Publ. Math. Debrecen
64/3-4 (2004), 269–284

On some sufficient conditions of supersolvability of finite groups

By YANMING WANG (Guangzhou) and YANGMING LI (Guangzhou)

Abstract. A subgroup H is said to be c-supplemented in a finite group G if there exists a subgroup K of G such that $HK = G$ and $H \cap K$ is contained in $\mathrm{Core}_G(H)$. We determine the structure of a finite group G with the minimal subgroups of the generalized Fitting subgroup of some normal subgroups of G c-supplemented in G, generalizing some known results.

1. Introduction

All groups considered in this paper will be finite. We use $M <\cdot G$ to indicate that M is a maximal subgroup of G.

We say, following [5], a subgroup H of a group G is c-supplemented (in G) if there exists a subgroup K of G such that $HK = G$ and $H \cap K \leq H_G = \mathrm{Core}_G(H)$, and K is called a c-supplement of H in G.

Recall that a subgroup H of G is said to be c-normal (in G) if there exists a normal subgroup N of G such that $HN = G$ and $H \cap N \leq H_G$ ([4]). A subgroup H is said to be complemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K = 1$. One can easily see that c-supplementation is a generalization of c-normality and complementation,

Mathematics Subject Classification: 20D10, 20D20.
Key words and phrases: c-supplemented, generalized Fitting subgroup, formation, supersolvable.
Project supported in part by NSFC, NSFG, Fund from Education Ministry of China and Advanced Center of ZSU.

that is, to remove the normal supplementation assumption in c-normality and to remove the trivial intersection assumption in complementation. There are examples to show that c-supplementation does not imply c-normality or complementation ([5]).

Let $\mathcal{F}$ be a class of groups. We call $\mathcal{F}$ a formation provided that (i) if $G \in \mathcal{F}$ and $H \lhd G$, then $G/H \in \mathcal{F}$, and (ii) if G/M and G/N are in $\mathcal{F}$, then $G/(M \cap N)$ is in $\mathcal{F}$ for normal subgroups M, N of G. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$ (see [1, Ch VI]). Throughout this paper $\mathcal{U}$ will denote the class of all supersolvable groups. Clearly, $\mathcal{U}$ is a saturated formation.

Let $\mathcal{P}$ be the set of prime numbers. A formation function is a function f defined on $\mathcal{P}$ such that $f(p)$ is a, possibly empty, formation. A chief factor H/K of a group G is f-central in G if $G/C_G(H/K) \in f(p)$ for all primes p dividing $|H/K|$. $\mathcal{F}$ is local if and only if there exists a formation function f such that $\mathcal{F}$ is the class of all groups with f-central factors. We write $\mathcal{F} = LF(f)$ and say that f is a local definition of $\mathcal{F}$. The Theorem of Gaschutz–Lubeseder–Schmid [1, IV 4.6] states that the non-empty saturated formations are the local ones. A chief factor H/K of a group G is said to be $\mathcal{F}$-central in G, $\mathcal{F}$ a saturated formation and f an integrated and local definition of $\mathcal{F}$, if H/K is f-central in G; H/K is $\mathcal{F}$-eccentric otherwise. A maximal subgroup M of G is called $\mathcal{F}$-normal in G if $G/M_G \in \mathcal{F}$ and $\mathcal{F}$-abnormal otherwise. M is said to be $\mathcal{F}$-critical in G if $Soc(G/M_G)$ is the unique minimal normal subgroup of G/M_G, M is $\mathcal{F}$-abnormal in G, and $G = MF'(G)$, where $F'(G)/\Phi(G) = \mathrm{Soc}(G/\Phi(G))$. By [13, Theorem 3.5], if G does not belong to $\mathcal{F}$, then G has an $\mathcal{F}$-critical maximal subgroup.

For a formation $\mathcal{F}$, each group G has a smallest normal subgroup N such that G/N is in $\mathcal{F}$. This uniquely determined normal subgroup of G is called the $\mathcal{F}$-residual subgroup of G and is denoted by $G^{\mathcal{F}}$.

Definition. Let p be a prime and G be a group. We define:

$$\mathcal{P}_p(G) = \{x \mid x \in G,\ |x| = p\},$$

$$\mathcal{P}(G) = \bigcup_{p \in \pi(G)} \mathcal{P}_p(G),$$

Let x be an element of G. We say that x is c-supplemented in G if $\langle x \rangle$ is c-supplemented in G.

2. Preliminary results

In this section, we give some results that are needed in this paper.

Lemma 2.1 (5, Lemma 2.1). *Let G be a group. Then*

(1) *If H is c-supplemented in G, $H \leq M \leq G$, then H is c-supplemented in M;*

(2) *Let $N \triangleleft G$ and $N \leq H$. Then H is C-supplemented in G if and only if H/N is c-supplemented in G/N;*

(3) *Let π be a set of primes. Let N be a normal π'-subgroup and let H be a π-subgroup of G. If H is c-supplemented in G, then HN/N is c-supplemented in G/N.*

(4) *Let $H \leq G$ and $L \leq \Phi(H)$. If L is c-supplemented in G, then $L \triangleleft G$ and $L \leq \Phi(G)$.*

Let G be a group. The generalized Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasinilpotent subgroup of G. $F^*(G)$ is an important subgroup of G and it is a natural generalization of $F(G)$. The definition and important properties can be found in [6, X 13]. We would like to give the following basic facts which we will use in our proof.

Lemma 2.2. *Let G be a group and M a subgroup of G.*

(1) *If M is normal in G, then $F^*(M) \leq F^*(G)$;*

(2) *$F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = \mathrm{soc}(F(G)C_G(F(G))/F(G))$;*

(3) *$F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is soluble, then $F^*(G) = F(G)$.*

(4) *$C_G(F^*(G)) \leq F(G)$;*

(5) *Suppose K is a subgroup of G contained in $Z(G)$, then $F^*(G/K) = F^*(G)/K$.*

Proof. (1)–(4) can be found in [6, X 13].

