Publ. Math. Debrecen
64/3-4 (2004), 285–298

On collapsing iteration semigroups of set-valued functions

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Abstract. We introduce the notion of collapsing iteration semigroup of set-valued functions and study conditions under which a certain family of set-valued functions, naturally occuring in iteration theory, is such a semigroup.

Introduction

Given a set X a function $F : (0,\infty) \times X \to 2^X$ is said to be *set-valued iteration semigroup* if

$$F(s+t,x) = F\big(t, F(s,x)\big) \quad \text{for } x \in X \quad \text{and } s,t \in (0,\infty)$$

(here and in the sequel we write $F(t,A)$ for the image $F(\{t\}\times A)$ of $\{t\}\times A$; see also Section 2). This notion was introduced and studied under various assumptions by A. Smajdor in [2]. In the present paper we propose a more general notion of *collapsing iteration semigroup* postulating that $F : (0,\infty) \times X \to 2^X$ satisfies the condition

$$F(s+t,x) \subset F\big(t, F(s,x)\big) \quad \text{for } x \in X \quad \text{and } s,t \in (0,\infty).$$

Both above definitions take pattern by the classical notion of iteration semigroup intensively studied, among others, by M. C. Zdun (see, for

Mathematics Subject Classification: Primary 39B12, 39B52, 26E25.
Key words and phrases: iteration theory, set-valued function, collapsing iteration semigroup.

instance, [3]). It is well-known and easy to check that if α is a bijection mapping X onto $\mathbb{R}$ then the function $f : (0,\infty) \times X \to X$, given by

$$f(t,x) = \alpha^{-1}\bigl(\alpha(x) + t\bigr),$$

is an iteration semigroup:

$$f(s+t,x) = f\bigl(t, f(s,x)\bigr) \quad \text{for } x \in X \quad \text{and } s,t \in (0,\infty).$$

More generally we have the following observation.

Remark 1. Let α be a bijection mapping a set X onto an interval $I \subset \mathbb{R}$ with the right endpoint $q \in (-\infty, +\infty]$. Assume that $q \in I$ whenever q is finite. Then the function $f : (0,\infty) \times X \to X$, given by

$$f(t,x) = \alpha^{-1}\bigl(\min\{\alpha(x) + t, q\}\bigr), \tag{1}$$

is an iteration semigroup.

Proof. Fix $s \in (0,\infty)$ and $x \in X$. If $\alpha(x) + s < q$ then

$$\min\bigl\{\min\{\alpha(x)+s, q\} + t, q\bigr\} = \min\{\alpha(x) + s + t, q\}$$

and if $\alpha(x) + s \geqslant q$ then

$$\begin{aligned}\min\bigl\{\min\{\alpha(x)+s, q\} + t, q\bigr\} &= \min\{q + t, q\} = q \\ &= \min\{\alpha(x) + s + t, q\}\end{aligned}$$

for every $t \in (0,\infty)$. Then

$$\begin{aligned} f(s+t,x) &= \alpha^{-1}\bigl(\min\{\alpha(x) + s + t, q\}\bigr) \\ &= \alpha^{-1}\bigl(\min\bigl\{\min\{\alpha(x)+s, q\} + t, q\bigr\}\bigr) \\ &= \alpha^{-1}\bigl(\min\left\{\alpha\left(\alpha^{-1}\left(\min\left\{\alpha(x)+s, q\right\}\right)\right) + t, q\right\}\bigr) \\ &= \alpha^{-1}\bigl(\min\{\alpha(f(s,x)) + t, q\}\bigr) = f\bigl(t, f(s,x)\bigr) \end{aligned}$$

for every $t \in (0,\infty)$ which gives the desired equality. □

It seems that for the first time this observation was made by M. C. Zdun (cf. [3, Theorems 5.1–8.1]). As he proved there, (1) with homeomorfic α is a general form of the so called continuous iteration semigroups on an interval (cf. also [1, Theorem 1]).

The aim of the present paper is to introduce a set-valued counterpart of (1) and to find conditions under which such set-valued functions are collapsing iteration semigroups.

