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The linear-affine functional equation and group actions

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Abstract. We investigate generalizations of the linear-affine functional equation

$$u(rx) = \alpha(r)u(x) + \beta(r)$$

usually studied for $r, x \in \mathbb{R}_{>0}$ or $r, x \in \mathbb{R}_{\geq 1}$, by introducing a group action of a group R on a set X on the left hand side, and by studying actions of affine or arbitrary (semi) groups on the right hand side of this equation.

1. Introduction

The linear-affine functional equation is the equation

$$u(rx) = \alpha(r)u(x) + \beta(r)$$

for the three unknown functions u, α, and β. It is usually studied for r, x being elements of the group $\mathbb{R}_{>0}$, for r, x belonging to the semigroup $\mathbb{R}_{\geq 1}$, but also on intervals, and under certain regularity conditions (like monotonicity, continuity, boundedness on an interval) on the real-valued functions u, α and β. (See [1], pp. 37–41 and 148–151, [2], [3], [4], Sections 2–4, [5], and [10].)

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We want to study the following generalization of the functional equation above: Let X be a set, R a multiplicative group acting on X, and let V be a linear space over the field K. We study the linear-affine functional equation

$$u(rx) = \alpha(r)u(x) + \beta(r), \qquad r \in R,\ x \in X \tag{1}$$

for the three unknown functions

$$u : X \to V \qquad \alpha : R \to K \qquad \beta : R \to V.$$

A solution of (1) is indicated as a triple (u, α, β).

A multiplicative group R with neutral element 1 acts on a set X, if there exists a mapping

$$* : R \times X \to X, \qquad *(r, x) = r * x$$

such that

$$(r_1 r_2) * x = r_1 * (r_2 * x), \qquad r_1, r_2 \in R,\ x \in X$$

and

$$1 * x = x, \qquad x \in X.$$

We usually write rx instead of $r * x$. The orbit $R(x)$ of $x \in X$ is defined as the set $\{rx \mid r \in R\}$. It is the equivalence class of x with respect to the equivalence relation

$$x_1 \sim_R x_2 :\Longleftrightarrow \exists\, r \in R : x_2 = rx_1.$$

The set of orbits (i.e. the set of equivalence classes) will be denoted by $R\backslash\backslash X = \{R(x) \mid x \in X\}$. A transversal $\mathcal{T}(R\backslash\backslash X)$ is a complete set of orbit representatives. The stabilizer of $x \in X$ is $R_x = \{r \in R \mid rx = x\}$ which is a subgroup of R. An element $x \in X$ is called a fixed point of $r \in R$ if $rx = x$.

The results concerning this generalization of the linear-affine functional equation will be presented as Lemma 1, Lemma 2, Lemma 4, Lemma 5, Lemma 7, Corollary 18, and Theorem 19. In a final step (cf. Section 3) we study the functional equation

$$u(rx) = \varphi(r)u(x), \qquad r \in R,\ x \in X \tag{12}$$

for the unknown functions

$$u : X \to Y \qquad \varphi : R \to S$$

where now X and Y are sets, R is a group acting on X, and S is a (semi)group acting on Y. It is not difficult to see that (1) is a special case of (12) (see Section 3).

The description of the general solution of (1) respectively (12) under the assumptions made in this paper is carried out in the terminology of group actions (or semigroup actions) and by certain constructions, rather than by explicit formulas. This seems to be natural in our context, cf. [7], where analogous methods were applied to the equation of the mean sun. See also [9] where the general solution of the translation equation is described in the same spirit.

2. First generalization by introducing a group action on the left hand side

2.1. The general solution of (1) in special situations.

Lemma 1. *Assume that $\alpha = 0$. The triple $(u, 0, \beta)$ is a solution of (1) if and only if there exists a vector $v_0 \in V$ such that $u(x) = \beta(r) = v_0$ for all $x \in X$ and $r \in R$.*

Proof. If $(u, 0, \beta)$ is a solution of (1), then $u(rx) = \beta(r)$ for all $x \in X$ and $r \in R$, whence $u(x) = u(rr^{-1}x) = \beta(r)$ for all $x \in X$ and $r \in R$.

Conversely, if $u = v_0$, $\beta = v_0$ for some $v_0 \in V$ and $\alpha = 0$, then (u, α, β) satisfies (1). □

Lemma 2. *Assume that u is a constant function, say $u = v_0 \in V$. The triple (u, α, β) is a solution of (1) if and only if $\beta(r) = (1 - \alpha(r))v_0$ for all $r \in R$.*

In order to construct all solutions (u, α, β) where u is constant, we may choose an arbitrary constant function $u = v_0 \in V$ and an arbitrary function α and determine β by $\beta(r) = (1-\alpha(r))v_0$. This gives all solutions of (1) in the present case.

If u is not constant, then for each solution (u, α, β) of (1) the function α is a group homomorphism from R to $K^* := K \setminus \{0\}$, and (α, β) satisfies

$$\beta(rt) = \alpha(r)\beta(t) + \beta(r), \qquad r, t \in R. \tag{2}$$

Proof. Only the last part of the lemma needs some justification. Assume that (u, α, β) is a solution of (1) and u is not constant. Then there exist $x, y \in X$ such that $u(x) \neq u(y)$. For $r, t \in R$ and $x \in X$ we obtain

$$\begin{aligned} u((rt)x) - u((rt)y) &= \alpha(rt)u(x) + \beta(rt) - \big(\alpha(rt)u(y) + \beta(rt)\big) \\ &= \alpha(rt)\big(u(x) - u(y)\big) \end{aligned}$$

and

$$u(r(tx)) - u(r(ty)) = \alpha(r)\big(u(tx) - u(ty)\big) = \alpha(r)\alpha(t)\big(u(x) - u(y)\big).$$

Thus we conclude that $\alpha(rt)(u(x) - u(y)) = \alpha(r)\alpha(t)(u(x) - u(y))$, and since $u(x) - u(y) \neq 0$ it follows that $\alpha(rt) = \alpha(r)\alpha(t)$ for all $r, t \in R$. If there were some $r_0 \in R$ such that $\alpha(r_0) = 0$, then $\alpha(r) = \alpha(r_0 r_0^{-1} r) = \alpha(r_0)\alpha(r_0^{-1} r) = 0$ for all $r \in R$, whence $\alpha = 0$. Then, according to Lemma 1, u is constant. This is a contradiction, thus $\alpha(r) \neq 0$ for all $r \in R$ which means that α is a group homomorphism. These homomorphism are also called (onedimensional) characters, but here we rather speak of homomorphisms.

Moreover, we obtain for $r, t \in R$ and $x \in X$ that

$$\begin{aligned} \alpha(rt)u(x) + \beta(rt) = u((rt)x) = u(r(tx)) &= \alpha(r)\big(\alpha(t)u(x) + \beta(t)\big) + \beta(r) \\ &= \alpha(r)\alpha(t)u(x) + \big(\alpha(r)\beta(t) + \beta(r)\big) \end{aligned}$$

which implies (2), since α is a homomorphism. □

From the last proof we obtain

Corollary 3. *If (u, α, β) is a solution of (1) where α is a homomorphism, then (α, β) satisfies (2).*

Lemma 4. *Assume that $\alpha = 1$. Let $R' := \langle\{r \in R \mid \exists\, x \in X : rx = x\}\rangle$, then R' is a normal subgroup of R, in short $R' \trianglelefteq R$.*

If $(u, 1, \beta)$ is a solution of (1), then β is a group homomorphism and $\ker \beta \geq R'$.

In order to construct all solutions $(u,1,\beta)$ *of* (1), *assume that* β *is a group homomorphism with* $\ker\beta \geq R'$. *If* u *takes arbitrary values* $u(x_0)\in V$ *for* x_0 *belonging to a transversal* $\mathcal{T}(R\backslash\backslash X)$ *and* $u(rx_0)$ *is defined as* $u(x_0)+\beta(r)$ *for all* $r\in R$, *then* $(u,1,\beta)$ *is a solution of* (1). *This gives all solutions of* (1) *in the present case.*

Proof. By definition R' is a subgroup of R. If r' belongs to R', then there exist $n\in\mathbb{N}$, $r_1,\ldots,r_n\in R$, and $x_1,\ldots,x_n\in X$ such that $r'=r_1\cdots r_n$ and $r_ix_i=x_i$ for $1\leq i\leq n$. (We need not explicitly consider the multiplicative inverse of the elements r_i in the representation of r', since, if $x\in X$ is a fixed point of $r\in R$, then $r^{-1}x=r^{-1}(rx)=x$, whence x is also a fixed point of r^{-1}.) In order to prove that R' is a normal subgroup of R, we show that $tR't^{-1}$ is a subset of R' for any $t\in R$. The product $tr't^{-1}$ which can be written as $(tr_1t^{-1})(tr_2t^{-1})\cdots(tr_nt^{-1})$ also belongs to R', since tr_it^{-1} has tx_i as a fixed point.