(5) Denote $F^*(G/K) = L/K$. Consider a chief series of G of the form

$$G = G_0 > \cdots > G_{m-1} > G_m = K > G_{m+1} > \cdots > G_n = 1.$$

By the definition of the generalized Fitting subgroup, $\forall x \in L$, $\overline{x} = xK$ induces an inner automorphism on the chief factor $\overline{G_{i-1}}/\overline{G_i} = (G_{i-1}/K)/(G_i/K)$ of G/K, for $i = 1, 2, \ldots, m$, thus x induces an inner automorphism on the chief factor G_{i-1}/G_i of G, for $i = 1, 2, \ldots, m$. Since $K \le Z(G)$, the automorphism induced by x on the chief factor G_{i-1}/G_i of G is identity, for $i = m+1, \ldots, n$, so x induces an inner automorphism on the chief factor G_{i-1}/G_i of G, for $i = 1, 2, \ldots, n$. Hence by [6, X, Lemma 13.1], x induces an inner automorphism on any chief factor of G. Thus $x \in F^*(G)$, i.e., $L \le F^*(G)$. Obviously $F^*(G) \le L$, hence $F^*(G/K) = F^*(G)/K$. □

Lemma 2.3. *Let G be a group. Assume that N is a normal subgroup of G ($N \ne 1$) and $N \cap \Phi(G) = 1$. Then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which are contained in $F(N)$. In particular, if $\Phi(G) = 1$, then $F(G)$ the direct product of minimal normal subgroups of G which are contained in $F(G)$.*

Proof. Refer to [1] Chapter 3. □

Next we give two properties of Fitting subgroup.

Lemma 2.4. *Suppose N, L are two normal subgroups of a group G.*

(1) *If L is nilpotent, then $F(NL) = F(N)L$.*

(2) *If $L \le \Phi(G)$, then $F(NL/L) = F(N)L/L$.*

Proof. (1) By the hypothesis, it is easy to see that $F(N)L$ is a nilpotent normal subgroup of NL, so $F(N)L \le F(NL)$. On the other hand, $F(NL) = F(NL) \cap NL = (F(NL) \cap N)L$, then $F(NL) \cap N$ is a nilpotent normal subgroup of N, thus is contained in $F(N)$. So $F(NL) \le F(N)L$, the equality holds.

(2) Denote $F(NL/L)=K/L$. Since $F(NL)/L$ is nilpotent, $F(NL)/L \le K/L$. On the other hand, K/L is nilpotent, $L \le \Phi(G)$, [1, p. 220, Satz 3.5] implies that K is nilpotent, thus $K \le F(NL)$, so $F(NL/L) = F(NL)/L = F(N)L/L$ by (1). □

Lemma 2.5. *Let G be a group with a normal subgroup N such that G/N is supersolvable. Suppose that $\Phi(G) = 1$ and $F^*(G) = F(G)$. If for any maximal subgroup M of G, either $F(N) \le M$ or $F(N) \cap M <\cdot F(N)$, then G is supersolvable.*

Proof. If $N = 1$, obviously the theorem holds. So assume that $N \ne 1$, then Lemma 2.2 implies that $F^*(N) \le F^*(G) = F(G)$, so $F^*(N) = F(N) \ne 1$. By Lemma 2.3 and the hypothesis, we have $F(N) = L_1 \times L_2 \times$

$\cdots \times L_s$, where L_i is a minimal normal subgroup of G contained in $F(N)$. Again by Lemma 2.3 we can write that $F(G) = F(N) \times H_1 \times H_2 \times \cdots \times H_r$, where H_i is a minimal normal subgroup of G contained in $F(G)$ but not contained in $F(N)$ and $F(G)$ is abelian. $\forall i$, $L_i \nleq \Phi(G)$ as $\Phi(G) = 1$, so there exists a maximal subgroup M_i of G such that $L_i \nleq M_i$. It follows that $G = L_i M_i$. Since L_i is abelian, $L_i \cap M_i \lhd L_i M_i = G$. The minimality of L_i implies that $L_i \cap M_i = 1$. Since $F(N) = F(N) \cap L_i M_i = L_i(F(N) \cap M_i)$, $F(N) \cap M_i < \cdot F(N)$ by hypotheses, then the nilpotence of $F(N)$ implies that $[F(N) : F(N) \cap M_i]$ is a prime.

Since $L_i \leq F(N), G = M_i F(N)$, thus $|L_i| = [G : M_i] = [F(N) : F(N) \cap M_i]$ is a prime. So $G/C_G(L_i)$ is abelian, then $G' \leq C_G(L_i)$, $\forall i$.

Since $H_j N/N$ is the minimal normal subgroup of the supersolvable group G/N, we have that $H_j \cong H_j N/N$ has also prime order. So $G/C_G(H_j)$ is abelian, then $G' \leq C_G(H_j)$. Therefore

$$G' \leq \left(\bigcap_{j=1}^{r} C_G(H_j) \right) \cap \left(\bigcap_{i=1}^{s} C_G(L_i) \right) = C_G(F(G)).$$

By Lemma 2.2(4), we have $C_G(F^*(G)) \leq F(G)$, thus $G' \leq F(G)$ by hypotheses.