1. Preliminaries

In what follows, given sets X, Y and a set-valued function $F : X \to 2^Y$, we define images and preimages by F putting

$$F(U) := \bigcup_{x \in U} F(x)$$

for every $U \subset X$ and

$$F^{-1}(V) := \{x \in X : F(x) \cap V \neq \emptyset\}$$

for every $V \subset Y$.

Fix a set X and a set-valued function $A : X \to 2^{\mathbb{R}}$ with non-empty values. Put

$$S := A(X) \quad \text{and} \quad q := \sup S.$$

Given $x, y \in X$ and $t \in (0, \infty)$ we say that $A(y)$ is *t-attainable from* $A(x)$ if

$$[A(x) + t] \cap A(y) \neq \emptyset.$$

If $x \in X$, $t \in (0, \infty)$ and

$$\inf A(x) + t \geqslant q$$

$A(x)$ is called *t-coming out.*

Throughout this paper we will always assume that

(H) for every $s, t \in (0, \infty)$ and $x, z \in X$ such that $A(z)$ is $(s + t)$-attainable from $A(x)$ there exists a $y \in X$ such that $A(y)$ is s-attainable from $A(x)$ and $A(z)$ is t-attainable from $A(y)$.

Proposition 1. (i) *If S is an interval then* (H) *holds.*

(ii) *Assume that all values of A are open sets. If* $(\inf S, \sup S) \subset \operatorname{cl} S$ *then* (H) *holds.*

(iii) *Assume that all values of A are intervals. If* (H) *holds then* $(\inf S, \sup S) \subset \operatorname{cl} S$.

Proof. (i) Assume that S is an interval. Fix $s, t \in (0, \infty)$ and $x, z \in X$ such that $A(z)$ is $(s + t)$-attainable from $A(x)$, that is

$$[A(x) + s] \cap [A(z) - t] \neq \emptyset.$$

Therefore there exists a real number u such that

$$u \in A(x) + s \quad \text{and} \quad u \in A(z) - t, \tag{2}$$

whence

$$u - s \in A(x) \subset S \quad \text{and} \quad u + t \in A(z) \subset S.$$

Since S is an interval and $u - s < u < u + t$ we have $u \in S$. Then there exists a $y \in X$ such that $u \in A(y)$. Hence and by (2) we get

$$A(y) \cap [A(x) + s] \neq \emptyset$$

and

$$[A(y) + t] \cap A(z) \neq \emptyset.$$

(ii) Now assume that $(\inf S, \sup S) \subset \operatorname{cl} S$. Fix $s, t \in (0, \infty)$ and $x, z \in X$ such that

$$[A(x) + s + t] \cap A(z) \neq \emptyset.$$

Then $[A(x) + s] \cap [A(z) - t]$ is a non-empty open subset of $(\inf S, \sup S)$. Since the latter is contained in $\operatorname{cl} S$ this means that

$$[A(x) + s] \cap [A(z) - t] \cap S \neq \emptyset$$

which completes the proof of (ii).

(iii) Assume that (H) holds. Suppose that $(\inf S, \sup S) \not\subset \operatorname{cl} S$. Then there exists $v \in (\inf S, \sup S)$ and an open interval G such that $v \in G$ and

$$G \cap S = \emptyset. \tag{3}$$

Since $\inf S < v < \sup S$ and all values of A are intervals there exist $x, z \in X$ such that

$$u < v \quad \text{for } u \in A(x) \quad \text{and} \quad v < w \quad \text{for } w \in A(z).$$

We can find $s, t \in (0, \infty)$ such that

$$\emptyset \neq [A(x) + s] \cap [A(z) - t] \subset G. \tag{4}$$

Obviously

$$[A(x) + s + t] \cap A(z) \neq \emptyset.$$

Then, by (H), there is a $y \in X$ with

$$[A(x)+s] \cap A(y) \neq \emptyset \quad \text{and} \quad [A(z)-t] \cap A(y) \neq \emptyset. \tag{5}$$

Since $A(x)+s$, $A(z)-t$ and $A(y)$ are intervals, by (4) and (5), we have

$$\big[[A(x)+s] \cap [A(z)-t]\big] \cap A(y) \neq \emptyset.$$

Then, by (4), $G \cap S \neq \emptyset$ which contradicts (3). □

The examples below show that none of the implications in Proposition 1 can be converted; also the assumptions made in (ii) and (iii) turn out to be essential.