Assume that $(u,1,\beta)$ is a solution of (1), then

$$u(rx)=u(x)+\beta(r),\qquad r\in R,\ x\in X.$$

For $r,t\in R$ and $x\in X$ we obtain

$$\begin{aligned}u(x)+\beta(rt)&=u((rt)x)=u(r(tx))=u(tx)+\beta(r)\\&=\bigl(u(x)+\beta(t)\bigr)+\beta(r),\end{aligned}$$

whence $\beta(rt)=\beta(r)+\beta(t)$. In other words, β is a group homomorphism. (Thus $\beta(r_1\cdots r_n)=\beta(r_1)+\cdots+\beta(r_n)$ for $n\in\mathbb{N}$, $r_1,\ldots,r_n\in R$.)

Let $r'=r_1\cdots r_n$ with $r_ix_i=x_i$ for some $x_i\in X$ for $1\leq i\leq n$. Consequently $r_i\in R'$ and $\beta(r_i)=0$ for $1\leq i\leq n$, since $u(x_i)+\beta(r_i)=u(r_ix_i)=u(x_i)$. Moreover, $\beta(r')=\beta(r_1\cdots r_n)=\beta(r_1)+\cdots+\beta(r_n)=0$ for all $r'\in R'$. Hence $\ker\beta\geq R'$.

Conversely assume that $R'\trianglelefteq R$ and β is a homomorphism whose kernel contains R'. Let x_0 be an arbitrary element of $\mathcal{T}(R\backslash\backslash X)$ and let $u(x_0)$ be an arbitrary element in V. If we define $u(rx_0):=u(x_0)+\beta(r)$, then we claim that u is well defined on the orbit $R(x_0)$. Assume that $r_1x_0=r_2x_0$ for $r_1,r_2\in R$, then $r_2^{-1}r_1x=x$ whence $r_2^{-1}r_1\in R'$ and consequently $0=\beta(r_2^{-1}r_1)=-\beta(r_2)+\beta(r_1)$ which leads to $\beta(r_1)=\beta(r_2)$ and $u(r_1x_0)=u(r_2x_0)$. If $x_1\in\mathcal{T}(R\backslash\backslash X)$ is different from x_0, then u can be determined on $R(x_1)$ independently from the values of u on $R(x_0)$. This way we determine functions u defined on X, which is the disjoint union of $R(x_0)$ for all $x_0 \in \mathcal{T}(R\backslash\backslash X)$, by choosing for each x_0 an arbitrary value $u(x_0)$ and by setting $u(rx_0) = u(x_0) + \beta(r)$ for $r \in R$. Finally we have to prove that for each of these u the triple $(u, 1, \beta)$ is a solution of (1). Take an arbitrary $x \in X$, then there exists some $r_0 \in R$ and a uniquely determined $x_0 \in \mathcal{T}(R\backslash\backslash X)$ such that $x = r_0 x_0$. For arbitrary $r \in R$ we have $u(rx) = u(r(r_0 x_0)) = u((rr_0)x_0) = u(x_0) + \beta(rr_0) = u(x_0) + \beta(r_0) + \beta(r) = u(r_0 x_0) + \beta(r) = u(x) + \beta(r)$. It is obvious that each solution $(u, 1, \beta)$ of (1) can be obtained in this way. $\square$

Lemma 5. *Assume that $\beta = 0$. If $(u, \alpha, 0)$ is a solution of (1) where u is not constant, then α is a group homomorphism from R to K^* and $\ker \alpha \geq R'_u := \langle \{r \in R \mid \exists\, x \in X : u(x) \neq 0 \text{ and } rx = x\} \rangle$. Moreover, for each orbit $\omega \in R\backslash\backslash X$, either $u(x) = 0$ for all $x \in \omega$ or $u(x) \neq 0$ for all $x \in \omega$.*

In order to construct all solutions $(u, \alpha, 0)$ of (1) where u is not constant, choose a subset X' of X as a nonempty union of R-orbits. Let $R'' := \langle \{r \in R \mid \exists\, x \in X' : rx = x\} \rangle$, and let $\alpha : R \to K^*$ be a homomorphism with $\ker \alpha \geq R''$. If $u(x) = 0$ for all $x \in X \setminus X'$ and if u takes arbitrary values $u(x_0) \in V \setminus \{0\}$ for all x_0 belonging to a transversal $\mathcal{T}(R\backslash\backslash X')$ and $u(rx_0)$ is defined as $\alpha(r)u(x_0)$ for all $r \in R$, then $(u, \alpha, 0)$ is a solution of (1). By this construction we obtain all solutions $(u, \alpha, 0)$ of (1) where u is not identically 0, which allows to determine all solutions where u is not constant.

Proof. Assume that $(u, \alpha, 0)$ is a solution of (1), then

$$u(rx) = \alpha(r)u(x), \qquad r \in R,\ x \in X.$$

If u is not constant, then according to Lemma 2 the function α is a group homomorphism from R to K^*. Using similar methods as in the proof of Lemma 4 we prove that R'_u is a normal subgroup of R and that $\ker \alpha \geq R'_u$. Moreover, if $u(x_0) = 0$ for some $x_0 \in X$, then $u(rx_0) = \alpha(r)u(x_0) = 0$ for all $r \in R$, whence u is 0 on the orbit $R(x_0)$.

Conversely, choose a nonempty subset X' of X. We want to determine all solutions (u, α, β) of (1) where u is 0 on $X \setminus X'$ and $u(x) \neq 0$ for all $x \in X'$. Then, necessarily X' is a union of R-orbits. Determine R'' depending on the action of R on X' as indicated above and let α be a homomorphism with $\ker\alpha \geq R''$. As a matter of fact, R'' is a normal subgroup of R. Using similar methods as in the proof of Lemma 4 we prove that if u takes arbitrary values $u(x_0) \in V \setminus \{0\}$ for all x_0 belonging to a transversal $\mathcal{T}(R\backslash\backslash X')$ and $u(rx_0)$ is defined as $\alpha(r)u(x_0)$ for all $r \in R$, then u is well defined on each orbit and the triple $(u, \alpha, 0)$ is a solution of (1). □

Finally we have to consider the case that u is not constant, $\alpha \neq 1$ is a homomorphism, and $\beta \neq 0$. First we study those solutions where β satisfies the property

$$\beta(rt) = \beta(tr), \qquad r, t \in R. \tag{K}$$

Lemma 6. *If (u, α, β) is a solution of* (1) *where u is not constant, then β satisfies* (K) *if and only if β satisfies the so called* Kannappan *condition (cf. for instance* [8], [6], [11]*)*

$$\beta(srt) = \beta(str), \qquad s, r, t \in R.$$

Proof. (K) follows from the Kannappan condition for $s = 1$.

Conversely, according to Lemma 2, the pair (α, β) satisfies (2). Since β satisfies (K), for any $s, r, t \in R$ we have

$$\beta(srt) = \alpha(s)\beta(rt) + \beta(s) = \alpha(s)\beta(tr) + \beta(s) = \beta(str). \qquad \square$$

Consequently we call (K) the Kannappan condition for β.

If β satisfies (K), then the set of all solutions of (1) is described in

Lemma 7. *If (α, β) is a solution of* (2) *where $\alpha \neq 1$ is a group homomorphism and β satisfies* (K)*, then for any $t \notin \ker\alpha$ and for all $r \in R$*

$$\beta(r) = (1 - \alpha(t))^{-1}(1 - \alpha(r))\beta(t)$$

holds.

If (u, α, β) is a solution of (1) *where u is not constant, $\alpha \neq 1$, $\beta \neq 0$, and β satisfies* (K)*, then there exists a vector $v_0 \in V \setminus \{0\}$ such that $\beta(r) = (1 - \alpha(r))v_0$ for all $r \in R$. Moreover, the function $U : X \to V$ defined by $U(x) := u(x) - v_0$ satisfies*

$$U(rx) = \alpha(r)U(x), \tag{3}$$

a functional equation, which was completely solved in Lemma 5. Thus $\ker\alpha \geq \langle\{r \in R \mid \exists\, x \in X : u(x) \neq v_0$ *and* $rx = x\}\rangle$.