For any maximal subgroup M of G, by hypotheses we have either $F(N) \leq M$ or $F(N) \cap M < \cdot F(N)$. If $F(N) \cap M < \cdot F(N)$, then $G = MF(N)$, thus $[G : M] = [F(N) : F(N) \cap M]$. The nilpotence of $F(N)$ implies $[F(N) : F(N) \cap M]$ is a prime, so $[G : M]$ is also a prime. If $F(N) \leq M$ but $F(G) \nleq M$, then there exists a minimal normal subgroup H_i such that $H_i \nleq M$, so $G = H_i M$. Thus $[G : M] = |H_i|$ is a prime. If $F(G) \leq M$, then $M \geq G'$, thus $M \lhd G$ and G/M has no trivial proper subgroup, so $|G : M|$ is also a prime. Therefore G is supersolvable by the well-known Huppert Theorem. □

Lemma 2.6. *Let G be a finite group. Suppose $G = PM$, where P is a normal p-subgroup of G and M is a maximal subgroup of G. Then $P \cap M \lhd G$.*

Proof. First we have $N_G(P \cap M) \geq M$ by the normality of P. Obviously $P \nleq M$, so $P \cap M < P$. Since P is a p-group, P has a subgroup P_1 such that $P \cap M$ a proper normal subgroup of P_1, so $N_G(P \cap M) \geq$

$\langle M, P_1 \rangle = G$ as M is a maximal subgroup of G and $P_1 \nleq M$. Thus $P \cap M \lhd G$. □

The following Theorem is a generalization of [14, Theorem 3.7].

Theorem 2.7. *Let G be a group with a normal subgroup N such that G/N is supersolvable. If every element of prime order of N is c-supplemented in G and N is quaternion-free, then G is supersolvable.*

Proof. Assume that the result is false and let G be a counterexample of minimal order.

(1) Every proper subgroups of G is supersolvable. Furthermore

  (a) There exists a normal Sylow p-subgroup of G such that $G = PR$ and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$;

  (b) If $p > 2$, then the exponent of P is p. When $p = 2$, the exponent of P is 2 or 4, especially in this case, every proper subgroup of G is nilpotent and $\Phi(P) \leq Z(G)$.

  Denote $K = G^{\mathcal{U}}$. Let M be a maximal subgroup of G. It is clear that $M/M \cap K$ is supersolvable and hence $M^{\mathcal{U}} \leq M \cap K$. By Lemma 2.1, every element of $\mathcal{P}(M^{\mathcal{U}})$ is c-supplemented in M, obviously, $M^{\mathcal{U}}$ is quaternion-free, so that M satisfies the hypotheses of G. The minimal choice of G yields that M is supersolvable. This holds for every maximal subgroup M of G. Hence we have that G is not supersolvable but every proper subgroup of G is supersolvable. [1, VI, §, Ex. 1] implies (1)(a) and (1)(b).

(2) $K = P$. Since G/P is supersolvable, we have that $K \leq P$. Then $K\Phi(P)/\Phi(P)$ is a normal subgroup of $G/\Phi(P)$ contained in $P/\Phi(P)$. Since $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, we have that either $K\Phi(P) = P$ or $K \leq \Phi(P)$. If $K < \Phi(P)$, then K is actually contained in $\Phi(G)$ and $G/\Phi(G)$ is supersolvable. Hence G is supersolvable, a contradiction, and so we have that $P = K$.

(3) $\Phi(P) \neq 1$. Otherwise P is elementary abelian and hence, by (1), every element of P lies in $\mathcal{P}(P)$. Our hypotheses claims that every element of P is c-supplemented in G. Let $1 \neq x \in P$. Then there exists $K \leq G$ such that $\langle x > K = G$ and $\langle x \rangle \cap K \leq \langle x \rangle_G$. Then $P = \langle x \rangle (P \cap K)$. Since P is abelian, we have that $P \cap K \lhd G$. By (1), P is a minimal

normal subgroup of G when $\Phi(P) = 1$. Therefore $P \cap K = 1$ or $P \leq M$. In both case, we have that $\langle x \rangle = P$ and therefore G is supersolvable, a contradiction.

(4) $p = 2$. Assume that $p > 2$. Then by (1)(b) every element of P is c-supplemented in G. Moreover, By Lemma 2.1(4), $\Phi(P)$ is contained in $\Phi(G)$ and $\Phi(P) \triangleleft G$. Next we see that the hypotheses of Theorem holds in $G/\Phi(P)$. Let $x \in P - \Phi(P)$. By hypotheses there exists a subgroup M of G such that $G = \langle x \rangle M$ and $\langle x \rangle \cap M \leq \langle x \rangle_G$. If $\langle x \rangle = \langle x \rangle_G$, then (1) implies that $P = \langle x \rangle \Phi(P) = \langle x \rangle$. Then G is supersolvable, a contradiction. And so we have that $\langle x \rangle \cap M = 1$. Hence M is a maximal subgroup of G because $o(x) = p$. This implies that $G/\Phi(P) = \langle x \rangle \Phi(P)/\Phi(P) \cdot M/\Phi(P)$ and $(\langle x \rangle \Phi(P)/\Phi(P)) \cap (M/\Phi(P)) = \overline{1}$ and $\langle x \rangle \Phi(P)/\Phi(P)$ is c-supplemented in $G/\Phi(P)$. Since $(G/\Phi(P))^{\mathcal{U}} \leq K\Phi(P)/\Phi(P)$, we have that $(G/\Phi(P))^{\mathcal{U}}$ is quaternion-free. The minimal choice of G (notice that $\Phi(P) \neq 1$ by (3)) implies that $G/\Phi(P)$ is supersolvable. Since $\Phi(P) \leq \Phi(G)$, we have that $G/\Phi(G)$ is supersolvable and so is G, a contradiction.

(5) Final contradiction.

