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A theorem on Besov–Nikol'skiĭ class

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Dedicated to Professor Lajos Tamássy on his 80th birthday

Abstract. Very recently S. Yu. Tikhonov proved a theorem which gives a necessary and sufficient condition in order that a function $f(x) \in L_p$ having quasi-monotone decreasing Fourier coefficients should belong to the Besov–Nikol'skiĭ class. In the present paper the analogue of his result is proved with function having Fourier coefficients of rest bounded variation.

1. Introduction

It is well known that there are a great number of theorems having conditions with monotone decreasing coefficients. It is also known that these theorems, or a part of them, have been generalized to quasi-monotonic, almost monotonic, quasi-positive or δ-quasi-monotonic sequences. Namely these sequences share many of the properties of decreasing sequences. See e.g. the paper of R. P. Boas, Jr. [2].

Recently in [5] we also defined a new class of sequences sharing also many good properties of decreasing sequences, and called it "*sequences*

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of rest bounded variation", briefly denoted by RBVS. We say that a null-sequence $\mathbf{c} := \{c_n\}$ is of rest bounded variation, or $\mathbf{c} \in$ RBVS, if $c_n \to 0$ and for any $m \in \mathbb{N}$

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \le K(\mathbf{c}) c_m \tag{1.1}$$

holds, where $K(\mathbf{c})$ is a constant depending only on $\mathbf{c}$. The definition (1.1) clearly yields that if $\mathbf{c} \in$ RBVS, then it is also *almost monotonic*, that is, for all $n \ge m$

$$c_n \le K(\mathbf{c}) c_m \tag{1.2}$$

stays. If a sequence $\mathbf{c}$ suffices (1.2), we denote by $\mathbf{c} \in$ AMS. This notion is due to S. N. Bernstein. If (1.2) is required only for $m \le n \le 2m$, then we say that the sequence $\mathbf{c}$ is *locally almost monotonic*, and denote by $\mathbf{c} \in$ LAMS. Unfortunately I do not know who investigated first these sequences. The following embedding relations are obvious

$$\text{MS} \subset \text{RBVS} \subset \text{AMS} \subset \text{LAMS}, \tag{1.3}$$

where MS denotes the monotone decreasing null-sequences. If we denote by QMDS the quasi-monotone decreasing sequences, defined by

$$c_{n+1} \le c_n \left(1 + \frac{\alpha}{n}\right), \quad \alpha > 0,\ n \ge n_0(\alpha),$$

or equivalently if

$$n^{-\beta} c_n \downarrow 0 \quad \text{for some } \beta;$$

then clearly only

$$\text{MS} \subset \text{QMDS} \subset \text{LAMS} \tag{1.4}$$

maintains.

The embedding relations (1.3) and (1.4) foreshadow that

$$\text{RBVS} \subset \text{QMDS}. \tag{1.5}$$

But this is not the case, (1.5) is not true, namely in [6] we showed that the classes QMDS and RBVS are not comparable. Thus it is a natural task to prove the analogues of the theorems having conditions of QMDS-type.

In this paper we shall prove such a result being the analogue of a theorem due to S. YU. Tikhonov [9] concerning the Besov–Nikol'skiĭ class.

To recall Tikhonov's theorem we need some notions and notations.

Let L_p $(1 < p < \infty)$ be the space of 2π-periodic, integrable functions $f(x)$ with the norm

$$\|f\|_p := \left(\int_0^{2\pi} |f(x)|^p dx\right)^{1/p},$$

and let

$$\omega_\beta(f,t)_p := \sup_{|h|\le t} \left\| \sum_{\nu=0}^{\infty} (-1)^\nu \binom{\beta}{\nu} f(x + (\beta-\nu)h) \right\|_p$$

the modulus of smoothness of order β $(\beta > 0)$ of $f \in L_p$.

A non-negative function $\alpha(t)$ satisfies σ-condition if $\int_0^1 \alpha(t) t^\sigma dt < \infty$.

We shall use the following notations: $L \ll R$ is there exists a positive constant K such that $L \le KR$; and if $L \ll R$ and $R \ll L$ hold simultaneously, then we write $L \asymp R$.