In order to construct all solutions (u,α,β) *of* (1) *with* u *not constant,* $\alpha \neq 1$, *and* $\beta \neq 0$, *satisfying* (K), *we determine all solutions* (U,α) *of* (3) *where* U *is not constant and* $\alpha \neq 1$. *Then for each* (U,α) *and for any vector* $v_0 \in V \setminus \{0\}$, *we get a solution* (u,α,β) *of* (1) *by setting* $u(x) := U(x) + v_0$ *for all* $x \in X$, *and* $\beta(r) := (1-\alpha(r))v_0$ *for all* $r \in R$. *These* (u,α,β) *are all solutions of* (1) *under the above conditions.*

Proof. Assume that (α,β) is a solution of (2) where $\alpha \neq 1$ is a group homomorphism and β satisfies (K), then from $\beta(rt) = \beta(tr)$ for $r,t \in R$ we obtain

$$\alpha(r)\beta(t) + \beta(r) = \alpha(t)\beta(r) + \beta(t)$$

and consequently

$$\bigl(1-\alpha(t)\bigr)\beta(r) = \bigl(1-\alpha(r)\bigr)\beta(t).$$

If t_0 denotes an element of R such that $\alpha(t_0) \neq 1$, this implies

$$\beta(r) = (1-\alpha(t_0))^{-1}(1-\alpha(r))\beta(t_0), \qquad r \in R.$$

Moreover, $\beta = 0$ if and only if $\beta(t_0) = 0$.

Assume that (u,α,β) is a solution of (1) where u is not constant, $\alpha \neq 1$, and $\beta \neq 0$, satisfying (K). For arbitrary $t_0 \in R \setminus \ker\alpha$ we deduce $\beta(t_0) \neq 0$ and $\beta(r) = (1-\alpha(r))v_0$ for $v_0 = (1-\alpha(t_0))^{-1}\beta(t_0)$. This allows to rewrite (1) as

$$u(rx) = \alpha(r)u(x) + (1-\alpha(r))v_0 = \alpha(r)\bigl(u(x)-v_0\bigr) + v_0.$$

Introducing $U : R \to V$ by $U(x) := u(x) - v_0$ we end up with the equation

$$U(rx) = \alpha(r)U(x) \tag{3}$$

with U not constant, which was solved in Lemma 5. Since (U,α) is a solution of (3) where U is not constant, $\ker\alpha \geq R'_U = \langle\{r \in R \mid \exists\, x \in X : U(x) \neq 0 \text{ and } rx = x\rangle\} = \langle\{r \in R \mid \exists\, x \in X : u(x) \neq v_0 \text{ and } rx = x\}\rangle$.

Conversely, if (U,α) is a solution of (3) where U is not constant, and if v_0 is an arbitrary element of $V \setminus \{0\}$, then the triple (u,α,β) with $u(x) = U(x) + v_0$ for $x \in X$, and $\beta(r) = (1-\alpha(r))v_0$ for $r \in R$ is a solution of (1) since

$$u(rx) = U(rx) + v_0 = \alpha(r)U(x) + v_0 = \alpha(r)\bigl(u(x)-v_0\bigr) + v_0$$

$$= \alpha(r)u(x) + \bigl(1-\alpha(r)\bigr)v_0 = \alpha(r)u(x) + \beta(r). \qquad \square$$

Hence, it is important to analyze, under which conditions β necessarily satisfies (K).

Lemma 8. *Assume that $\alpha \neq 1$ is a group homomorphism and that (α, β) satisfies* (2).

There exists a function u and some $x_0 \in X$ such that (u, α, β) is a solution of (1) *where u is constant on the orbit $R(x_0)$ if and only if β satisfies* (K).

Proof. If (u, α, β) is a solution of (1) and (2) where α is a homomorphism and where u is constant on the orbit $R(x_0)$, then $u(x_0) = u(rx_0) = \alpha(r)u(x_0) + \beta(r)$ for all $r \in R$, whence $\beta(r) = (1-\alpha(r))u(x_0)$. For that reason $\beta(rt) = (1-\alpha(rt))u(x_0) = (1-\alpha(r)\alpha(t))u(x_0) = (1-\alpha(tr))u(x_0) = \beta(tr)$ for all $r, t \in R$. Thus β satisfies (K).

Conversely assume that β satisfies (K). Since $\alpha \neq 1$, according to Lemma 7 $\beta(r) = (1-\alpha(r))v_0$ with $v_0 = (1-\alpha(t_0))^{-1}\beta(t_0)$ for some $t_0 \notin \ker\alpha$ and for all $r \in R$. Define $u(x) = v_0$ for all $x \in X$, then u is constant on any orbit of R, and $\alpha(r)u(x)+\beta(r) = \alpha(r)v_0+(1-\alpha(r))v_0 = v_0 = u(rx)$ for all $r \in R$ and $x \in X$. Consequently (u, α, β) is a solution of (1). $\square$

Corollary 9. *If there exists an orbit $\omega \in R\backslash\backslash X$ of cardinality* 1, *then for each solution (u, α, β) of* (1), *where α is a homomorphism, the function β satisfies* (K).

Proof. If $\alpha = 1$ and (u, α, β) is a solution of (1), then according to Lemma 4 the function β is a homomorphism, whence it satisfies (K). If $\alpha \neq 1$ is a homomorphism, then our claim follows directly from Lemma 8. $\square$

An interesting application of the last Corollary is described in

Example 10. If $X = R$ and R acts by conjugation on itself, then (1) means

$$u(rtr^{-1}) = \alpha(r)u(t) + \beta(r), \qquad r, t \in R. \tag{4}$$

Since the conjugacy class of 1 consists of only one element, for each solution (u, α, β) of (4) where α is a homomorphism the function β satisfies (K).

By setting $t = r$ we derive from (4) that

$$(1-\alpha(r))u(r) = \beta(r),$$

whence $\beta(r) = 0$ for all $r \in \ker\alpha$.

If $\alpha = 1$, then $\beta = 0$ and u is constant on each conjugacy class in R.

If $\alpha \neq 1$, we get for $r \notin \ker\alpha$

$$u(r) = (1-\alpha(r))^{-1}\beta(r).$$

For $t = 1$ we also derive from (4) that

$$u(1) = (1-\alpha(r))^{-1}\beta(r), \qquad r \notin \ker\alpha,$$

thus

$$u(r) = u(1), \qquad r \notin \ker\alpha.$$

Moreover, u restricted to the center $C(R) := \{r \in R \mid rt = tr \text{ for all } t \in R\}$ is constant. Assuming in contrary that there exist two elements r_1 and r_2 in $C(R)$ with $u(r_1) \neq u(r_2)$, then it follows from Lemma 7 (for $v_0 = u(r_1)$, say) that $\ker\alpha \geq \left\{t \in R \mid tr_2t^{-1} = r_2\right\} = R$, which is a contradiction to $\alpha \neq 1$. Hence $u(1) = v_0$, $u(r) = v_0$ for all $r \in C(R)$ and for all $r \notin \ker\alpha$. As a first consequence we derive that

$$\beta(r) = (1-\alpha(r))v_0, \qquad r \in R.$$

As a second consequence we see that only the values $u(r)$ for $r \in R' := \ker\alpha\backslash C(R)$ can be different from v_0. The set R' decomposes into conjugacy classes $C_R(t)$, on each of which the function u is uniquely determined by the single value $u(t)$ of the representative t of the conjugacy class $C_R(t)$ and by the formula $u(rtr^{-1}) = \alpha(r)u(t) + (1-\alpha(r))v_0$ for all $r \in R$ which is a direct consequence of (4).

2.2. The general solution of (1) for $\alpha \neq 1$, $\beta \neq 0$ and u not a constant function.

As we already saw, it is crucial to determine all solutions (α, β) of (2) where α is a group homomorphism. Some properties of β are collected in

Lemma 11. *If* (α, β) *is a solution of* (2) *where* $\alpha : R \to K^*$ *is a group homomorphism, then*

- $\beta(1) = 0$
- $\beta(r^n) = (1-\alpha(r))^{-1}(1-\alpha(r)^n)\beta(r)$ *for all* $n \in \mathbb{Z}$ *and* $r \notin \ker\alpha$
- $\beta(r^n) = n\beta(r)$ *for all* $n \in \mathbb{Z}$ *and* $r \in \ker\alpha$
- $\beta(r^{-1}) = -\alpha(r)^{-1}\beta(r)$ *for* $r \in R$
- *for* $n \in \mathbb{N}$ *and* $r_1, \ldots, r_n \in R$ *we have*

$$\beta(r_1 \cdots r_n) = \sum_{j=1}^{n} \left(\prod_{i=1}^{j-1} \alpha(r_i) \right) \beta(r_j)$$

- $\alpha(r)\big(\beta(r^{-1}tr) - \beta(t)\big) = \beta(rt) - \beta(tr)$ *for all* $r, t \in R$.

Now we are going to interpret the functional equation (2) for $\alpha : R \to K^*$, being a homomorphism, and $\beta : R \to V$ in the language of Linear Algebra, i.e. in terms of certain automorphisms of the vector space $V \times K$.