If $\exp(P) = 2$, then P is elementary abelian, contrary to (3), so $\exp(P) = 4$. Thus the order of every element of $P - \Phi(P)$ is 4. Pick $a, b \in P - \Phi(P)$ such that the order of $c = [a, b]$ is 2. Denote $R = \langle a, b \rangle / \langle a^2 b^2, ca^2 \rangle$, then $\overline{a^2} = \overline{b^2} = [\overline{a}, \overline{b}] = \overline{c} \neq 1$, $o(\overline{a}) = o(\overline{b}) = 4$, $o(\overline{c}) = 2$. So R is quaternion and a section of P, it is contrary to the hypothesis that K is quaternion-free. These complete our proof. □

Lemma 2.8. *Suppose N is a normal subgroup of a group G. If every element of $\mathcal{P}(F(N))$ is c-supplemented in G and $F(N)$ is quaternion-free, then for any maximal subgroup M of G, there holds either $F(N) \leq M$ or $F(N) \cap M < \cdot F(N)$.*

Proof. For any maximal subgroup M of G, we first indicate that as in the proof of Lemma 2.3 when $F(N) \nleq M$, $F(N) \cap M < \cdot F(N)$ is equivalent to $[G : M]$ is a prime.

Suppose that $F(N) \nleq M$, then there exists a prime p such that $O_p(N) \nleq M$, therefore $G = O_p(N)M$. We go through our discussion with two cases.

(i) p is an odd prime.

Assume that there exists an element of order p of $O_p(N)$, x_1 say, such that x_1 is not normal in G. Since x_1 is c-supplemented in G by the hypotheses, there exists a subgroup of G, M_1 say, such that $G = \langle x_1\rangle M_1$ and $\langle x_1\rangle \cap M_1 = 1$. Then we have that $O_p(N) = \langle x_1\rangle(O_p(N) \cap M_1)$, where $O_p(N) \cap M_1$ is normal in G by Lemma 2.6, so we can write

$$G = \langle x_1\rangle[(O_p(N) \cap M_1)M],$$

where $[(O_p(N) \cap M_1)M$ is a subgroup of G containing the maximal subgroup M. If $(O_p(N) \cap M_1)M = M$, then $G = \langle x_1\rangle M$, so $[G : M]$ is a prime. Thus the lemma holds. So we assume $(O_p(N) \cap M_1)M = G$, denote $P_1 = O_p(N) \cap M_1$, thus $G = P_1M$, P_1 is normal in G by Lemma 2.6 and $x_1 \notin P_1$.

If there exists an element of order p of P_1, x_2 say, such that x_2 is not normal in G, then there exists a subgroup of G, M_2 say, such that $G = \langle x_2\rangle M_2$ and $\langle x_2\rangle \cap M_2 = 1$ as x_2 is c-supplemented in G by the hypotheses. Then $P_1 = \langle x_2\rangle(P_1 \cap M_2)$, $P_2 \cap M_2$ is normal in G by Lemma 2.6, so we can write $G = \langle x_2\rangle[(P_1 \cap M_2)M]$, where $[(P_1 \cap M_2)M$ is a subgroup of G containing the maximal subgroup M. If $(P_1 \cap M_2)M = M$, then $G = \langle x_2\rangle M$. It follows that $[G : M]$ is a prime. Thus the lemma holds. So assume $(P_1 \cap M_2)M = G$, denote $P_2 = P_1 \cap M_2$, thus $G = P_2M$, P_2 is a normal subgroup of G contained in $O_p(N)$ by Lemma 2.6 and $x_1, x_2 \notin P_2$.

Repeating this method, at end we have that either $[G : M]$ is a prime or there exists a normal p-subgroup of G contained in $O_p(N)$, P say, such that $G = PM$ and every element of order p of P is normal in G. In particular, P is a PN-group ([11]).

If P has an element x of order p such that $x \notin M$, then $G = \langle x\rangle M$ as M is a maximal subgroup of G and x is normal in G. Thus $[G : M]$ is a prime, the lemma holds. So we can assume that $\Omega_1(P) \leq M$. Consider the factor group $G/\Omega_1(P) = P/\Omega_1(P)\cdot M/\Omega_1(P)$. $\forall x\Omega_1(P) \in \Omega_1(P/\Omega_1(P)) = \Omega_2(P)/\Omega_1(P)$, then $x^p \in \Omega_1(P)$, thus $\langle x^p\rangle \lhd G$ by the properties of P. So $\forall g \in G$, there exists an integer i such that $(x^p)^g = (x^p)^i = (x^i)^p = (x^g)^p$. Since x^g, x^i lie in $\Omega_2(P)$ which is a PN-group with exponent of p^2, now [11, Th 1(iii)] implies that $(x^gx^{-i})^p = 1$, so $x^gx^{-i} \in \Omega_1(P)$, therefore $\langle x\Omega_1(P)\rangle \lhd G/\Omega_1(P)$. If $\langle x\Omega_1(P)\rangle \nleq M/\Omega_1(P)$, then $G/\Omega_1(P) = \langle x\Omega_1(P)\rangle M/\Omega_1(P)$, it follows that $[G : M] = [G/\Omega_1(P) : M/\Omega_1(P)]$ is a prime, the lemma holds. Thus we can assume that $\Omega_1(P/\Omega_1(P)) \leq M/\Omega_1(P)$. This leads that $\Omega_2(P) \leq M$.

Again consider the factor group $G/\Omega_2(P) = P/\Omega_2(P) \cdot M/\Omega_2(P)$. Using the same method as in above, we get that either $[G : M]$ is a prime or $\Omega_3(P) \leq M$, ..., repeating the method above, at end we have that $[G : M]$ is a prime as P is finite. So the lemma holds in this case.

(ii) $p = 2$.

Suppose that M_2 is a Sylow 2-subgroup of M, then $O_2(N)M_2$ is a Sylow 2-subgroup of G as $G = O_2(N)M$. Suppose that H is a maximal subgroup of $O_2(N)M_2$ containing M_2, then $H = H \cap (O_2(N)M_2) = (H \cap O_2(N))M_2$.