Example 1. Let X be an arbitrary set and $A : X \to 2^{\mathbb{R}}$ be defined by

$$A(x) = (0,2) \cup (3,5).$$

Obviously $S = (0,2) \cup (3,5)$ and

$$(\inf S, \sup S) = (0,5) \not\subset \operatorname{cl} S.$$

We will show that (H) holds. Notice that

$$\big\{t \in (0,\infty) : [A(x_1)+t] \cap A(x_2) \neq \emptyset\big\} = (0,5)$$

for every $x_1, x_2 \in X$. Fix $x, z \in X$ and let $s, t \in (0,\infty)$ be such that

$$[A(x)+s+t] \cap A(z) \neq \emptyset.$$

Then $s+t \in (0,5)$ whence $s, t \in (0,5)$. Therefore, taking any $y \in X$, we have $[A(x)+s] \cap A(y) \neq \emptyset$ and $A(y) \cap [A(z)-t] \neq \emptyset$.

Example 2. Let $X = \{1,2,3\}$ and $A : X \to 2^{\mathbb{R}}$ be defined by

$$A(x) = \begin{cases} (0,2) & \text{for } x = 1, \\ (2,3) & \text{for } x = 2, \\ \{3\} & \text{for } x = 3. \end{cases}$$

Obviously $S = (0,2) \cup (2,3]$ and $(\inf S, \sup S) = (0,3) \subset \operatorname{cl} S$. Notice that $[A(1)+2] \cap A(3) \neq \emptyset$ but $A(3)-1 = \{2\}$ and $2 \notin S$. Then $[A(3)-1] \cap S = \emptyset$ and, consequently, (H) does not hold.

Corollary 1. (i) *Assume that A is single-valued. Then* (H) *holds if and only if S is an interval.*

(ii) *Assume that all values of A are open intervals. Then* (H) *holds if and only if* $(\inf S, \sup S) \subset \operatorname{cl} S$.

Proof. (i) Assume (H) and take $u, w \in S$ such that $u < w$. Therefore there exist points $x, z \in X$ such that $A(x) = \{u\}$, $A(z) = \{w\}$. Fix a $v \in (u, w)$ and put

$$s := v - u, \qquad t := w - v.$$

Obviously $A(x) + s + t = A(z)$, i.e. $A(z)$ is $(s+t)$-attainable from $A(x)$. Then, by (H), there exists a $y \in X$ such that $[A(x) + s] \cap A(y) \neq \emptyset$, i.e. $A(x) + s = A(y)$. On the other hand, $A(x) + s = \{v\}$. Thus $A(y) = \{v\}$ and, consequently, $v \in S$. This means that S is an interval and by Proposition 2(i) completes the proof of (i).

The second assertion follows immediately from Proposition 2(ii) and (iii). □

For every $x \in X$ define

$$\tau(x) := \sup\bigl\{t \in [0, \infty) : [A(x) + t] \cap S \neq \emptyset\bigr\}.$$

Theorem 1. *Let $x \in X$. If $t < \tau(x)$ then $[A(x) + t] \cap S \neq \emptyset$ and if $t > \tau(x)$ then $[A(x) + t] \cap S = \emptyset$ for every $t \in (0, \infty)$.*

Proof. To prove the first claim it suffices to show that if for some $t \in \bigl(0, \tau(x)\bigr)$ condition $[A(x) + t] \cap S \neq \emptyset$ holds, then $[A(x) + s] \cap S \neq \emptyset$ for every $s \in (0, t)$. To this aim let $t \in \bigl(0, \tau(x)\bigr)$ satisfy $[A(x) + t] \cap S \neq \emptyset$ and fix an $s \in (0, t)$. Then there exists a $z \in X$ such that

$$[A(x) + t] \cap A(z) \neq \emptyset.$$

Let $u := t - s$. Then $u \in (0, \infty)$ and

$$[A(x) + s + u] \cap A(z) \neq \emptyset.$$

By (H) there exists a $y \in X$ such that

$$[A(x) + s] \cap A(y) \neq \emptyset.$$

Thus we have shown that

$$[A(x)+s]\cap S\neq\emptyset,$$

which completes the proof in this case.