For any pair (α, β) where $\alpha : R \to K^*$ and $\beta : R \to V$, and for any $r \in R$ the mapping

$$\Phi_r^{(\alpha,\beta)} : V \times K \to V \times K \quad \text{with} \quad \Phi_r^{(\alpha,\beta)}(v,\lambda) = (\alpha(r)v + \lambda\beta(r), \lambda),$$
$$(v,\lambda) \in V \times K$$

is a vector space automorphism of $V \times K$. In order to show that $\Phi_r^{(\alpha,\beta)}$ is a homomorphism fix $r \in R$. For $a = \alpha(r)$, $b = \beta(r)$ we have $\Phi_r^{(\alpha,\beta)}(v,\lambda) = (av+\lambda b, \lambda)$. This is a homomorphism since $(v,\lambda) \mapsto av+\lambda b$ and $(v,\lambda) \mapsto \lambda$ are linear. Moreover, $\Phi_r^{(\alpha,\beta)}$ is injective, since $\Phi_r^{(\alpha,\beta)}(v_1,\lambda_1) = \Phi_r^{(\alpha,\beta)}(v_2,\lambda_2)$ implies $(av_1 + \lambda_1 b, \lambda_1) = (av_2 + \lambda_2 b, \lambda_2)$. Hence $\lambda_1 = \lambda_2$. Consequently $av_1 = av_2$, which leads to $v_1 = v_2$ since $a = \alpha(r) \neq 0$. Finally it is surjective, since for arbitrary $(v_2,\lambda_2) \in V \times K$ let $\lambda_1 = \lambda_2$ and $v_1 = a^{-1}(v_2 - \lambda_1 b)$, then $(v_1,\lambda_1) \in V \times K$ and $\Phi_r^{(\alpha,\beta)}(v_1,\lambda_1) = (v_2,\lambda_2)$.

All solutions of (2) are implicitly described in

Lemma 12. 1. *Assume that* $\alpha(r) \neq 0$ *for all* $r \in R$. *The pair* (α, β) *is a solution of* (2) *and* α *is a homomorphism if and only if the mapping*

$$\Phi^{(\alpha,\beta)} : R \to \operatorname{Aut}(V \times K), \qquad r \mapsto \Phi^{(\alpha,\beta)}(r) := \Phi_r^{(\alpha,\beta)}$$

is a group homomorphism.

2. *Let* $(b_i)_{i\in I}$ *be a basis of* V*, and write*

$$\beta(r) = \sum_{i\in I}{}^{*} \beta_i(r) b_i, \qquad r \in R$$

as a formally infinite sum with coefficient functions $\beta_i : R \to K$ *for* $i \in I$*. Then* (α, β) *is a solution of* (2) *and* α *is a group homomorphism if and only if for all* $i \in I$ *the mappings*

$$R \ni r \mapsto \begin{pmatrix} \alpha(r) & \beta_i(r) \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(K) \tag{5}$$

are group homomorphisms from R *into the general linear group of* 2×2*-matrices over* K*.*

Proof. Assume that (α, β) satisfies (2) and α is a homomorphism. Choose $r, t \in R$ and $(v, \lambda) \in V \times K$. Then

$$\begin{aligned}
\Phi_{rt}^{(\alpha,\beta)}(v, \lambda) &= \big(\alpha(rt)v + \lambda\beta(rt), \lambda\big) = \big(\alpha(r)\alpha(t)v + \lambda(\alpha(r)\beta(t) + \beta(r)), \lambda\big) \\
&= \big(\alpha(r)(\alpha(t)v + \lambda\beta(t)) + \lambda\beta(r), \lambda\big) \\
&= \Phi_r^{(\alpha,\beta)}(\alpha(t)v + \lambda\beta(t), \lambda) = \Phi_r^{(\alpha,\beta)}(\Phi_t^{(\alpha,\beta)}(v, \lambda)),
\end{aligned}$$

whence $\Phi_{rt}^{(\alpha,\beta)} = \Phi_r^{(\alpha,\beta)} \circ \Phi_t^{(\alpha,\beta)}$.

Assume conversely that the mapping $\Phi^{(\alpha,\beta)}$ is a group homomorphism, then for all $(v, \lambda) \in V \times K$ and for all $r, t \in R$ we have

$$\Phi_{rt}^{(\alpha,\beta)}(v, \lambda) = (\Phi_r^{(\alpha,\beta)} \circ \Phi_t^{(\alpha,\beta)})(v, \lambda).$$

This means

$$(\alpha(rt)v + \lambda\beta(rt), \lambda) = \big(\alpha(r)(\alpha(t)v + \lambda\beta(t)) + \lambda\beta(r), \lambda\big).$$

For $\lambda = 0$ and $v \neq 0$ we obtain $\alpha(rt)v = \alpha(r)\alpha(t)v$, whence α is a homomorphism. Moreover, for $\lambda \neq 0$ we obtain $\lambda\beta(rt) = \alpha(r)\lambda\beta(t) + \lambda\beta(r)$, whence (α, β) satisfies (2).

Moreover, it is clear that the mapping

$$\Phi : \big\{(\alpha, \beta) \mid \alpha \in (K^*)^R,\ \beta \in V^R\big\} \to \Big\{\Phi^{(\alpha,\beta)} \mid \alpha \in (K^*)^R,\ \beta \in V^R\Big\},$$

$$\Phi(\alpha, \beta) := \Phi^{(\alpha,\beta)}$$

is a bijection.

Using the basis $(b_i)_{i\in I}$ of V, each element v of V can be written as a formally infinite sum

$$v = \sum_{i\in I}{}^{*} v_i b_i, \qquad v_i \in K$$

with only finitely many $v_i \neq 0$. Expressing also β in terms of its coefficient functions β_i we obtain

$$\Phi_{rt}^{(\alpha,\beta)}(v,\lambda) = \Bigl(\sum_{i\in I}{}^{*}(\alpha(rt)v_i + \lambda\beta_i(rt))b_i, \lambda\Bigr)$$

and

$$\Phi_r^{(\alpha,\beta)}(\Phi_t^{(\alpha,\beta)}(v,\lambda)) = \Bigl(\sum_{i\in I}{}^{*}\Bigl(\alpha(r)(\alpha(t)v_i + \lambda\beta_i(t)) + \lambda\beta_i(r)\Bigr)b_i, \lambda\Bigr).$$

Comparing coefficients we derive that $\Phi^{(\alpha,\beta)}$ is a group homomorphism if and only if the mapping

$$R \ni r \mapsto \begin{pmatrix} \alpha(r) & \beta_i(r) \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(K) \tag{5}$$

is a group homomorphism for all $i \in I$. For $\lambda = 0$ we obtain that α is a homomorphism, and then for $\lambda = 1$ we obtain that $\beta_i(rt) = \alpha(r)\beta_i(t) + \beta_i(r)$. $\square$

This way we described all solutions (α,β) of (2) where α is a homomorphism. Here we assume the homomorphisms (5) as known. This procedure is similar to what happens usually in the investigation of certain functional equations where additive and generalized exponential functions are considered as known.

Finally for a given solution (α,β) of (2) we analyse under which conditions there exists a function u such that (u,α,β) is a solution of (1).

Lemma 13. *If (u,α,β) is a solution of (1) where α is a group homomorphism, then for $x \in X$ and $r \in R_x$ (the stabilizer of x) we have*

$$\beta(r) = (1-\alpha(r))u(x).$$

Hence, $\beta(r) = 0$ for all $r \in R_x \cap \ker\alpha$ and $u(x) = (1-\alpha(r))^{-1}\beta(r)$ for all $r \in R_x \setminus \ker\alpha$.

Proof. Choose $x \in X$ and assume that r belongs to the stabilizer of x, then

$$u(x) = u(rx) = \alpha(r)u(x) + \beta(r),$$

whence $\beta(r) = (1 - \alpha(r))u(x)$. If moreover $r \in R_x \cap \ker \alpha$ then $\beta(r) = 0$. For any $r \in R_x \setminus \ker \alpha$ the function $r \mapsto (1 - \alpha(r))^{-1}\beta(r)$ takes the same value $u(x)$. □

This motivates us to introduce a function $\gamma : R \setminus \ker \alpha \to V$ by

$$\gamma(r) := (1 - \alpha(r))^{-1}\beta(r). \tag{6}$$

From the last lemma we derive that γ is constant on $R_x \setminus \ker \alpha$ for all $x \in X$.