$\forall q \neq 2$, $\forall M_q \in S_q(M)$, since $O_2(N)M_q/O_2(N)$ is supersolvable, every element of $\mathcal{P}(O_2(N))$ is c-supplemented in G, thus is c-supplemented in $O_2(N)M_q$. Since $O_2(N) \subseteq F(N)$, $O_2(N)$ is quaternion-free by hypotheses, now Lemma 2.7 implies that $O_2(N)M_q$ is supersolvable, thus $O_2(N)M_q = O_2(N) \times M_q$. So $O^2(M) \leq C_G(O_2(N))$, therefore $(H \cap O_2(N))M_{2'}$ is a group, so is $(H \cap O_2(N))M$.

Since $M_2 \cap O_2(N) \leq M \cap (H \cap O_2(N)) \leq M \cap O_2(N) \leq O_2(N) \cap M_2$, we have that $M \cap O_2(N) = M_2 \cap O_2(N)$. Thus $(H \cap O_2(N)) \cap M = H \cap (O_2(N) \cap M) = H \cap (O_2(N) \cap M_2) = O_2(N) \cap M_2 = O_2(N) \cap M$. Therefore

$$|(H \cap O_2(N))M| = \frac{|H \cap O_2(G)| \cdot |M|}{|(H \cap O_2(N)) \cap M|} < \frac{|O_2(N)| \cdot |M|}{|M \cap O_2(N)|} = |G|,$$

so $(H \cap O_2(N))M < G$, this implies $(H \cap O_2(N))M = M$ as M is a maximal subgroup of G, so $H \cap O_2(N) \leq M$, $H \cap O_2(N) \leq M \cap O_2(N)$. This time,

$$[G : M] = \frac{|O_2(N)|}{|O_2(N) \cap M|} \leq \frac{|O_2(N)|}{|O_2(N) \cap H|} = |HO_2(N)/H| \leq 2,$$

as H is a maximal subgroup of Sylow 2-subgroup of G. Therefore $[G : M] = 2$ is a prime.

These complete the proof of the lemma. $\square$

3. Main results

Theorem 3.1. *Suppose G is a solvable group with a normal subgroup N such that G/N is supersolvable. If every element of $\mathcal{P}(F(N))$ is c-supplemented in G and $F(N)$ is quaternion-free, then G is supersolvable.*

Proof. Since G is solvable, $F^*(G){=}F(G)$ by Lemma 2.2. If $\Phi(G){=}1$, then By Lemma 2.5 and Lemma 2.8, G is supersolvable. Assume that $\Phi(G) \neq 1$. We consider the factor group $\overline{G} = G/\Phi(G)$. Since $\Phi(\overline{G}) = 1$, $F^*(\overline{G}) = F(\overline{G})$ as $\overline{G}$ is solvable. For any maximal subgroup $M/\Phi(G)$ of $\overline{G}$, obviously M is also a maximal subgroup of G, by Lemma 2.8 we have either $M \geq F(N)$ or $M \cap F(N)$ is a maximal subgroup of $F(N)$. If $F(N) \leq M$, then $F(\overline{N}) = F(N\Phi(G)/\Phi(G)) = F(N\Phi(G))/\Phi(G) = F(N)\Phi(G)/\Phi(G) \leq M/\Phi(G)$ by Lemma 2.4. If $F(N) \cap M < \cdot F(N)$, then $[F(N) : F(N) \cap M]$ is a prime. Since $[F(\overline{N}) : F(\overline{N}) \cap \overline{M}] = [F(N)\Phi(G) : F(N)\Phi(G) \cap M] = [F(N)\Phi(G) : (F(N) \cap M)\Phi(G)] = [F(N) : F(N) \cap M]$ is a prime, $F(\overline{N}) \cap \overline{M}$ is a maximal subgroup of $F(\overline{N})$. Therefore $\overline{G}$ satisfies the hypotheses of Lemma 2.5, thus $\overline{G}$ is supersolvable, so G is supersolvable. $\square$

We want to delete the hypotheses of solvability of G in Theorem 3.1, but we should replace the Fitting subgroup $F(N)$ with the generalized Fitting subgroup $F^*(N)$.

Theorem 3.2. *Suppose G is a group with a normal subgroup N such that G/N is supersolvable. $F^*(N)$ is the generalized Fitting subgroup of N. If every element of prime order of $F^*(N)$ is c-supplemented in G and $F^*(N)$ is quaternion-free, then G is supersolvable.*

Proof. Assume that the theorem is false and let G be a counterexample of minimal order. Then we have:

(1) Every proper normal subgroup of G is supersolvable.

If H is a proper normal subgroup of G, then we have that $H/H \cap N \cong HN/N$, thus $H/H \cap N$ is supersolvable. Since $F^*(H \cap N) \leq F^*(N)$ by Lemma 2.2, we have that every element of $\mathcal{P}^*(F(H \cap N))$ is c-supplemented in G, thus is c-supplemented in H by Lemma 2.1. So H satisfies the hypotheses of the theorem. The minimal choice of G implies that H is supersolvable.

Since $F(N) \leq F^*(N)$, Lemma 2.8 and Theorem 3.1 imply that:

(2) $\forall M < \cdot G$, there holds either $F(N) \leq M$ or $F(N) \cap M < \cdot F(N)$. Furthermore, G is not solvable.

(3) $G = G' = O^q(G) = N$, $\forall q \in \pi(G)$.

If $G' < G$, then G' is supersolvable by (1). Since G/G' is abelian, we have that G is solvable, contrary to (2).