The second assertion follows directly from the definition of $\tau(x)$. □

Lemma 1. *For every* $x\in X$

$$\tau(x)=q-\inf A(x).$$

Proof. Fix an $x\in X$ and suppose that $\tau(x)<q-\inf A(x)$. Then $\tau(x)\neq\infty$ and $\tau(x)+\inf A(x)<q$ whence there exist s,t such that $s\in A(x)+\tau(x)$, $t\in S$ and $s<t$. Thus

$$t=s+(t-s)\in A(x)+\big(\tau(x)+(t-s)\big),$$

contrary to Theorem 1.

Now suppose that $\tau(x)>q-\inf A(x)$ for an $x\in X$. Therefore $\inf A(x)\neq-\infty$ and there exists a u such that $q<u<\inf A(x)+\tau(x)$. We can find an $s\in\big(0,\tau(x)\big)$ such that $u=\inf A(x)+s$. Obviously $u\notin S$. Then for every $v\in A(x)+s$ we have $v\geqslant u$ and, consequently, $v\notin S$ which contradicts Theorem 1. □

Corollary 2. *For every* $x\in X$

$$\big[A(x)+\tau(x)\big]\cap S\subset\{q\}.$$

Proof. By Lemma 1 and definition of the number q we have

$$\big[A(x)+\tau(x)\big]\cap S=\big[A(x)+q-\inf A(x)\big]\cap S\subset[q,+\infty]\cap(-\infty,q]=\{q\}$$

for every $x\in X$. □

Let $e:(0,\infty)\times X\to[0,\infty)$ be a function defined by

$$e(t,x):=\sup\{s\in[0,t]:[A(x)+s]\cap S\neq\emptyset\}.$$

Lemma 2. *For every* $t\in(0,\infty)$ *and* $x\in X$

$$e(t,x)=\min\{t,\tau(x)\}.$$

Proof. If $t \leqslant \tau(x)$ then, by Theorem 1,

$$[A(x)+s] \cap S \neq \emptyset$$

for every $s \in [0,t)$ whence

$$e(t,x) = \sup\{s \in [0,t] : [A(x)+s] \cap S \neq \emptyset\} = t = \min\{t, \tau(x)\}.$$

If $t > \tau(x)$ then, again by Theorem 1, we have

$$\begin{aligned} e(t,x) &= \sup\big\{s \in [0,t] : [A(x)+s] \cap S \neq \emptyset\big\} \\ &= \sup\big\{s \in [0,\tau(x)] : [A(x)+s] \cap S \neq \emptyset\big\} = \tau(x) = \min\{t, \tau(x)\}. \end{aligned}$$

□

2. Collapsing iteration semigroups

In what follows, given a set-valued function $F : (0,\infty) \times X \to 2^X$, we put

$$F(t,U) := F\big(\{t\} \times U\big) = \bigcup_{x \in U} F(t,x)$$

whenever $U \subset X$; moreover, we will write $F^t(x)$ and $F^t(U)$ instead of $F(t,x)$ and $F(t,U)$, respectively.

Remark 2. If $F : (0,\infty) \times X \to 2^X$ is a single-valued collapsing iteration semigroup then

$$F(s+t,x) = F\big(t, F(s,x)\big)$$

for every $s,t \in (0,\infty)$ and $x \in X$, that is F generates a classical iteration semigroup.

