normal subgroup of G, it always follows that $G \in \mathfrak{F}$.

In [4] the following result is obtained.

Theorem 2.1 (L. A. Shemetkov)**.** *A formation $\mathfrak{F}$ is $\mathfrak{N}_p$-saturated iff it is p-solubly saturated.*

We also need the following result.

Theorem 2.2 (see [5], Theorem 1)**.** *Let $\mathfrak{F}$ be a formation, and let N and K be some normal subgroups of G such that $K \subseteq N$, $N/K \in \mathfrak{F}$, and $K \in \mathfrak{F}$. Assume further that for each prime divisor p of $|K|$ one of the following conditions is satisfied:*

1) *$\mathfrak{F}$ is p-saturated, and a Sylow p-subgroup of K is contained in $\Phi(G)$;*
2) *$\mathfrak{F}$ is $\mathfrak{N}_p$-saturated, and a Sylow p-subgroup of K is contained in the Frattini subgroup of the p-soluble radical of G.*

Then $N \in \mathfrak{F}$.

Suppose that a non-empty formation $\mathfrak{F}$ is solubly saturated, i.e. it is p-solubly saturated for every prime p. Then $\mathfrak{F}$ can be defined by a function

$$f : \{\text{simple groups}\} \to \{\text{formations}\}$$

such that $f(G_1) = f(G_2)$ if $G_1 \simeq G_2$; L. A. Shemetkov calls that function a composition satellite. We write $\mathfrak{F} = CF(f)$ if $\mathfrak{F}$ is the class of groups G such that every chief factor H/K of G is f-central, i.e. $G/C_G(H/K) \in f(A)$, where $H/K = A \times \cdots \times A$; in this case they say that f is a composition satellite of $\mathfrak{F}$. It is known that a non-empty formation is solubly saturated iff it has a composition satellite (see [1], Theorem IV.4.17; a generalised version is in [5], Lemma 7). A composition satellite f of $\mathfrak{F} = CF(f)$ is called: 1) integrated if $f(A) \subseteq \mathfrak{F}$ for any simple group A; 2) semi-integrated if for any simple group A either $f(A) \subseteq \mathfrak{F}$ or $f(A)$ is the class $\mathfrak{E}$ of all groups. It will be convenient to assume that $f(H) = f(A)$ if every composition factor of H is isomorphic to a simple group A. Following [3], we give a definition of an f-centralizer. Let K, M and N be normal subgroups of G, and $M \supseteq N$. Assume that M/N is characteristically simple. They say that K acts f-centrally on M/N if $K/C_K(M/N) \in f(M/N)$. Then an f-centralizer $C_G^f(M/N)$ of M/N is the product of all the normal subgroups of G that act f-centrally on M/N.

Lemma 2.1 ([6], Lemma 15.6). *Every non-empty solubly saturated formation* $\mathfrak{F}$ *has a unique maximal semi-integrated composition satellite* f, *and for every simple group* H *one of the following statements holds:*
1) $|H| = p$ *is a prime, and* $f(p) = \mathfrak{N}_p f(p)$; 2) $f(H) = \mathfrak{F}$; 3) $f(H) = \mathfrak{E}$.

Lemma 2.2 ([6], Lemma 17.7). *Let* f *be a maximal semi-imtegrated satellite of a Fitting formation* $\mathfrak{F}$. *Then* $f(A)$ *is a Fitting formation for every simple group* A.

The following result is due to P. Hall.

Lemma 2.3 (see [7], Theorem 9.9 and Lemma 9.3). *Let* A *be an automorphism group of a group* G. *Assume that* A *acts trivially on each* A*-chief factor of* G. *If the Fitting subgroup* $F(G)$ *of* G *is a* π*-group, then* A *is a nilpotent* π*-group.*

Lemma 2.4 (see [6], Lemma 15.4). *Let* $\mathfrak{F} = CF(f)$ *be a solubly saturated formation. Let* M *be a normal subgroup of a group* G. *If* $M \in \mathfrak{F}$, *then* M *acts* f*-centrally on each* G*-chief factor of* M.

Lemma 2.5 ([6], Lemma 15.10). *Let* $\mathfrak{F} = CF(f)$ *be a Fitting formation, and* f *be semi-integrated. Let* H/K *be a chief factor of* G, *and* N *be a normal subgroup of* G *contained in* $C_G^f(H/K)$. *Then* N *acts* f*-centrally on* H/K.

3. Main results

We introduce the two following definitions.

Definition 1. If $\mathfrak{F} = CF(f)$ then f^c is a composition satellite such that $f^c(A) = f(A)$ if $f(A) \neq \emptyset$, and $f^c(A) = \mathfrak{E}$ if $f(a) = \emptyset$.

Definition 2. A chief factor H/K of a group G is called:

1) an s-Frattini chief factor if H/K is contained in the Frattini subgroup of the soluble radical of G/K;

2) a ps-Frattini chief factor if H/K is a q-group for some prime q, and H/K is contained in the Frattini subgroup of the q-soluble radical of G/K.