If $O^q(G) < G$, then $O^q(G)$ is supersolvable by (1), thus G is solvable as $G/O^q(G)$ is a q-group, contrary to (2).

Again If $N < G$, then N is supersolvable by (1). Since G/N is supersolvable, we have that G is solvable, contrary to (2) too.

(4) $F^*(G) = F^*(N) = F(G) = F(N) < G$.

If $F^*(G) = G$, then every element of prime order or order 4 of G is c-supplemented in G. Lemma 2.7 implies that G is supersolvable, a contradiction. So $F^*(G) < G$, $F^*(G)$ is supersolvable by (1). In particular, $F^*(G) = F^*(F^*(G)) = F(F^*(G)) \leq F(G)$. So we have that $F^*(G) = F(G)$.

(5) $\Phi(G) = 1$.

Assume that $\Phi(G) \neq 1$. Suppose Q is any Sylow q-subgroup of $\Phi(G)$, for every element x of $\mathcal{P}^*(Q)$, x is c-supplemented in G by hypotheses, thus there exists a subgroup L of G such that $G = \langle x \rangle L$ and $\langle x \rangle \cap L \leq \langle x \rangle_G$. Since $x \in \Phi(G)$, we have that $G = L$ and $\langle x \rangle$ is normal in G. Thus $G/C_G(\langle x \rangle)$ is abelian, this implies that $G = G' \leq C_G(\langle x \rangle)$, i.e., G centralizes every element of $\mathcal{P}^*(Q)$. By Huppert's result ([1, Satz IV 5.12]), we get every q'-element of G centralizes Q. Thus $G/C_G(Q)$ is a q-group. Then $G = O^q(G) \leq C_G(Q)$, so $Q \leq Z(G)$, therefore $\Phi(G) \leq Z(G)$.

Consider the factor group $\overline{G} = G/\Phi(G)$. Since $\Phi(\overline{G}) = 1$, and by Lemma 2(5), $F^*(\overline{G}) = F^*(G)/\Phi(G) = F(G)/\Phi(G) = F(G/\Phi(G)) = F(\overline{G})$. For any maximal subgroup $M/\Phi(G)$ of $\overline{G}$, obviously M is also a maximal subgroup of G, so it holds that $F(N) \leq M$ or $F(N) \cap M < \cdot F(N)$ by (2). If $F(N) \leq M$, then $F(\overline{N}) = F(N\Phi(G)/\Phi(G)) = F(N\Phi(G))/\Phi(G) = F(N)\Phi(G)/\Phi(G) \leq M/\Phi(G)$. If $F(N) \cap M < \cdot F(N)$, then $[F(N) : F(N) \cap M]$ is a prime. Since $[F(\overline{N}) : F(\overline{N}) \cap \overline{M}] = [F(N)\Phi(G) : F(N)\Phi(G) \cap M] = [F(N)\Phi(G) : (F(N) \cap M)\Phi(G)] = [F(N) : F(N) \cap M]$ is a prime, $F(\overline{N}) \cap \overline{M}$ is a maximal subgroup of $F(\overline{N})$. We have either $M/\Phi(G) \geq F(G)/\Phi(G) = F(G/\Phi(G))$ or $M/\Phi(G) \cap F(G/\Phi(G)) = M \cap F(G)/\Phi(G)$ is a maximal subgroup of $F(G/\Phi(G))$. So $\overline{G}$ satisfies the hypotheses of Lemma 2.5, thus $\overline{G}$ is supersolvable, therefore G is supersolvable, a contradiction.

(6) The final contradiction.

By (1), (2), (4) and (5), G satisfies the hypotheses of Lemma 2.5, G is supersolvable. The final contradiction.

These complete the proof of our theorem. □

Corollary 3.3. *Let G be a group. If every element of prime order of $F^*(G)$ is c-supplemented in G and $F^*(G)$ is quaternion-free, then G is supersolvable.*

4. Generalization to formations

The following is a special case of [12, Theorem 2]:

Lemma 4.1. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Assume G is a group with a normal subgroup N of prime order such that $G/N \in \mathcal{F}$, then $G \in \mathcal{F}$.*

First we generalize Theorem 2.7 with formation, it is also a generalization of [14, Theorem 3.7] with c-supplementment.

Theorem 4.2. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Assume G is a group with a normal subgroup N such that $G/N \in \mathcal{F}$. If every element of $\mathcal{P}(N)$ is c-supplemented in G and N is quaternion-free, then G belongs to $\mathcal{F}$.*

Proof. Without losing generality we can assume $N = G^{\mathcal{F}}$.