Remark 3. If S is an interval, $q \in S$ whenever q is finite and A is a single-valued bijection mapping X onto S then

$$A^{-1}\big(A(x) + e(t,x)\big) = A^{-1}\big(\min\{A(x)+t, q\}\big)$$

for every $t \in (0,\infty)$ and $x \in X$.

Proof. Assume that S is a interval and $A : X \to S$ is a single-valued bijection. Fix $t \in (0,\infty)$ and $x \in X$. Then, by Lemmas 2 and 1 and the equality $\inf\{A(x)\} = A(x)$, we have

$$\begin{aligned} A(x) + e(t,x) &= A(x) + \min\{t, \tau(x)\} \\ &= \min\big\{A(x) + t, A(x) + \tau(x)\big\} \\ &= \min\big\{A(x) + t, A(x) + q - \inf\{A(x)\}\big\} \\ &= \min\{A(x) + t, q\} \end{aligned}$$

which gives the desired property. □

Now we pass to the problem of finding conditions under which a set-valued function of the form

$$F(t,x) := A^{-1}\big(A(x) + e(t,x)\big) \tag{6}$$

is a collapsing iteration semigroup.

Lemma 3. *Let $F : (0,\infty)\times X \to 2^X$ be given by* (6) *and let $t \in (0,\infty)$ and $x \in X$. If $t < \tau(x)$ then*

$$F(t,x) = A^{-1}\big(A(x) + t\big) \neq \emptyset$$

and if $t \geqslant \tau(x)$ then

$$F(t,x) = \begin{cases} A^{-1}(\{q\}), & \text{if } q \in S \text{ and } \inf A(x) \in A(x); \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. The first assertion follows directly from Lemma 2 and Theorem 1. So assume that $t \geqslant \tau(x)$. Again by Lemma 2 we have

$$F^t(x) = A^{-1}\big(A(x) + \tau(x)\big) = A^{-1}\big((A(x) + \tau(x)) \cap S\big).$$

If $q \in S$ and $\inf A(x) \in A(x)$ then, by Lemma 1 and Corollary 2,

$$F^t(x) = A^{-1}(\{q\}).$$

If either $q \notin S$, or $\inf A(x) \notin A(x)$ then, again by Lemma 1 and Corollary 2, we have

$$\big[A(x) + \tau(x)\big] \cap S = \emptyset$$

whence $F^t(x) = \emptyset$. □

Proposition 2. *Let* $F : (0,\infty) \times X \to 2^X$ *be given by* (6), $x \in X$, *and* $s,t \in (0,\infty)$. *If* $s+t \leqslant \tau(x)$ *then*

$$F^{s+t}(x) \subset F^t\big(F^s(x)\big). \tag{7}$$

Proof. Assume that $s+t \leqslant \tau(x)$. To show the inclusion (7) fix a $z \in F^{s+t}(x)$. By (6) and Lemma 2 we have

$$z \in F^{s+t}(x) = A^{-1}\big(A(x)+(s+t)\big)$$

that is

$$A(z) \cap \big[A(x)+(s+t)\big] \neq \emptyset.$$

Thus, by virtue of (H), there exists a $y \in X$ such that

$$A(z) \cap [A(y)+t] \neq \emptyset \quad \text{and} \quad A(y) \cap [A(x)+s] \neq \emptyset.$$

According to Theorem 1 we have $t \leqslant \tau(y)$, whence

$$z \in A^{-1}\big(A(y)+t\big) = F^t(y) \quad \text{and} \quad y \in A^{-1}\big(A(x)+s\big) = F^s(x)$$

that is $z \in F^t\big(F^s(x)\big)$. □

Now we are in position to formulate the main result of the paper.

Theorem 2. *Let* $F : (0,\infty) \times X \to 2^X$ *be given by* (6). *Then* F *is a collapsing iteration semigroup if and only if either* $q \notin S$, *or, for every* $x \in X$ *and* $s,t \in (0,\infty)$ *such that* $s+t > \tau(x)$ *and* $A(x)$ *has the smallest element, at least one of the following conditions holds:*

(i) *there exists a* $y \in F(s,x)$ *such that* $A(y)$ *is* t*-coming out and has the smallest element,*

(ii) *for every* $z \in A^{-1}(\{q\})$ *there exists a* $y \in F(s,x)$ *such that* $A(z)$ *is* t*-attainable from* $A(y)$.