Lemma 14. *Assume (u, α, β) is a solution of (1) where α is a group homomorphism, and that x and y are two different elements of X, such that $(R_x \cap R_y) \setminus \ker \alpha \neq \emptyset$. Then γ is constant on $(R_x \cup R_y) \setminus \ker \alpha$.*

Proof. According to Lemma 13, the function γ is constant both on $R_x \setminus \ker \alpha$ and $R_y \setminus \ker \alpha$. Since $(R_x \setminus \ker \alpha) \cap (R_y \setminus \ker \alpha)$ which equals $(R_x \cap R_y) \setminus \ker \alpha$ is not empty by assumption, the function γ is constant on $(R_x \setminus \ker \alpha) \cup (R_y \setminus \ker \alpha) = (R_x \cup R_y) \setminus \ker \alpha$. □

Lemma 15. *Let (α, β) be a solution of (2) where α is a group homomorphism, let $x_0 \in X$, and assume that there exists a vector $v_0 \in V$ such that*

$$\alpha(r)v_0 + \beta(r) = v_0, \qquad r \in R_{x_0}. \tag{7}$$

Then the function $u : R(x_0) \to V$ defined by

$$u(rx_0) = \alpha(r)v_0 + \beta(r), \qquad r \in R \tag{8}$$

satisfies (1) on the orbit $R(x_0)$. It is the unique solution (u, α, β) on $R(x_0)$ with $u(x_0) = v_0$.

Condition (7) is equivalent to

$$\begin{aligned} \beta(r) &= 0, \qquad r \in R_{x_0} \cap \ker \alpha \\ \gamma(r) &= v_0, \qquad r \in R_{x_0} \setminus \ker \alpha. \end{aligned} \tag{9}$$

Proof. First we show that u satisfies (1) on $R(x_0)$. For $x \in R(x_0)$ there exists some $r_0 \in R$ such that $x = r_0 x_0$. The function u is well defined, since if $r_0 x_0 = r_1 x_0$ for some $r_1 \in R$, then $r_1^{-1} r_0$ belongs to R_{x_0}, whence $\alpha(r_1^{-1} r_0) v_0 + \beta(r_1^{-1} r_0) = v_0$. Since (α, β) is a solution of (2) and α is a homomorphism, we obtain from the last equality that $\alpha(r_1)^{-1} \alpha(r_0) v_0 + \alpha(r_1)^{-1} \beta(r_0) + \beta(r_1)^{-1} = v_0$. An application of the fourth item of Lemma 11 yields finally $\alpha(r_1)^{-1}(\alpha(r_0)v_0 + \beta(r_0) - \beta(r_1)) = v_0$, whence $\alpha(r_0)v_0 + \beta(r_0) = \alpha(r_1)v_0 + \beta(r_1)$. This shows that the value of $u(x)$ does not depend on the special choice of r_0 or r_1.

For $r \in R$ and for $x = r_0 x_0 \in R(x_0)$ we obtain

$$\begin{aligned} u(rx) &= u(r(r_0 x_0)) = u((r r_0) x_0) = \alpha(r r_0) v_0 + \beta(r r_0) \\ &= \alpha(r)\alpha(r_0) v_0 + \alpha(r)\beta(r_0) + \beta(r) \\ &= \alpha(r)\Big(\alpha(r_0) v_0 + \beta(r_0)\Big) + \beta(r) = \alpha(r) u(x) + \beta(r). \end{aligned}$$

This shows that (u, α, β) satisfies (1) for all $x \in R(x_0)$.

If (u, α, β) is a solution of (1) with $u(x_0) = v_0$, then

$$u(r x_0) = \alpha(r) u(x_0) + \beta(r) = \alpha(r) v_0 + \beta(r),$$

whence the restriction of u to $R(x_0)$ is of the form (7).

If (α, β) and v_0 satisfy (7), then $\beta(r) = (1 - \alpha(r)) v_0$, whence (9) is satisfied. Conversely, if β and γ satisfy (9), then $\beta(r) = (1 - \alpha(r)) v_0$ for all $r \in R_{x_0}$, consequently (7) is satisfied. □

Remark 16. Assume that R acts transitively on X, in other words $R(x_0) = X$ for $x_0 \in X$. If (u, α, β) is a solution of (1) where α is a group homomorphism, then

$$\beta(r) = 0, \qquad r \in \left(\bigcup_{t \in R} t R_{x_0} t^{-1} \right) \cap \ker \alpha,$$

and γ is constant on $t R_{x_0} t^{-1} \setminus \ker \alpha$ for any $t \in R$.

If (α, β) is a solution of (2) where α is a group homomorphism, and (9) is satisfied for some $v_0 \in V$, then for u given by (8) the triple (u, α, β) is a solution of (1). We obtain, in the present situation, all solutions (u, α, β) of (1) by this construction.

Proof. Since (u, α, β) is a solution of (1) and since $R(x_0) = X$, according to Lemma 15, the function β vanishes on $R_{x_0} \cap \ker \alpha$. But X can also be expressed as the orbit $R(tx_0)$ for any $t \in R$, whence $\beta(r) = 0$ for all $r \in R_{tx_0} \cap \ker \alpha$ which equals $tR_{x_0}t^{-1} \cap \ker \alpha$. The same argument shows that γ is constant on $tR_{x_0}t^{-1} \setminus \ker \alpha$. □

If R acts transitively on X, then we determined in the last remark the set of all solutions (u, α, β) of (1). Finally we still have to deal with the situation that X decomposes into several orbits. We have to analyse whether different orbits impose new conditions on u.

Let

$$T := \{r \in R \mid \exists\, x \in X : rx = x\} \tag{10}$$

be the set of all elements of the group R which have at least one fixed point. The set of all fixed points of r is indicated as X_r, thus $X_r = \{x \in X \mid rx = x\}$. For $r \in T \setminus \ker \alpha$ let

$$U(r) := \left(\bigcup_{x \in X_r} R_x \right) \setminus \ker \alpha. \tag{11}$$

Then $r \in U(r) \subseteq T \setminus \ker \alpha$.

If (u, α, β) is a solution of (1) where α is a group homomorphism, then $\beta(r) = 0$ for $r \in T \cap \ker \alpha$ and γ is constant on each $U(r)$ for $r \in T \setminus \ker \alpha$. In general, some of these sets $U(r)$ will have non-empty intersection. Now we want to determine the maximal subsets of $T \setminus \ker \alpha$ on which γ is constant. For doing this, we introduce a relation on $T \setminus \ker \alpha$. Two elements $r, t \in T \setminus \ker \alpha$ are in relation $r \simeq t$ if there exists an integer $n \in \mathbb{N}_0$ and a sequence $r_0 = r,\ r_1, \ldots,\ r_n = t$ in $T \setminus \ker \alpha$, such that the intersection $U(r_i) \cap U(r_{i+1})$ is not empty for $0 \leq i < n$. It is easy to prove that $\simeq$ is an equivalence relation. The equivalence class of r is indicated as $[r]$. Then $r \in U(r) \subseteq [r]$ for all $r \in T \setminus \ker \alpha$.

Lemma 17. *Let α be a group homomorphism from R to K^*. A function $\gamma : T \setminus \ker \alpha \to V$ is constant on $U(r)$ for all $r \in T \setminus \ker \alpha$ if and only if γ is constant on each equivalence class $[r]$.*

Proof. Assume that γ is constant on $U(r)$ for all $r \in T \setminus \ker \alpha$. Choose $r_0 \in R$ and assume that $r_0 \simeq t$. Then there exist $n \in \mathbb{N}_0$ and a sequence $r_0, r_1, \dots, r_n = t$ in $T \setminus \ker\alpha$, such that $U(r_i) \cap U(r_{i+1}) \neq \emptyset$. For that reason $\gamma(r_0) = \gamma(r_1) = \cdots = \gamma(r_n) = \gamma(t)$, whence γ is constant on the equivalence class $[r_0]$.

Obviously, if γ is constant on $[r]$, then it is constant on the subset $U(r)$ of $[r]$. $\square$

Using the notation just introduced, we collect some necessary conditions on solutions (u, α, β) of (1) in

Corollary 18. *If (u, α, β) is a solution of (1) where α is a group homomorphism, then $\beta(r) = 0$ for $r \in T \cap \ker\alpha$ and γ is constant on each class $[r]$ for $r \in T \setminus \ker\alpha$.*

Finally, if (α, β) is a solution of (2) where α is a group homomorphism we want to determine whether it is possible to find a function u such that (u, α, β) is a solution of (1). And if so we want to construct all functions u with this property. This is done in

Theorem 19. *Assume that (α, β) is a solution of (2) where α is a group homomorphism, and assume that R acts on X. Let T be given by (10). For $r \in T \setminus \ker\alpha$ let $U(r)$ be determined by (11). These sets determine an equivalence relation on $T \setminus \ker\alpha$ by setting $r \simeq t$ if there exist $n \in \mathbb{N}_0$ and $r_0 = r, r_1, \dots, r_n = t \in T \setminus \ker\alpha$ such that $U(r_i) \cap U(r_{i+1}) \neq \emptyset$ for $0 \leq i < n$.*

Moreover, assume that $\beta(r) = 0$ for all $r \in T \cap \ker\alpha$, and that γ, given by (6), is constant on each equivalence class $[r] = \{t \in T \setminus \ker\alpha \mid r \simeq t\}$. Then for any function $u : R \to V$ determined in the following way the triple (u, α, β) is a solution of (1). For each $x_0 \in \mathcal{T}(R\backslash\backslash X)$, a transversal of the R-orbits on X, determine u on the orbit $R(x_0)$ by:

- *If the stabilizer R_{x_0} is a subgroup of $\ker\alpha$, then choose $u(x_0)$ as an arbitrary element of V.*
- *If $R_{x_0} \not\subseteq \ker\alpha$, then choose an arbitrary element r_0 of $R_{x_0} \setminus \ker\alpha$ and set $u(x_0) = \gamma(r_0)$.*
- *Set $v_0 = u(x_0)$. Determine u on the other elements of the orbit $R(x_0)$ by (8).*

Proof. The sets T, $U(r)$ and the equivalence relation $\simeq$ just depend on the group action of R on X. Choose some $x_0 \in \mathcal{T}(R\backslash\backslash X)$.