A function $f \in L_p$ $(1 < p < \infty)$ belongs to the Besov–Nikol'skiĭ class $\mathrm{BN}(\alpha, \beta, \psi, p, \theta, k)$ if

$$\left(\int_0^\delta \alpha(t)\omega^\theta_{k+\beta}(f,t)_p dt + \delta^{\beta\theta} \int_\delta^1 \alpha(t) t^{-\beta\theta} \omega^\theta_{k+\beta}(f,t)_p dt\right)^{\frac{1}{\theta}} \ll \psi(\delta)$$

holds, where $\beta, k, \theta > 0$, $\alpha(t)$ satisfies σ-condition with $\sigma = k\theta$, and $\psi(\delta)$ is non-negative continuous function with the properties $\psi(\delta_1) \ll \psi(\delta_2)$, $0 \le \delta_1 \le \delta_2 \le 1$; $\psi(2\delta) \ll \psi(\delta)$, $0 \le \delta \le 1/2$.

For the sake of simplicity, we shall recall the theorem of Tikhonov in the case if $f(x)$ is an even function; and later on we also consider only even functions, namely the proof for even and odd functions is the same, and the general case can be proved by using the identity

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}.$$

Tikhonov's theorem reads as follows.

Theorem A. *Let β, k and θ be positive numbers, let the function $\alpha(t)$ satisfy σ-condition with $\sigma = k\theta$, and denote $\lambda_\nu := \int_{1/\nu+1}^{1/\nu} \alpha(t) dt$.*

If the Fourier coefficients a_n of $f \in L_p$ $(1 < p < \infty)$ belong to QMDS *and*

$$\sum_{\nu=1}^{n-1} \lambda_\nu \asymp n\,\lambda_n \asymp n^{k\theta} \int_0^{1/n} \alpha(t) t^{k\theta} dt \tag{1.6}$$

maintains, then $f \in \mathrm{BN}(\alpha, \beta, \psi, p, \theta, k)$ if and only if

$$\left(n^{-\beta\theta} \sum_{\nu=1}^{n} a_\nu^\theta \lambda_\nu \nu^{\theta(\beta+1)-\frac{\theta}{p}} + \sum_{\nu=n+1}^{\infty} a_\nu^\theta \lambda_\nu \nu^{\theta\left(1-\frac{1}{p}\right)} \right)^{\frac{1}{\theta}} \ll \psi\left(\frac{1}{n}\right) \tag{1.7}$$

holds for all $n \in \mathbb{N}$.

2. Result

We prove the following analogue of Theorem A.

Theorem. *The assertions of Theorem A also hold with the assumption $\{a_n\} \in$ RBVS in place of $\{a_n\} \in$ QMDS.*

3. Lemmas

In order to verify our theorem we need three lemmas, two of them are the analogues of the lemmas used by Tikhonov [9].

Lemma 1 ([3])**.** *Let $a_n \ge 0$, $\lambda_n \ge 0$ and $p \ge 1$. Let $\nu_1 < \cdots < \nu_n < \ldots$ denote the indices for which $\lambda_{\nu_n} > 0$. Let N denote the number of the positive terms of the sequence $\{\lambda_n\}$, provided this number is finite; in the contrary case set $N = \infty$. Set $\nu_0 = 0$, and if $N < \infty$ then $\nu_{N+1} = \infty$. Using the notations*

$$A_{m,n} := \sum_{i=m}^{n} a_i \quad \text{and} \quad \Lambda_{m,n} := \sum_{i=m}^{n} \lambda_i,$$

we have the following inequalities:

$$\sum_{n=1}^{\infty} \lambda_n A_{1,n}^p \ll \sum_{n=1}^{N} \lambda_{\nu_n}^{1-p} \Lambda_{\nu_n,\infty}^p A_{\nu_{n-1}+1,\nu_n}^p, \tag{3.1}$$

$$\sum_{n=1}^{\infty} \lambda_n A_{n,\infty}^p \ll \sum_{n=1}^{N} \lambda_{\nu_n}^{1-p} \Lambda_{1,\nu_n}^p A_{\nu_n,\nu_{n+1}-1}^p. \tag{3.2}$$

If $0 < p \leq 1$ *then the converse inequalities hold.*

Lemma 2 ([7])**.** *If* $f \in L_p,\ 1 < p < \infty$, *and the Fourier coefficients* $\{a_n\}$ *of* f *belongs to* RBVS, *then*

$$\omega_\beta\left(f, \frac{1}{n}\right)_p \asymp n^{-\beta}\left\{\sum_{\nu=1}^{n} a_\nu^p \nu^{p(\beta+1)-2}\right\}^{1/p} + \left\{\sum_{\nu=n+1}^{\infty} a_\nu^p \nu^{p-2}\right\}^{1/p}. \tag{3.3}$$

We note that this lemma in [7] as the generalization of a theorem (there denoted by Theorem B) was proved but only for positive integers $\beta = k$. However the same proof can be used with any positive β in place of k, therefore we omit the details.