Before proving Theorem 2 we will derive the following consequence of it.

Theorem 3. *Let* $F : (0,\infty) \times X \to 2^X$ *be given by* (6). *Every of the following conditions is sufficient for* F *to be a collapsing iteration semigroup:*

(i) $q \notin S$; (ii) $q = \infty$;

(iii) *no value of A has the smallest element;*

(iv) *A is single-valued.*

Proof. It follows immediately from Theorem 2 that each of the conditions (i)–(iii) is sufficient for F to be a collapsing iteration semigroup.

Assume that A is single-valued and $q \in S$. Fix $s, t \in (0, \infty)$ and $x \in X$ such that $s + t > \tau(x)$. Then, by Lemma 1, we have

$$s + t > q - \inf A(x).$$

Of course $\inf A(x) \neq -\infty$ and

$$\inf A(x) + s + t > q. \tag{8}$$

First assume that $s < \tau(x)$. Thus, on account of Lemma 3,

$$F^s(x) = A^{-1}(A(x) + s) \neq \emptyset.$$

Fix a $y \in F^s(x)$. Then $A(y) \cap [A(x) + s)] \neq \emptyset$, that is $A(x) + s = A(y)$ whence, by (8), $\inf A(y) + t > q$ which means that $A(y)$ is t-coming out.

Now consider the case $s \geqslant \tau(x)$. Then, by Lemma 3 and the assumptions,

$$F^s(x) = A^{-1}(\{q\}) \neq \emptyset.$$

Taking any $y \in F^s(x)$ we have $A(y) = \{q\}$, whence

$$\inf A(y) + t = q + t > q.$$

To complete the proof it is enough to use Theorem 2. □

Proof of Theorem 2. Assume that F is a collapsing iteration semigroup and $q \in S$. Fix $x \in X$ and $s, t \in (0, \infty)$ such that

$$s + t > \tau(x) \quad \text{and} \quad \inf A(x) \in A(x).$$

Obviously $A^{-1}(\{q\}) \neq \emptyset$, so we can fix a $z \in A^{-1}(\{q\})$. By Lemma 3

$$F^{s+t}(x) = A^{-1}(\{q\}).$$

Thus $z \in F^{s+t}(x)$ and, consequently, there exists a $y \in F^s(x)$ such that

$$z \in F^t(y). \tag{9}$$

At first assume that $t \geqslant \tau(y)$. Then, by Lemma 1,

$$\inf A(y) + t \geqslant q.$$

Since $F^t(y) \neq \emptyset$ it follows from Lemma 3 that

$$F^t(y) = A^{-1}(\{q\}) \quad \text{and} \quad \inf A(y) \in A(y).$$

In such a way we have come to condition (i).

If $t < \tau(y)$ then, by Lemma 3,

$$F^t(y) = A^{-1}(A(y) + t)$$

whence, by (9),

$$A(z) \cap [A(y) + t] \neq \emptyset$$

and, consequently, (ii) holds true.

To prove the converse fix $x \in X$ and $s, t \in (0, \infty)$. If $s + t \leqslant \tau(x)$ the assertion follows from Proposition 2 so assume that $s + t > \tau(x)$. If $q \notin S$ then, by Lemma 3, $F^{s+t}(x) = \emptyset$ and (7) holds true. Thus we can assume that $q \in S$. To prove (7) fix a $z \in F^{s+t}(x)$. Since $F^{s+t}(x) \neq \emptyset$ we have, by Lemma 3,

$$F^{s+t}(x) = A^{-1}(\{q\}) \quad \text{and} \quad \inf A(x) \in A(x).$$

In particular,

$$z \in A^{-1}(\{q\}). \tag{10}$$

Assume (i). Then there exists a $y \in F^s(x)$ such that

$$\inf A(y) + t \geqslant q \quad \text{and} \quad \inf A(y) \in A(y).$$

Thus, according to Lemma 1, $\tau(y) = q - \inf A(y) \leqslant t$. Consequently, it follows from Lemma 3 that $F^t(y) = A^{-1}(\{q\})$ whence, by (10), we have $z \in F^t(y) \subset F^t(F^s(x))$.