Suppose that the result is false and let G be a counterexample of minimal order. Then $G \notin \mathcal{F}$ and $G^{\mathcal{F}} \neq 1$. Suppose that $G^{\mathcal{F}}$ is not 2-nilpotent. Then $G^{\mathcal{F}}$ has a subgroup K such that K is not 2-nilpotent but every proper subgroup of K is 2-nilpotent. Then by [1, IV satz 5.4], $K = K_2K_q$ where K_2 is a normal Sylow 2-subgroup and K_q is a non-normal cyclic Sylow q-subgroup for some odd prime q. Let $\mathcal{H}$ be the saturated formation of 2-nilpotent groups. Then $K^{\mathcal{H}}$ is contained in K_2 and every chief factor of K below $K^{\mathcal{H}}$ is $\mathcal{H}$-eccentric by [14, Lemma 3.6]. Let E be a minimal normal subgroup of K contained in $Z(K^{\mathcal{H}})$. If $EK_q < K$, then E is central in K and so $\mathcal{H}$-central, a contradiction. Hence $K^{\mathcal{H}} = K_2$ is a minimal normal subgroup of K. Pick an element x of order 2 in K_2, then x is c-supplemented in G by hypotheses, so there exists a subgroup H of G such that $G = \langle x\rangle H$ and $\langle x\rangle \cap H \leq \langle x\rangle_G$. If $\langle x\rangle_G = \langle x\rangle$, then obviously $K_2 = \langle x\rangle$. If $\langle x\rangle_G = 1$, then $[G : H] = 2$, so H is normal in G, so $K_2 \cap H$ is normal in K. Thus $K_2 \cap H = 1$ or K_2 by the minimality of K_2. If $K_2 \cap H = 1$, then $K_2 = K_2 \cap G = K_2 \cap (\langle x\rangle H) = \langle x\rangle(K_2 \cap H) = \langle x\rangle$. If $K_2 \cap H = K_2$, then $K_2 \leq H$, so $\langle x\rangle \cap H = \langle x\rangle = \langle x\rangle_G$, a contradiction. So we have that K_2 is a cyclic group of order 2. Then $[K : K_q] = 2$, so K_q is normal in K. Thus K is 2-nilpotent, a contradiction. Consequently $G^{\mathcal{F}}$ is 2-nilpotent. In particular, $G^{\mathcal{F}}$ is solvable. Let M be a maximal subgroup of G such that $G = MG^{\mathcal{F}}$ and $G/M_G \notin \mathcal{F}$. Then $G = MF(G)$. By [12, Proposition 1], $G^{\mathcal{F}}$ is a p-group for some prime p. By [15, Theorem 4.2], $p = 2$ and $(G^{\mathcal{F}})' = Z(G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$. Moreover $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ by [12, Proposition 1]. If $\exp(G^{\mathcal{F}}) = 2$, then every element of $G^{\mathcal{F}}$ is c-supplemented in G by hypotheses, now [15, Theorem 4.2] implies that $G \in \mathcal{F}$, a contradiction. So $\exp(G^{\mathcal{F}}) = 4$. Then the order of every element of $G^{\mathcal{F}} - \Phi(G^{\mathcal{F}})$ is 4. Pick $a, b \in G^{\mathcal{F}} - \Phi(G^{\mathcal{F}})$ such that the order of $c = [a, b]$ is 2. Denote $R = \langle a, b\rangle/\langle a^2b^2, ca^2\rangle$, then $\overline{a^2} = \overline{b^2} = [\overline{a}, \overline{b}] = \overline{c} \neq 1, o(\overline{a}) = o(\overline{b}) = 4, o(\overline{c}) = 2$. So R is quaternion and a section of $G^{\mathcal{F}}$, it is contrary to the hypothesis that $G^{\mathcal{F}}$ is quaternion-free. These complete our proof. □

For purpose of giving generalizations of Theorem 3.1 and Theorem 3.2 in word of formations, we give a lemma which is a generalization of Lemma 2.5.

Lemma 4.3. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Assume G is a group with a solvable normal subgroup such $G/N \in \mathcal{F}$. If for any maximal subgroup M of G, either $F(N) \leq M$ or $F(N) \cap M <\cdot F(N)$, then $G \in \mathcal{F}$.*

PROOF. We prove the lemma by induction on $|G|$.

Case 1. $N \cap \Phi(G) \neq 1$.

Denote $N_1 = N \cap \Phi(G)$ and $\overline{G} = G/N_1$. Then $F(N/N_1) = F(N)/N_1$ by Lemma 2.4. Since $(G/N_1)/(N/N_1) \cong G/N$, we have that $(G/N_1)/(N/N_1)$ lies in $\mathcal{F}$ and N/N_1 is solvable. Suppose that M/N_1 is an arbitrary maximal subgroup of G/N_1, then $M <\cdot G$, so it holds either $F(N) \leq M$ or $F(N) \cap M <\cdot F(N)$ by hypotheses. If $F(N) \leq M$, then $F(\overline{N}) = F(N/N_1) =$

$F(N)/N_1 \leq M/N_1$. If $F(N) \cap M < \cdot F(N)$, then $[F(N) : F(N) \cap M]$ is a prime. Since $[F(\overline{N}) : F(\overline{N}) \cap \overline{M}] = [F(N) : F(N) \cap M]$ is a prime, we have that $F(\overline{N}) \cap \overline{M}$ is a maximal subgroup of $F(\overline{N})$. Therefore $\overline{G}$ satisfies the hypotheses of the Theorem, thus $\overline{G} \in \mathcal{F}$ by induction, so $G \in \mathcal{F}$ as $\mathcal{F}$ is saturated.

Case 2. $N \cap \Phi(G) = 1$.

By Lemma 2.3, we have we have $F(N) = L_1 \times L_2 \times \cdots \times L_s$, where L_i is a minimal normal subgroup of G contained in $F(N)$. $\forall i$, $L_i \nleq \Phi(G)$ as $N \cap \Phi(G) = 1$, so there exists a maximal subgroup M_i of G such that $L_i \nleq M_i$. It follows that $G = L_i M_i$. Since L_i is abelian, we have that $L_i \cap M_i \lhd L_i M_i = G$. The minimality of L_i implies that $L_i \cap M_i = 1$. Since $F(N) = F(N) \cap L_i M_i = L_i(F(N) \cap M_i)$, $F(N) \cap M_i < \cdot F(N)$ by hypotheses, the nilpotence of $F(N)$ implies that $[F(N) : F(N) \cap M_i]$ is a prime.

Since $L_i \leq F(N)$, we have that $G = M_i F(N)$ and thus $|L_i| = [G : M_i] = [F(N) : F(N) \cap M_i]$ is a prime. So $G/C_G(L_i)$ is abelian, so $G/C_G(L_i) \in \mathcal{F}$. Then $G/C_G(F(N)) = G/\bigcap_{i=1}^{s} C_G(L_i) \in \mathcal{F}$ as $\mathcal{F}$ is a formation. So $G/C_N(F(N)) = G/N \cap C_G(F(N)) \in \mathcal{F}$. Since N is solvable by hypotheses, we have that $C_N(F(N)) \leq F(N)$, so we get that $G/F(N) \in \mathcal{F}$.