In [2] we considered intersections of f-centralizers of non-s-Frattini chief factors. Now we are going to study intersections of non-ps-Frattini chief factors. The considered situation is more general in view of the following example.

Example. Let A be a non-abelian simple group, and $|A|$ is not divisible by a prime p. Consider a wreath product H of A with a group of order p. The group H is p-nilpotent and its soluble radical is the identity. By Theorem B.11.8 in [1], there exists a group G with a minimal normal p-subgroup L such that $L \subseteq \Phi(G)$, and $G/L \simeq H$. Evidently, L is ps-Frattini. Since the soluble radical of G is equal to L, it follows that L is non-s-Frattini. So, this example shows that the set of non-s-Frattini chief factors is wider than the set of non-ps-Frattini chief factors.

The main result of this paper is the following.

Theorem 3.1. *Let $\mathfrak{F}$ be a Fitting formation, and $\mathfrak{F} = CF(f)$, where f is a semi-integrated composition satellite. Then $\mathfrak{H} = CF(f^c)$ is a Fitting formation containing $\mathfrak{F}$, and $G_{\mathfrak{H}}$ coincides with the intersection of f^c-centralizers of all non-ps-Frattini chief factors of G.*

Proof. We prove that $\mathfrak{H}$ is a Fitting formation. By Lemma 2.1, $\mathfrak{F}$ has a unique maximal semi-integrated composition satellite F, and for every simple group A one of the following conditions is satisfied: 1) $|A| = p$ is a prime, and $F(A) = \mathfrak{N}_p f(A)$; 2) $F(A) \in \{\mathfrak{F}, \mathfrak{E}\}$. By Lemma 2.2, all the values of F are Fitting formations. Let R be a normal subgroup of a group H in $\mathfrak{H}$. By induction, we can assume that H has the only minimal normal subgroup N contained in R. If $f^c(N) = \mathfrak{E}$, then from $R/N \in \mathfrak{H}$ it follows that $R \in \mathfrak{H}$. Assume that $f^c(N) = f(N) \neq \mathfrak{E}$. Then $H/C_H(N)$ is contained in $f(N) \subseteq \mathfrak{F}$. Since $\mathfrak{F}$ is a Fitting formation, we have $R/C_R(N) \in \mathfrak{F}$. If N is non-abelian, then $C_H(N) = 1$, and therefore $R \in \mathfrak{F}$. Assume that N is a p-group. Then $F(N) = \mathfrak{N}_p f(N)$ is a Fitting formation, and from $H/C_H(N) \in F(A)$ it follows that $R/C_R(N) \in \mathfrak{N}_p f(N)$. Since irreducible automorphism groups of p-groups have no non-identity normal p-subgroups, it follows that every R-chief factor of N is f-central in R, i.e. $R \in \mathfrak{H}$. So, we proved that $\mathfrak{H}$ is closed under taking normal subgroups.

Now we consider a group $H = R_1 R_2$, where R_1 and R_2 are normal $\mathfrak{H}$-subgroups of H. We prove that $H \in \mathfrak{H}$. It is true if $R_1 \cap R_2 = 1$

because $\mathfrak{H}$ is a formation. Suppose that $R_1 \cap R_2 \neq 1$. Let N be a minimal normal subgroup of H contained in $R_1 \cap R_2$. By induction, N is a unique minimal normal subgroup of H. If $f^c(N) = \mathfrak{E}$, then from $H/N \in \mathfrak{H}$ it follows that $H \in \mathfrak{H}$. Assume that $f^c(N) = f(N) \neq \mathfrak{E}$, i.e. $f(N) \subseteq \mathfrak{F}$. If N is non-abelian, then $C_H(N) = 1$, and the Fitting subgroup of N is the identity. Since every R_i-chief factor of N is f-central in R_i, we obtain $R_i \in \mathfrak{F}$, $i = 1, 2$ (here, we use Lemma 2.3). Hence, $H = R_1R_2 \in \mathfrak{F} \subseteq \mathfrak{H}$. Now assume that N is a p-group, p a prime. Since every R_i-chief factor of N is f-central in R_i, we obtain that

$$R_i/C_{R_i}(N) \in \mathfrak{N}_p f(N) = F(N), \quad i = 1, 2.$$

Since $F(N)$ is a Fitting formation, $H/C = (R_1C/C)(R_2C/C)$ belongs to $F(N)$, where $C = C_H(N)$. Clearly, every H-chief factor of N is f-central in H, i.e. $H \in \mathfrak{F} \subseteq \mathfrak{H}$. So, we proved that $\mathfrak{H}$ is a Fitting formation.

Let D be the intersection of f^c-centralizers of all non-*ps*-Frattini chief factors of G. By Lemma 2.4, $G_{\mathfrak{H}}$ is contained in D. In order to prove an inclusion $D \subseteq G_{\mathfrak{H}}$ we need to prove that $D \in \mathfrak{H}$. Let N be a minimal normal subgroup of G contained in D. By Lemma 2.5, D acts f^c-centrally on each non-*ps*-Frattini chief factor of G. By induction, $D/N \in \mathfrak{H}$. If N is non-*ps*-Frattini, D acts f^c-centrally on N, and we have $D \in \mathfrak{H}$. Suppose that N is *ps*-Frattini. It means that N is a p-group, and N is contained in the Frattini subgroup of the p-soluble radical of G. By Theorem 2, $D \in \mathfrak{H}$. Theorem is proved. □

Using Lemma 2.4 we immediately obtain the following.