Lemma 3. *Let* $\beta,\ k$ *and* θ *be positive numbers, let the function* $\alpha(t)$ *satisfy* σ*-condition with* $\sigma = k\theta$. *If* $\delta \in \left(\frac{1}{n+1}, 1\right]$, $n \in \mathbb{N}$, *and* $f \in L_p$ $(1 < p < \infty)$ *possesses Fourier coefficients* $\{a_n\} \in$ RBVS, *then the inequalities*

$$\begin{aligned}
&n^{-\beta\theta}\left[\sum_{\nu=1}^{n} a_\nu^p\, \lambda_\nu^{\frac{p}{\theta}}\, \nu^{\left(\beta+1+\frac{1}{\theta}\right)p-2}\right]^{\frac{\theta}{p}} + \left[\sum_{\nu=n+1}^{\infty} a_\nu^p\, \lambda_\nu^{\frac{p}{\theta}}\, \nu^{p\left(1+\frac{1}{\theta}\right)-2}\right]^{\frac{\theta}{p}} \\
&\quad \ll \int_0^\delta t^{-k\theta-1} \int_0^t \alpha(u)\, u^{k\theta}\, du\, \omega_{k+\beta}^\theta(f,t) dt \\
&\quad\quad + \delta^{\beta\theta} \int_\delta^1 t^{-\beta\theta-1} \int_t^1 \alpha(u) du\, \omega_{k+\beta}^\theta(f,t)_p dt \\
&\quad \ll n^{-\beta\theta} \sum_{\nu=1}^{n} a_\nu^\theta\, \lambda_\nu^\theta\, \nu^{\theta(\beta+1)-\frac{\theta}{p}} + \sum_{\nu=n+1}^{\infty} a_\nu^\theta\, \lambda_\nu^\theta\, \nu^{\theta-\frac{\theta}{p}}
\end{aligned}$$

hold for any $\theta \in (0, p]$, *and their opposities hold if* $\theta \in [p, \infty)$.

In this form Lemma 3 is new, we do not prove it after all, namely Tikhonov proved it in [9] for $\{a_n\} \in$ QMDS. Carefully analyzing his proof we see that the condition $\{a_n\} \in$ QMDS is employed only by the inequality (3.3); which originally in [8] for monotonic coefficients was proved, and Tikhonov extended it to $\{a_n\} \in$ QMDS; furthermore at some calculations is used merely the locally almost monotonic property of the quasi-monotone decreasing sequences, however this property maintains if the sequence in question belongs to RBVS. Thus since in [7] we verified the inequality (3.3) for any $\{a_n\} \in$ RBVS, to repeat Tikhonov's proof seems to be superfluous.

4. Proof of theorem

Our proof mainly follows the lines that of Theorem A using the modified lemmas; and at some places it is simplified.

Making use of the conditions (1.6) and $\frac{1}{n+1} < \delta \le \frac{1}{n}$ it is easy to see that

$$\begin{aligned} I_\delta &:= \int_0^\delta \alpha(t)\,\omega_{k+\beta}^\theta(f,t)_p dt + \delta^{\beta\theta} \int_\delta^1 \alpha(t) t^{-\beta\theta}\,\omega_{k+\beta}^\theta(f,t)_p dt \\ &\ll \int_0^{1/n} t^{-k\theta-1} \int_0^t \alpha(u)\,u^{k\theta} du\,\omega_{k+\beta}^\theta(f,t)_p dt \\ &\quad + n^{-\beta\theta} \int_{1/n}^1 t^{-\beta\theta-1} \int_t^1 \alpha(u) du\,\omega_{k+\beta}^\theta(f,t)_p dt. \end{aligned}$$

Hence, if $\theta \in (0,p]$, by Lemma 3 (with $\delta = \frac{1}{n}$) and utilizing again the assumptions (1.6) we obtain that

$$I_\delta \ll n^{-\beta\theta} \sum_{\nu=1}^n a_\nu^\theta\,\lambda_\nu\,\nu^{\beta\theta+\theta-\frac{\theta}{p}} + \sum_{\nu=n+1}^\infty a_\nu^\theta\,\lambda_\nu\,\nu^{\theta-\frac{\theta}{p}}. \tag{4.1}$$