If $R_{x_0} \subseteq \ker\alpha$, then choose $u(x_0)$ as an arbitrary element of V. If $R_{x_0} \not\subseteq \ker\alpha$, then there exists some $r_0 \in R_{x_0} \setminus \ker\alpha$. Consequently $R_{x_0} \setminus \ker\alpha \subseteq [r_0]$, whence γ is constant on $R_{x_0} \setminus \ker\alpha$ and the value of $u(x_0)$ does not depend on the special choice of r_0.

If we put $v_0 = u(x_0)$ then in both cases (9) is satisfied. If $\hat{u}$ denotes the restriction of u to $R(x_0)$ determined by (8), then according to Lemma 15 the triple $(\hat{u}, \alpha, \beta)$ is a solution of (1) restricted to $R(x_0)$.

A triple (u, α, β) is a solution of (1) if and only if the restriction of u to any orbit $R(x_0)$ together with (α, β) is a solution of (1) restricted to $R(x_0)$. Hence, any function u, independently determined on each orbit $R(x_0)$ as described above, is a solution of (1). □

Remark 20. Using the notation from above, it is easy to prove that if (α, β) is a solution of (2) where $\alpha : R \to K^*$ is a homomorphism, then γ satisfies the functional equation

$$\gamma(rtr^{-1}) = \alpha(r)\gamma(t) + \beta(r), \qquad r, t \in T \setminus \ker\alpha.$$

3. Arbitrary (semi)group actions on the right hand side of (1)

Now the vector space on the right hand side of (1) will be replaced by an arbitrary set Y on which a (semi)group S acts. We want to solve the functional equation

$$u(rx) = \varphi(r)u(x), \qquad r \in R,\ x \in X \tag{12}$$

for the two unknown functions

$$u : X \to Y, \qquad \varphi : R \to S.$$

A solution of (12) is indicated as a pair (u, φ).

We show now that (12) *is a generalization of* (1). In the previous section we were dealing with the situation Y being a K-vector space and S being the affine semigroup $\mathrm{Aff}(V)$ or the affine group $\mathrm{Aff}(V)^*$. They are given by the two sets

$$\mathrm{Aff}(V) = \{(a, v) \mid a \in K,\ v \in V\} \text{ and}$$

$$\mathrm{Aff}(V)^* = \{(a,v) \mid a \in K^*,\ v \in V\}$$

together with the multiplication

$$(a_1,v_1)(a_2,v_2) = (a_1a_2, a_1v_2 + v_1), \qquad (a_1,v_1),(a_2,v_2) \in \mathrm{Aff}(V).$$

Their action on V is described by

$$(a,v) * w = aw + v, \qquad (a,v) \in \mathrm{Aff}(V),\ w \in V.$$

In that situation the mapping φ is given by $\varphi(r) = (\alpha(r), \beta(r))$ for $r \in R$. Hence, it is clear that (12) generalizes (1).

To begin with we give a formal definition of an action of a semigroup on a set. It naturally generalizes the notion of group actions. A multiplicative semigroup S with neutral element 1 acts on a set X if there exists a mapping

$$* : S \times X \to X, \qquad *(s,x) \mapsto s * x$$

such that

$$(s_1s_2) * x = s_1 * (s_2 * x), \qquad s_1, s_2 \in S,\ x \in X$$

and

$$1 * x = x, \qquad x \in X.$$

Usually we write sx instead of $s * x$. In general we cannot speak of orbits under this action. The stabilizer of x is indicated by $S_x = \{s \in S \mid sx = x\}$. It is a subsemigroup of S.

First we want to determine necessary conditions on u and φ for being a solution of (12). If (u,φ) is a solution of (12), then for all $r_1, r_2 \in R$ and $x \in X$ we have

$$\varphi(r_1r_2)u(x) = u((r_1r_2)x) = u(r_1(r_2x)) = \varphi(r_1)\varphi(r_2)u(x). \qquad (13)$$

Let $S' := \langle \varphi(R) \rangle$, in other words S' is the subsemigroup of S generated by $\varphi(R)$. It is the set

$$S' = \left\{ \prod_{i=1}^{n} \varphi(r_i) \mid n \in \mathbb{N},\ r_i \in R \right\}.$$

The element $\varphi(1)$ belongs to S', and it satisfies $\varphi(1)u(x) = u(x)$ for all $x \in X$.

We define an equivalence relation on S' (depending on a solution u) by

$$s_1 \sim s_2 :\Longleftrightarrow s_1 u(x) = s_2 u(x), \qquad \forall x \in X.$$

From (13) it follows immediately that

$$\varphi(r_1 r_2) \sim \varphi(r_1)\varphi(r_2), \qquad r_1, r_2 \in R.$$

The equivalence class of $s \in S'$ is denoted by $\bar{s}$, and $(S'/\sim)$ stands for the set of all equivalence classes with respect to $\sim$.

Lemma 21. *The set $(S'/\sim)$ together with the multiplication $\bar{s}_1 \cdot \bar{s}_2 := \overline{s_1 s_2}$ is a semigroup with neutral element $\overline{\varphi(1)}$.*

Proof. First we have to show that the composition defined on $(S'/\sim)$ is well defined, i.e. it does not depend on the particular choice of the representative s of $\bar{s}$. Assume that $t_1 \in \bar{s}_1$ and $t_2 \in \bar{s}_2$. Then

$$\overline{t_1 t_2} = \left\{ s \in S' \mid s \sim t_1 t_2 \right\} = \left\{ s \in S' \mid t_1 t_2 u(x) = s u(x),\ \forall\, x \in X \right\}.$$

Choose $x \in X$. From the choice of t_1 and t_2 it follows immediately that $t_1 u(x) = s_1 u(x)$ and $t_2 u(x) = s_2 u(x)$ for all $x \in X$. Moreover, since $s_2 \in S'$ there exist $n \in \mathbb{N}$ and $r_1, \dots, r_n \in R$ such that $s_2 = \varphi(r_1) \cdots \varphi(r_n)$. Hence,

$$\begin{aligned}
(t_1 t_2) u(x) &= t_1 (t_2 u(x)) = t_1 (s_2 u(x)) \\
&= t_1 \left(\prod_{i=1}^{n} \varphi(r_i) \right) u(x) = t_1 u \left(\prod_{i=1}^{n} r_i x \right) \\
&= s_1 u \left(\prod_{i=1}^{n} r_i x \right) = s_1 \left(\prod_{i=1}^{n} \varphi(r_i) \right) u(x) = s_1 s_2 u(x).
\end{aligned}$$

This equality holds for any $x \in X$, whence $t_1 t_2 \sim s_1 s_2$ and consequently $\overline{t_1 t_2} = \overline{s_1 s_2}$.