Corollary 3.1. *If $\mathfrak{F} = CF(f)$ is a Fitting formation containing the class $\mathfrak{N}$ of nilpotent groups. If f is semi-integrated, then for every group G the following statements hold:*

1) *$G_{\mathfrak{F}}$ coincides with the intersection of f-centralizers of all chief factors of G;*

2) *$G_{\mathfrak{F}}$ coincides with the intersection of f-centralizers of all non-s-Frattini chief factors of G;*

3) *$G_{\mathfrak{F}}$ coincides with the intersection of f-centralizers of all non-ps-Frattini chief factors of G.*

A group G is called quasinilpotent if $G = HC_G(H/K)$ for each chief factor H/K of G. The class $\mathfrak{N}^*$ of quasinilpotent groups is a solubly saturated Fitting formation. Moreover, $\mathfrak{N}^* = CF(f)$, where f is a satellite such that $f(A) = (1)$ if A is abelian, and $f(A) = \mathrm{form}\,(A)$ if A is a non-abelian simple group.

Corollary 3.2. *Let $\mathfrak{N}^* = CF(f)$, and f be semi-integrated. Then the $\mathfrak{N}^*$-radical R of a group G satisfies the following conditions:*

1) *R coincides with the intersection of f-centralizers of all chief factors of G;*
2) *R coincides with the intersection of f-centralizers of all non-s-Frattini chief factors of G;*
3) *R coincides with the intersection of f-centralizers of all non-ps-Frattini chief factors of G.*

A formation is called saturated if it is p-saturated for every prime p. By Gaschütz–Lubeseder–Schmid theorem ([1, Theorem IV.4.6]), every saturated formation $\mathfrak{F}$ is defined by a local satellite $f : \{\text{primes}\} \to \{\text{formations}\}$, and they write $\mathfrak{F} = LF(f)$. We can consider this local satellite as a composition satellite assuming $f(H) = \bigcap_{p|\,|H|} f(H)$ for every group H.

Definition 3. Let $\mathfrak{F} = LF(f)$ be a saturated formation.

(1) A local satellite f is called semi-integrated if for each prime p either $f(p) \subseteq \mathfrak{F}$ or $f(p) = \mathfrak{E}$;

(2) f^l is a local satellite such that $f^l(p) = f(p)$ if $f(p) \neq \emptyset$, and $f^l(p) = \mathfrak{E}$ if $f(p) = \emptyset$.

Theorem 3.2. *Let $\mathfrak{F} = LF(f)$ be a saturated Fitting formation, and the local satellite f be semi-integrated. Then the following statements hold:*

(1) *$\mathfrak{H} = LF(f^l)$ is a Fitting formation containing $\mathfrak{F}$;*

(2) *if $\mathfrak{F} \supseteq \mathfrak{N}$, then $G_{\mathfrak{F}}$ coincides with the intersection of f-centralizers of all non-Frattini chief factors of G.*

Proof. In the considered case, we have $LF(f) = CF(f)$ and $LF(f^l) = CF(f^l)$. Therefore, we can apply Theorem 3.1. So, $\mathfrak{H} = LF(f^l)$ is a Fitting formation. Suppose that $\mathfrak{F}$ contains the formation $\mathfrak{N}$ of nilpotent groups. Let D_1 be the intersection of f-centralizers of all non-Frattini chief factors of G, and D_2 be the intersection of f-centralizers of all non-ps-Frattini chief factors of G. By Lemma 2.4, we have $G_{\mathfrak{F}} \subseteq D_1 \subseteq D_2$. By Theorem 3.1, $D_2 = G_{\mathfrak{F}}$. Hence, $G_{\mathfrak{F}} = D_1$. Theorem is proved. □

A group G is called a pd-group if p divides $|G|$.

Corollary 3.3 ([1], p. 45)**.** *The p-nilpotent radical $F_p(G)$ of every group G coincides with the intersection of centralizers of all non-Frattini chief pd-factors of G.*

A group G is called p-decomposable if it has a normal Sylow p-subgroup and a normal Hall p'-subgroup. The formation of p-decomposable groups has a local satellite f such that $f(p) = (1)$, and $f(q)$ is the class of p'-groups for every prime $q \neq p$. Clearly, the following assertion is true,

Corollary 3.4. *Let $\mathfrak{F}$ be the formation of p-decomposable groups, and G a group. Then $G_{\mathfrak{F}} = \bigcap C^*_G(H/K)$, where H/K ranges over all non-Frattini chief factors of G, and $C^*_G(H/K) = C_G(H/K)$ if H/K is a pd-group, and $C^*_G(H/K)/K = O_{p'}(G/C_G(H/K))$ if p does not divide $|H/K|$.*

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