Consider the factor group G/L_1. Since $(G/L_1)/(F(N)/L_1) \cong G/F(N)$, we have that $(G/L_1)/(F(N)/L_1)$ lies in and $F(N)/L_1$ is solvable. Suppose that M/L_1 is an arbitrary maximal subgroup of G/L_1, then $M < \cdot G$, so it holds either $F(N) \leq M$ or $F(N) \cap M < \cdot F(N)$ by hypotheses. If $F(N) \leq M$, then $F(F(N)/L_1)) = F(N)/L_1) \leq M/N_1$. If $F(N) \cap M < \cdot F(N)$, then $[F(N) : F(N) \cap M]$ is a prime. Since $[F(F(N)/L_1) : F(F(N)/L_1) \cap M/L_1] = [F(N)/L_1 : F(N)/L_1 \cap M/L_1] = [F(N) : F(N) \cap M]$ is a prime, we have that $F(F(N)/L_1) \cap M/L_1$ is a maximal subgroup of $F(F(N)/L_1)$. Therefore $G/L_1, F(N)/L_1$ satisfy the hypotheses of the Theorem, thus $G/L_1 \in \mathcal{F}$ by induction. If $F(N)$ contains another minimal normal subgroup of G, L_2 say, then $G/L_2 \in \mathcal{F}$, then $L_1 \cap L_2 = 1$, it follows that $G = G/(L_1 \cap L_2) \in \mathcal{F}$. The lemma holds. So we assume that $F(N) = L_1$, then $G/L_1 \in \mathcal{F}$, now applying Lemma 4.1, we get $G \in \mathcal{F}$.

These complete our proof. $\square$

Theorem 4.4. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Assume G is a group with a solvable normal subgroup such $G/N \in \mathcal{F}$. If every* *element of $\mathcal{P}(F(N))$ is c-supplemented in G and $F(N)$ is quaternion-free, then G belongs to $\mathcal{F}$.*

Proof. By Lemma 2.8 and hypotheses, we have that for any maximal subgroup M, there holds either $F(N) \leq M$ or $F(N) \cap M < \cdot F(N)$. Now applying Lemma 4.3, it follows that $G \in \mathcal{F}$. $\square$

Theorem 4.5. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Assume G is a group with a normal subgroup such $G/N \in \mathcal{F}$. If every element of $\mathcal{P}(F^*(N))$ is c-supplemented in G and $F^*(N)$ is quaternion-free, then G belongs to $\mathcal{F}$.*

Proof. By the hypotheses every element of $\mathcal{P}(F^*(N))$ is c-suplemented in G, thus is c-supplemented in N, and $F^*(N)$ is quaternion-free by hypotheses. Now Corollary 3.3 implies that N is supersolvable, so $F^*(N) = F(N)$. Then $G \in \mathcal{F}$ by Theorem 4.4. $\square$

References

[1] B. Huppert, Endliche Gruppen I, *Springer-Verlag*, *Berlin*, 1968.

[2] J. Buckley, Finite groups whose minimal subgroups are normal, *Math. Z.* **116** (1970), 15–17.

[3] A. Shaalan, The influence of π-quasinormality of some subgroups, *Acta. Math. Hungar.* **56** (1990), 287–293.

[4] Y. Wang, C-normality of groups and its properties, *Journal of Algebra* **180** (1996), 954–965.

[5] Y. Wang, Finite groups with some subgroups of Sylow subgroups c-supplemented, *Journal of Algebra* **224** (2000), 467–478.

[6] B. Huppert and N. Blackburn, Finite Groups III, *Springer-Verlag*, *Berlin, New York*, 1982.

[7] K. Doerk and T. O. Hawkes, Finite Soluble Groups, *De Gruyter*, *Berlin*, 1992.

[8] Y. Wang, The influence of minimal subgroups on the structure of finite groups, *Acta Math. Sinica,* English series **16** (2000), 63–70.

[9] M. Weinstein (ed.), Between nilpotent and solvable, *Passaic*, 1982.

[10] A. Ballester-Bolinches, Y. Wang and G. Xiuyun, C-supplemented subgroups of finite groups, *Glasgow Math. J.* **42** (2000), 383–389.

[11] J. Buckley, Finite groups whose minimal subgroups are normal, *Math. Z.* **116** (1970), 15–17.

[12] A. Ballester-Bolinches and M. C. Pedraza Aguilera, On minimal subgroups of finite groups, *Acta Math. Hungar.* **73** (1996), 335–342.

[13] A. Ballester-Bolinches, $\mathcal{H}$-normalizers and local definitions of saturated formations of finite groups, *Israel J. Math.* **67** (1989), 312–326.

[14] A. Ballester-Bolinches and Y. Wang, Finite groups with c-normal minimal subgroups, *Journal of Pure and Applied Algebra* **153** (2000), 121–127.

[15] Y. Wang and Y. Li, Finite groups with c-supplemented minimal subgroups, *Algebra Colloquium, (accepted).*

YANMING WANG
LINGNAN COLLEGE AND SCHOOL OF MATHEMATICS
ZHONGSHAN UNIVERSITY
GUANGZHOU, 510275
CHINA

E-mail: stswym@zsu.edu.cn

YANGMING LI
DEPARTMENT OF MATHEMATICS
GUANGDONG COLLEGE OF EDUCATION
GUANGZHOU, 510303
CHINA

E-mail: liyangming@gdei.edu.cn

(Received May 7, 2002; revised July 2, 2003)