If $\theta \in [p,\infty)$ then on the basis of the properties of $\omega_{k+\beta}(f,t)$ it is clear that

$$I_\delta \asymp \sum_{\nu=n+1}^\infty \lambda_\nu\,\omega_{k+\beta}^\theta \left(f, \frac{1}{\nu}\right)_p + n^{-\beta\theta} \sum_{\nu=1}^n \lambda_\nu\,\nu^{\beta\theta}\,\omega_{k+\beta}^\theta \left(f, \frac{1}{\nu}\right)_p =: I_1 + I_2.$$

To estimate I_1 we use (3.3). Thus

$$I_1 \asymp \sum_{\nu=n+1}^\infty \lambda_\nu\,\nu^{-(k+\beta)\theta} \left[\sum_{\ell=1}^n a_\ell^p\,\ell^{(k+\beta+1)p-2} \right]^{\frac{\theta}{p}}$$

$$+\sum_{\nu=n+1}^{\infty}\lambda_\nu\,\nu^{-(k+\beta)\theta}\left[\sum_{\ell=n+1}^{\nu}a_\ell^p\,\ell^{(k+\beta+1)p-2}\right]^{\frac{\theta}{p}}$$

$$+\sum_{\nu=n+1}^{\infty}\lambda_\nu\left[\sum_{\ell=\nu+1}^{\infty}a_\ell^p\,\ell^{p-2}\right]^{\frac{\theta}{p}}=:I_{11}+I_{12}+I_{13}.$$

In order to estimate I_1 from above we make some auxiliary estimations. Using the assumptions given in (1.6) elementary consideration leads to

$$\begin{aligned}\sum_{\nu=n}^{\infty}\lambda_\nu\,\nu^{-k\theta}&\ll\lambda_n\,n^{1-k\theta}\ll\lambda_n\,n^{-k\theta}\sum_{\ell=n/2}^{n}1\\&\ll n^{-k\theta}\sum_{\ell=n/2}^{n}\lambda_\ell\ll\sum_{\ell=n/2}^{n}\frac{\lambda_\ell}{\ell^{k\theta}}.\end{aligned}\tag{4.2}$$

To estimate the sum

$$\sum_{\ell=1}^{n}a_\ell^p\,\ell^{(k+\beta+1)p-2}$$

we make blocks $\sum_{\ell=1}^{n}=\sum_{\ell=1}^{n/4}+\sum_{\ell=n/4+1}^{n/2}+\sum_{\ell=n/2+1}^{n}$ and use the condition $\{a_n\}\in$ LAMS. It is easy to see that the second block multiplied by a constant is greater than the third one, consequently

$$\sum_{\ell=1}^{n}a_\ell^p\,\ell^{(k+\beta+1)p-2}\ll\sum_{\ell=1}^{n/2}a_\ell^p\,\ell^{(k+\beta+1)p-2}.$$

Utilizing this inequality and (4.2) we obtain that

$$I_{11}\ll n^{-\beta\theta}\left[\sum_{\ell=1}^{n/2}a_\ell^p\,\ell^{(k+\beta+1)p-2}\right]^{\frac{\theta}{p}}\sum_{\ell=n/2}^{n}\frac{\lambda_\ell}{\ell^{k\theta}}$$

$$\ll n^{-\beta\theta}\sum_{\ell=1}^{n}\frac{\lambda_\ell}{\ell^{k\theta}}\left[\sum_{\nu=1}^{\ell}a_\nu^p\,\nu^{(k+\beta+1)p-2}\right]^{\frac{\theta}{p}}.$$

Now, by (3.1), regarding $\theta\ge p$, (4.2) and (1.6), we get that

$$I_{11}\ll n^{-\beta\theta}\sum_{\ell=1}^{n}\left(\frac{\lambda_\ell}{\ell^{k\theta}}\right)^{1-\frac{\theta}{p}}\left[\sum_{m=\ell}^{\infty}\frac{\lambda_m}{m^{k\theta}}\right]^{\frac{\theta}{p}}a_\ell^\theta\,\ell^{(k+\beta+1)\theta-\frac{2\theta}{p}}$$

$$\ll n^{-n\theta} \sum_{\ell=1}^{n} a_\ell^\theta \, \lambda_\ell \, \ell^{\beta\theta+\theta-\frac{\theta}{p}}.$$