Finally assume (ii). Then $[A(y) + t] \cap A(z) \neq \emptyset$ for a $y \in F^s(x)$, i.e.

$$z \in A^{-1}\big(A(y) + t\big). \tag{11}$$

In particular, the set $A^{-1}(A(y) + t)$ is non-empty, whence, by Theorem 1, $t \leqslant \tau(y)$ and, on account of (6) and Lemma 2,

$$A^{-1}\big(A(y) + t\big) = F^t(y).$$

Thus, by (11),

$$z \in F^t(y) \subset F^t\big(F^s(x)\big)$$

which completes the proof. □

Example 3. Let $X = [0,1]$ and $A : X \to 2^{\mathbb{R}}$ be defined by

$$A(x) := [x-1, x+1].$$

Obviously $S = [-1, 2]$ and $q = 2$. Then, by Proposition 1(i), A satisfies condition (H). According to Lemma 1

$$\tau(x) = 3 - x \tag{12}$$

for every $x \in X$. Let $F : (0,\infty) \times X \to 2^X$ be given by (6). We will find the explicite formula for F.

Fix $t \in (0,\infty)$ and $x \in X$. If $t < \tau(x)$ then, by Lemma 3,

$$\begin{aligned}
F^t(x) &= A^{-1}\big(A(x) + t\big) = A^{-1}\big([x-1, x+1] + t\big) \\
&= A^{-1}\big([x+t-1, x+t+1]\big) \\
&= \big\{y \in [0,1] : [y-1, y+1] \cap [x+t-1, x+t+1] \neq \emptyset\big\} \\
&= \big\{y \in [0,1] : x+t-1 \leqslant y+1 \text{ and } y-1 \leqslant x+t+1\big\} \\
&= [0,1] \cap \big[x+t-2, x+t+2\big] \\
&= \big[\max\{x+t-2, 0\}, 1\big].
\end{aligned}$$

If $t \geqslant \tau(x)$ then, again by Lemma 3,

$$F^t(x) = A^{-1}(\{2\}) = \{1\}.$$

Consequently, we have

$$F^t(x) = \begin{cases} [\max\{x+t-2, 0\}, 1], & \text{if } x+t < 3, \\ \{1\}, & \text{if } x+t \geqslant 3, \end{cases} \tag{13}$$

for every $t \in (0,\infty)$ and $x \in X$. Observe that

$$1 \in F^t(x) \tag{14}$$

for every $t \in (0,\infty)$ and $x \in X$.

Now fix $s,t \in (0,\infty)$ and $x \in X$. If $s+t > \tau(x)$ then, by (13) and (14), we have $F^{s+t}(x) = \{1\} \subset F^t\big(F^s(x)\big)$. Thus Proposition 2 implies that F is a collapsing iteration semigroup. Observe that F does not satisfy any of the assumptions (i)–(iv) of Theorem 3. Consequently, none of those conditions, which are sufficient for F to be a collapsing iteration semigroup, is necessary.

Observe also that $\tau(1) = 2$ and, by (13), $F^2(1) = \{1\}$ and $F^1(y) = [0,1]$ for $y \in [0,1]$. On the other hand, again by (13),

$$F^1\big(F^1(1)\big) = \bigcup_{y\in[0,1]} F^1(y) = \bigcup_{y\in[0,1]} [0,1] = [0,1].$$

Consequently, $F^1\big(F^1(1)\big) \not\subset F^2(1)$. This shows that F does not satisfy the condition

$$F^t\big(F^s(x)\big) \subset F^{s+t}(x) \quad \text{for } s,t \in (0,\infty) \quad \text{and } x \in X,$$

which could serve as a definition of expanding iteration semigroup.

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(Received September 9, 2002; revised July 7, 2003)