Next we show that $\overline{\varphi(1)}$ is the neutral element of $(S'/\sim)$. By definition, for $s \in S'$ we have $\bar{s}\overline{\varphi(1)} = \overline{s\varphi(1)}$ and $\overline{\varphi(1)}\bar{s} = \overline{\varphi(1)s}$. Applying $s\varphi(1)$ to $u(x)$ for any $x \in X$ we obtain $s\varphi(1)u(x) = su(1x) = su(x)$, whence $\overline{s\varphi(1)} = \bar{s}$. Since $s \in S'$ there exist $n \in \mathbb{N}$ and $r_1, \dots, r_n \in R$ such that $s = \varphi(r_1) \cdots \varphi(r_n)$. Consequently

$$\varphi(1) s u(x) = \varphi(1) \left(\prod_{i=1}^{n} \varphi(r_i) \right) u(x) = \varphi(1) u \left(\prod_{i=1}^{n} r_i x \right)$$

$$= u\biggl(1\prod_{i=1}^{n} r_i x\biggr) = \biggl(\prod_{i=1}^{n}\varphi(r_i)\biggr)u(x) = su(x)$$

whence also $\overline{\varphi(1)s} = \bar{s}$. This finishes the proof that $\overline{\varphi(1)}$ is the neutral element of $(S'/\sim)$. □

Thus we derive that

$$\overline{\varphi(r_1r_2)} = \overline{\varphi(r_1)}\,\overline{\varphi(r_2)}, \qquad r_1, r_2 \in R. \tag{14}$$

Lemma 22. *The semigroup $(S'/\sim)$ acts in a natural way on $Y' := u(X)$, namely as*

$$* : (S'/\sim) \times Y' \to Y', \qquad \bar{s} * y = sy.$$

Proof. First we have to show that this action is well defined, i.e. it does not depend on the special choice of s in $\bar{s}$ and that the mapping $*$ maps $(S'/\sim) \times Y'$ to Y'.

Assume that t also belongs to $\bar{s}$, then $s \sim t$, whence $su(x) = tu(x)$ for all $x \in X$, thus $sy = ty$ for all $y \in Y'$. Let $y \in Y'$ and $s \in S'$, then there exist some $x \in X$ such that $y = u(x)$ and $r_1, \dots, r_n \in R$ such that $s = \prod_{i=1}^{n}\varphi(r_i)$. For that reason, $sy = \prod_{i=1}^{n}\varphi(r_i)u(x) = u\Bigl(\prod_{i=1}^{n} r_i x\Bigr) \in Y'$.

Finally it is left to the reader to prove that $\bar{s}_1 * (\bar{s}_2 * y) = \overline{s_1 s_2} * y$ and $\overline{\varphi(1)} * y = y$ for all $\bar{s}_1, \bar{s}_2 \in (S'/\sim)$ and all $y \in Y'$. □

Lemma 23. *The mapping $\psi : R \to (S'/\sim)$ defined by $\psi(r) := \overline{\varphi(r)}$ is a homomorphism which maps $1 \in R$ to $\overline{\varphi(1)}$.*

Proof. For $r_1, r_2 \in R$ we derive from (14) that

$$\psi(r_1r_2) = \overline{\varphi(r_1r_2)} = \overline{\varphi(r_1)}\,\overline{\varphi(r_2)} = \psi(r_1)\psi(r_2).$$

By definition ψ maps the neutral element 1 of R to the neutral element $\overline{\varphi(1)}$ of $(S'/\sim)$. □

Lemma 24. *For $x \in X$ we have $\psi(R_x) \subseteq (S'/\sim)_{u(x)}$.*

Proof. If r belongs to the stabilizer R_x of $x \in X$, then $u(x) = u(rx) = \varphi(r)u(x)$, whence $\varphi(r)$ belongs to the stabilizer of $u(x)$ in S. Consequently, $\varphi(R_x)$ is contained in $S_{u(x)} \cap S'$. If $\pi : S' \to (S'/\sim)$ denotes the natural projection $\pi(s) = \bar{s}$, then

$$\psi(R_x) = (\pi \circ \varphi)(R_x) \subseteq \pi(S_{u(x)} \cap S') = (S'/\sim)_{u(x)}. \qquad \square$$

In Lemma 23 and Lemma 24 we have described necessary conditions on (u, φ) or $\psi = \pi \circ \varphi$ for being solutions of (12). In the next theorem we will prove that they are also sufficient conditions. This means that they can be used to describe the set of all solutions of (12).

Theorem 25. *Assume that $\mathcal{T}(R\backslash\backslash X)$ is given as $\{x_i \mid i \in I\}$. Let $(y_i)_{i\in I}$ be a family in Y and let S' be a subsemigroup of S with a particular element $e \in S'$ such that*

$$ey = y \qquad \forall y \in Y' := \left\{sy_i \mid s \in S',\ i \in I\right\}.$$

Introduce an equivalence relation on S' by

$$s_1 \sim s_2 :\Longleftrightarrow s_1 y = s_2 y, \qquad \forall y \in Y'.$$

Then $(S'/\sim)$ is a semigroup with neutral element $\bar{e}$. It acts in a natural way on Y', namely by $\bar{s} * y = sy$. Choose a homomorphism $\psi : R \to (S'/\sim)$ with the properties

$$\psi(1) = \bar{e}, \qquad \psi(R_{x_i}) \subseteq (S'/\sim)_{y_i}, \qquad i \in I.$$

If we define $\varphi : R \to S'$ by $\varphi(r) \in \psi(r)$ and $u : X \to Y$ by $u(rx_i) = \psi(r)y_i$, then u is well defined and (u, φ) is a solution of (12). Especially $u(x_i) = y_i$ for all $i \in I$. We obtain all solutions of (12) in this way.

Proof. Evidently $\sim$ is an equivalence relation on S'. The particular element e corresponds to the element $\varphi(1)$ in our previous considerations. It follows as in the proof of Lemma 21 that $(S'/\sim)$ is a semigroup with neutral element $\bar{e}$. The same arguments as in the proof of Lemma 22 can be used to show that this semigroup acts on Y'.

Assume that ψ is a homomorphism with the required properties and that $\varphi(r)$ is a representative of $\psi(r)$ for all $r \in R$. In order to show that u is well defined, i.e. its definition does not depend on the special choice of r representing an element rx_i of the orbit $R(x_i)$ assume that $rx_i = r_1 x_i$. Then $r_1^{-1} r x_i = x_i$, whence $r_1^{-1} r$ belongs to R_{x_i} and by assumption $\psi(r_1^{-1} r) \in (S'/\sim)_{y_i}$ which is equivalent to $\psi(r_1^{-1} r) y_i = y_i$. Multiplying this equation from the left with $\psi(r_1)$ and using the homomorphism property of ψ we get

$$\psi(r_1 r_1^{-1} r) y_i = \psi(r_1) y_i$$

from which we immediately deduce that $\psi(r)y_i = \psi(r_1)y_i$ and $u(rx_i) = u(r_1x_i)$, whence u is well defined on each orbit $R(x_i)$.

Moreover $u(x_i) = u(1x_i) = \psi(1)y_i = \bar{e}y_i = y_i$ for all $i \in I$.

Finally we have to prove that (u, φ) is a solution of (12). Assume that $x \in X$ is of the form r_0x_i for some $r_0 \in R$ and a uniquely determined $x_i \in \mathcal{T}(R\backslash\backslash X)$. Then for arbitrary $r \in R$ we have by definition of u

$$\begin{aligned} u(rx) &= u(rr_0x_i) = \psi(rr_0)y_i = \psi(r)\psi(r_0)y_i \\ &= \psi(r)u(r_0x_i) = \varphi(r)u(x), \end{aligned}$$

which finishes the proof. □

If the semigroup S acting on the right hand side is actually a group, then we can also proceed similar as in [7]. This second approach allows to replace certain equivalence relations by computations in factor groups.

Remark 26. If (u, φ) is a solution of (12) then

$$\varphi(R_x) \subseteq S_{u(x)}$$

and from

$$\varphi(r_1r_2)u(x) = u(r_1r_2x) = \varphi(r_1)\varphi(r_2)u(x), \qquad r_1, r_2 \in R,\ x \in X$$

we obtain that

$$(\varphi(r_1)\varphi(r_2))^{-1}\varphi(r_1r_2) \in \bigcap_{x\in X} S_{u(x)}, \qquad r_1, r_2 \in R.$$

Moreover, the intersection $\tilde{S} := \bigcap_{x\in X} S_{u(x)}$ is a normal subgroup of $S' := \langle\varphi(R)\rangle$. The factor group $S'/\tilde{S}$ acts in a natural way on $Y' := u(X)$ and the mapping $\psi : R \to S'/\tilde{S}$ given by $\psi(r) := \varphi(r)\tilde{S} = \overline{\varphi(r)}$ is a group homomorphism. Furthermore, $\psi(R_x) \subseteq (S'/\tilde{S})_{u(x)}$.

Conversely, assume that $\mathcal{T}(R\backslash\backslash X)$ is given as $\{x_i \mid i \in I\}$. Let $(y_i)_{i\in I}$ be a family in Y and let S' be a subgroup of S. Moreover, let N be a normal subgroup of S' such that $N \subseteq S_{y_i}$ for all $i \in I$. Then S'/N acts in a natural way on $Y' := \{sy_i \mid s \in S',\ i \in I\}$ namely by $\bar{s} * y = sy$. Choose a homomorphism $\psi : R \to S'/N$ with

$$\psi(R_{x_i}) \subseteq (S'/N)_{y_i}, \qquad i \in I.$$

If we define $\varphi : R \to S'$ by $\varphi(r) \in \psi(r)$ and $u : X \to Y$ by $u(rx_i) = \psi(r)y_i$, then u is well defined and (u, φ) is a solution of (12). Especially $u(x_i) = y_i$ for all $i \in I$.