A similar arguing yields that

$$I_{12} \ll \sum_{\nu=n+1}^{\infty} a_\nu^\theta \, \lambda_\nu \, \nu^{\theta-\frac{\theta}{p}}.$$

To estimate I_{13} we apply (3.2) and (1.6), thus

$$I_{13} \ll \sum_{\nu=n+1}^{\infty} \lambda_\nu^{1-\frac{\theta}{p}} \left[\sum_{m=1}^{\nu} \lambda_m \right]^{\frac{\theta}{p}} a_\nu^\theta \, \nu^{\theta-\frac{2\theta}{p}} \ll \sum_{\nu=n+1}^{\infty} a_\nu^\theta \, \lambda_\nu \, \lambda^{\theta-\frac{\theta}{p}}.$$

To estimate I_2 we use again (3.3), whence

$$I_2 \asymp n^{-\beta\theta} \sum_{\nu=1}^{n} \lambda_\nu \, \nu^{-k\theta} \left[\sum_{m=1}^{\nu} a_m^p \, m^{(k+\beta+1)p-2} \right]^{\frac{\theta}{p}}$$

$$+ n^{-\beta\theta} \sum_{\nu=1}^{n} \lambda_\nu \, \nu^{\beta\theta} \left[\sum_{m=\nu+1}^{n} a_m^p \, m^{p-2} \right]^{\frac{\theta}{p}}$$

$$+ n^{-\beta\theta} \sum_{\nu=1}^{n} \lambda_\nu \, \nu^{\beta\theta} \left[\sum_{m=n+1}^{\infty} a_m^p \, m^{p-2} \right]^{\frac{\theta}{p}} =: I_{21} + I_{22} + I_{23}.$$

Fortunately we can use the same discussing as in estimating I_{11}, thus we have

$$I_{21} \ll n^{-\beta\theta} \sum_{m=1}^{n} a_m^\theta \, \lambda_m \, m^{\beta\theta+\theta-\frac{\theta}{p}}.$$

In the proof of the estimation I_{22} we employ (3.2), thus

$$I_{22} \ll n^{-\beta\theta} \sum_{\nu=1}^{n} (\lambda_\nu \, \nu^{\beta\theta})^{1-\frac{\theta}{p}} \left[\sum_{m=1}^{\nu} \lambda_m \, m^{\beta\theta} \right]^{\frac{\theta}{p}} a_\nu^\theta \, \nu^{\theta-\frac{2\theta}{p}}$$

$$\ll n^{-\beta\theta} \sum_{\nu=1}^{n} a_\nu^\theta \, \lambda_\nu \, \nu^{\beta\theta+\theta-\frac{\theta}{p}}.$$

Clearly

$$I_{23} \ll \sum_{\nu=1}^{n} \lambda_\nu \left[\sum_{m=n+1}^{\infty} a_m^p \; m^{p-2} \right]^{\frac{\theta}{p}}$$

and hence by (1.6)

$$I_{23} \ll n\, \lambda_n \left[\sum_{m=n+1}^{\infty} a_m^p \, m^{p-2} \right]^{\frac{\theta}{p}}.$$

An elementary consideration yields that if

$$\sum_{\nu=1}^{n} \lambda_\nu \ll n\, \lambda_n$$

then there exists a positive ε such that for any $m \geq n$

$$n^{1-\varepsilon}\lambda_n \ll m^{1-\varepsilon}\lambda_m \tag{4.3}$$

holds. To be correct this is a special case of a more general result of Bari and Stečkin [1], see also the Lemma in [4] . Let us fix such an $\varepsilon > 0$ having the property (4.3) and denote $\gamma := \frac{\varepsilon}{\theta}$.

An application of the Hölder inequality gives

$$\sum_{m=n+1}^{\infty} a_m^p \, m^{p-2} \ll \left(\sum_{m=n+1}^{\infty} \left\{ a_m^p \, m^{p-1+\gamma p-\frac{p}{\theta}} \right\}^{\frac{\theta}{p}} \right)^{\frac{p}{\theta}}$$
$$\times \left(\sum_{m=n+1}^{\infty} \left\{ m^{-\gamma p+\frac{p}{\theta}-1} \right\}^{\frac{\theta}{\theta-p}} \right)^{\frac{\theta-p}{\theta}}.$$

Here the second factor is less than equal to $K\, n^{-\frac{\varepsilon\, p}{\theta}}$. Consequently, by (4.3),

$$I_{23} \ll n^{1-\varepsilon}\lambda_n \sum_{m=n+1}^{\infty} a_m^\theta \, m^{\theta-1+\varepsilon-\frac{\theta}{p}} \ll \sum_{m=n+1}^{\infty} a_m^\theta \, \lambda_m \, m^{\theta-\frac{\theta}{p}}.$$

Collecting our partial results we get that the estimation (4.1) is true for $\theta \in [p, \infty)$ as well.

Herewith it is verified that the condition (1.7) suffices to $f \in \mathrm{BN}(\alpha, \beta, \psi, p, \theta, k)$.

To prove that the condition (1.7) is also necessary; in the case $\theta \in [p, \infty)$, it is enough to consider Lemma 3; and if $\theta \in (0, p]$ we use that $I_1 + I_2 \gg I_{11} + I_{12} + I_{21}$ and estimate these sums from below one by one. It is clear that

$$I_{11} \gg \sum_{\nu=n+1}^{2n} \lambda_\nu \, \nu^{-(k+\beta)\theta} \left[\sum_{m=n/2}^{n} a_m^p \, m^{(k+\beta+1)p-2} \right]^{\frac{\theta}{p}}.$$

Since $\{a_n\} \in \mathrm{LAMS}$

$$\sum_{n=n/2}^{n} a_m^p \, m^{(k+\beta+1)p-2} \gg a_n^p \, n^{(k+\beta+1)p-1},$$

consequently,

$$I_{11} \gg \sum_{\nu=n+1}^{2n} \lambda_\nu \, \nu^{-(k+\beta)\theta} a_n^\theta \, n^{(k+\beta+1)\theta - \frac{\theta}{p}},$$

again relying on $\{a_n\} \in \mathrm{LAMS}$ we have that

$$I_{11} \gg \sum_{\nu=n+1}^{2n} a_\nu^\theta \, \lambda_\nu \, \nu^{\theta\left(1-\frac{1}{p}\right)}.$$

Similarly, regarding $\{a_n\} \in \mathrm{LAMS}$,

$$\begin{aligned} I_{12} &\gg \sum_{\nu=2n+1}^{\infty} \lambda_\nu \, \nu^{-(k+\beta)\theta} \left[\sum_{m=n+1}^{\nu} a_m^p \, m^{(k+\beta+1)p-2} \right]^{\frac{\theta}{p}} \\ &\gg \sum_{\nu=2n+1}^{\infty} \lambda_\nu \, \nu^{-(k+\beta)\theta} \left[\sum_{m=\nu/2}^{\nu} a_m^p \, m^{(k+\beta+1)p-2} \right]^{\frac{\theta}{p}} \\ &\gg \sum_{\nu=2n+1}^{\infty} a_\nu^\theta \, \lambda_\nu \, \nu^{\theta\left(1-\frac{1}{p}\right)}. \end{aligned}$$

Next we use the reverse of (3.1) and the second assertion of (1.6). Thus

$$
\begin{aligned}
I_{21} &\gg n^{-\beta\theta} \sum_{\nu=1}^{n} \lambda_\nu \, \nu^{-k\theta} \left[\sum_{m=1}^{\nu} a_m^p \, m^{(k+\beta+1)p-2} \right]^{\frac{\theta}{p}} \\
&\gg n^{-\beta\theta} \sum_{\nu=1}^{n} \left(\lambda_\nu \, \nu^{-k\theta}\right)^{1-\frac{\theta}{p}} \left(a_\nu^p \, \nu^{(k+\beta+1)p-2}\right)^{\frac{\theta}{p}} \left(\sum_{n=\nu}^{\infty} \lambda_n \, n^{-k\theta} \right)^{\frac{\theta}{p}} \\
&\gg n^{-\beta\theta} \sum_{\nu=1}^{n} a_\nu^\theta \, \lambda_\nu \, \nu^{(\beta+1)\theta-\frac{\theta}{p}}.
\end{aligned}
$$

Summing up we get that

$$
I_1 + I_2 \gg n^{-\beta\theta} \sum_{\nu=1}^{n} a_\nu^\theta \, \lambda_\nu \, \nu^{(\beta+1)\theta-\frac{\theta}{p}} + \sum_{\nu=n+1}^{\infty} a_\nu^\theta \, \lambda_\nu \, \nu^{\theta\left(1-\frac{1}{p}\right)},
$$

and this proves the necessity of (1.7) for $0 < \theta \le p$.

Herewith the proof is complete.

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