We leave the details to the reader.

These general results can easily be rewritten for the particular action of the affine (semi)group on a vector space V. Assume that the semigroup $S = \mathrm{Aff}(V)$ acts on the K-vector space V, then the stabilizer of $w \in V$ is given by

$$\mathrm{Aff}(V)_w = \{(a, (1-a)w) \mid a \in K\}\,.$$

If we are interested in solutions (u, φ) of (12) where u is constant, then it is enough to consider the case that $|Y'| = 1$, in other words $Y' = \{w\}$ for some $w \in V$, whence $y_i = w$ for all $i \in I$. Moreover, $S' \subseteq \mathrm{Aff}(V)_w$, whence for all $s_1, s_2 \in S'$ we have $s_1 w = w = s_2 w$. Consequently $s_1 \sim s_2$ for all $s_1, s_2 \in S'$ and $S'/\sim = \{\bar{s}\}$ for any $s \in S'$. As a matter of fact, $\psi : R \to (S'/\sim)$ is the constant mapping $\psi = \bar{s}$ and $\varphi(r) = (\alpha(r), \beta(r)) \in \psi(r) = \bar{s}$ for all $r \in R$. That's why, $\alpha : R \to K$ is an arbitrary mapping and $\beta : R \to V$ is given by $\beta(r) = (1 - \alpha(r))w$.

Finally, we consider the case that $|Y'| > 1$. Then there exist at least two different vectors w and w' in Y'. Let S' be a subsemigroup of S as indicated in Theorem 25. Two elements s_1, s_2 of S' are equivalent if and only if $s_1 = s_2$. (If $s_1 \sim s_2$, then $s_1 w = s_2 w$ and $s_1 w' = s_2 w'$. Since s_i belongs to $\mathrm{Aff}(V)$, it is of the form (a_i, v_i) for $i = 1, 2$. Thus we obtain $a_1 w + v_1 = a_2 w + v_2$ and $a_1 w' + v_1 = a_2 w' + v_2$. Subtracting the second equation from the first one, we get $a_1(w - w') = a_2(w - w')$ which implies $a_1 = a_2$, since $w - w' \neq 0$. Then we also obtain that $v_1 = v_2$, consequently $s_1 = s_2$.) Hence, $S'/\sim$ can be identified with S'. Moreover, it was assumed that there exists a particular element $e \in S'$ such that e applied to any element of Y' does not change it. Hence $ew = w$ and $ew' = w'$, what immediately implies that $e = (1, 0)$. Since the equivalence classes on S' are trivial, φ can be identified with ψ which is a homomorphism from R to S'. Thus $\varphi(r) = \psi(r) = (\alpha(r), \beta(r))$. Since ψ is a homomorphism we deduce

$$(\alpha(r_1 r_2), \beta(r_1 r_2)) = (\alpha(r_1)\alpha(r_2), \alpha(r_1)\beta(r_2) + \beta(r_1)).$$

If there exists an $r_0 \in R$ such that $\alpha(r_0) = 0$, then $\alpha(r) = 0$ for all

$r \in R$, which is a contradiction to $\psi(1) = (1, 0)$. Consequently α is a homomorphism from R to K^* and (α, β) satisfies (2).

We are able to deduce from Theorem 25 and the related lemmas concerning the solution of (12) some additional facts in the special case where $S = \mathrm{Aff}(V)$. We collect them in

Remark 27. Let $\mathcal{T}(R\backslash\backslash X)$ be given by $\{x_i \mid i \in I\}$ and let $(w_i)_{i\in I}$ be a family in V. The homomorphism ψ satisfies $\psi(R_{x_i}) \subseteq S'_{w_i}$ for all $i \in I$ if and only if the function γ given by (6) is constant on each equivalence class $[r]$ for $r \in T \setminus \ker\alpha$, where T is given by (10), $\gamma(r) = w_i$ for $r \in R_{x_i} \setminus \ker\alpha$ for $i \in I$, and $\beta(r) = 0$ for all $r \in T \cap \ker\alpha$.

Proof. Assume that $\psi(R_{x_i}) \subseteq S'_{w_i}$ which is included in the stabilizer of w_i in $\mathrm{Aff}(V)$, given by $\{(a, (1-a)w_i) \mid a \in K\}$. If $r \in R_{x_i} \setminus \ker\alpha$, then $\psi(r) = (\alpha(r), (1-\alpha(r))w_i)$, whence $\gamma(r) = (1-\alpha(r))^{-1}(1-\alpha(r))w_i = w_i$. Next we prove the claim that $\psi(R_{gx_i}) \subseteq S'_{\psi(g)w_i}$ for all $i \in I$ and $g \in R$. Since the stabilizer of gx_i equals $gR_{x_i}g^{-1}$ we have

$$\begin{aligned}\psi(R_{gx_i}) &= \psi(gR_{x_i}g^{-1}) = \psi(g)\psi(R_{x_i})\psi(g^{-1})\\ &\subseteq \psi(g)S'_{w_i}\psi(g^{-1}) \subseteq S'_{\psi(g)w_i}.\end{aligned}$$

The last inclusion follows from the fact that

$$\begin{aligned}(\psi(g)s\psi(g^{-1})) * (\psi(g)w_i) &= \psi(g)s\psi(g^{-1}g) * w_i\\ &= \psi(g)s\psi(1) * w_i = \psi(g)w_i\end{aligned}$$

for all $s \in S'_{w_i}$. If $r \in T \cap \ker\alpha$, then there exists some $x \in X$ such that $rx = x$. Moreover, there exist $g_0 \in R$ and $i_0 \in I$ such that $x = g_0x_{i_0}$, whence $r \in R_{g_0x_{i_0}}$. For that reason $\psi(r)$ belongs to $S'_{\psi(g_0)w_{i_0}}$, thus $\psi(r) = (\alpha(r), (1-\alpha(r))\psi(g_0)w_{i_0}) = (1, 0)$, and consequently $\beta(r) = 0$.

Finally we have to prove that γ is constant on each equivalence class $[r]$ for $r \in T \setminus \ker\alpha$. If $r_0 \in T \setminus \ker\alpha$, then there exists some $x \in X$ such that $r_0x = x$. Moreover, there exist $g_0 \in R$ and $i_0 \in I$ such that $x = g_0x_{i_0}$, whence $r_0 \in R_{g_0x_{i_0}}$. For that reason $\psi(r_0)$ belongs to $S'_{\psi(g_0)w_{i_0}}$, thus $\psi(r_0) = (\alpha(r_0), (1-\alpha(r_0))\psi(g_0)w_{i_0})$ and $\beta(r_0) = (1-\alpha(r_0))\psi(g_0)w_{i_0}$ and finally $\gamma(r_0) = \psi(g_0)w_{i_0}$. In other words, $\gamma(r_0)$ does not depend on r_0, consequently $\gamma(r) = \psi(g_0)w_{i_0}$ for all $r \in R_{g_0x_{i_0}} = R_x$. Thus γ is constant on each $R_x \setminus \ker\alpha$ for all $r \in R$. In the next step we prove that γ is constant on $U(r_0)$. If $r \in U(r_0)$, then there exists some $x' \in X_{r_0}$ such that

$r \in R_{x'}$. Moreover $r_0 \in (R_x \cap R_{x'}) \setminus \ker \alpha$. For that reason $\gamma(r) = \psi(g_0) w_{i_0}$ for all $r \in (R_x \cup R_{x'}) \setminus \ker \alpha$ and consequently $\gamma(r) = \psi(g_0) w_{i_0}$ for all $r \in U(r_0)$. Again this holds for any $r_0 \in T \setminus \ker \alpha$. Finally assume that $r \in [r_0]$, then there exist $n \in \mathbb{N}$ and $r_1, \ldots, r_n = r \in T \setminus \ker \alpha$ such that $U(r_i) \cap U(r_{i+1}) \neq \emptyset$ for all $0 \leq i < n$, whence γ is constant on the equivalence class $[r_0]$. This holds for all $r_0 \in T \setminus \ker \alpha$.

Conversely, assume that $r \in R_{x_i}$. If $r \in \ker \alpha$, then $\beta(r) = 0$, whence $\psi(r) = (1, 0) \in S'_{w_i}$. If $r \notin \ker \alpha$, then $w_i = \gamma(r) = (1 - \alpha(r))^{-1} \beta(r)$, whence $\beta(r) = (1 - \alpha(r)) w_i$. Consequently $\alpha(r) w_i + \beta(r) = w_i$ which proves that $\psi(r) = (\alpha(r), \beta(r)) \in S'_{w_i}$. □

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