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Submanifolds of special Finsler manifolds

By ALY A. TAMIM (Giza)

Abstract. In this paper we study various properties of totally umbilic submanifolds of a Finsler manifold. We also investigate Finsler submanifolds in the case where the ambient manifold is of scalar curvature, partially isotropic, Landsberg, semi-C-reducible or $S4$-like manifold.

Introduction

It is well known that the theory of Finsler submanifolds have not yet been studied in depth because of the tremendous computations involved and the lack of symmetry of the horizontal second fundamental form. In spite of these obstacles, several authors have made important contributions to this subject from various points of view (see e.g. [1], [2], [6]). The most natural idea is to study the induced and intrinsic connections and establish some tensor equations relating the properties of the embedded manifold to those of the ambient manifold.

In this paper, we try to overcome the above mentioned obstacles and continue developing our foregoing studies of Finsler submanifolds ([11], [13], [15], [16]) in such a way that we can extend some well-known results on Riemannian submanifolds to the Finslerian case and also find new results particular to Finsler submanifolds.

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Special attention is paid to the interesting class of totally h-umbilic submanifolds. Finsler submanifolds of Landsberg, semi-C-reducible and $S4$-like manifolds are also investigated.

1. Basic concepts and definitions

Let V be a differentiable manifold of dimension $n \geq 2$ and of class C^∞. We will denote by $\pi_V : TV \longrightarrow V$ (resp. $\pi : \mathcal{T}V \longrightarrow V$) the tangent bundle of V (resp. the subbundle of nonzero vectors tangent to V). Let $\mathfrak{X}(V)$ be the set of all smooth vector fields on V and $\mathfrak{F}(V)$ the ring of all real-valued smooth functions on V. Let $\pi^{-1}(TV) \longrightarrow \mathcal{T}V$ be the pullback bundle associated with π and TV. Sections of $\pi^{-1}(TV)$ are called π-vector fields and will be denoted by barred symbols, $\mathfrak{X}(\pi(V))$ is the $\mathfrak{F}(\mathcal{T}V)$-module of π-vector fields. The canonical vector field is the π-vector field ϑ defined by $\vartheta(u) = (u, u)$ for all $u \in \mathcal{T}V$. We have the bundle morphisms ρ and γ defined by $\rho = (\pi_{\mathcal{T}V}, d\pi)$ and $\gamma(u, v) = j_u(v)$ where j_u is the natural isomorphism between the tangent spaces $T_{\pi_V(v)}V$ and $T_u(T_{\pi_V(v)}V)$.

Let ∇ be a linear connection in $\pi^{-1}(TV)$. We associate to ∇ the map $K = \nabla\vartheta$. The connection ∇ is said to be regular [5] if $T_u(\mathcal{T}V) = V_u(\mathcal{T}V) \oplus H_u(\mathcal{T}V)$ for every $u \in \mathcal{T}V$, where $V_u(\mathcal{T}V)$ and $H_u(\mathcal{T}V) := \operatorname{Ker} K_u$ are respectively the vertical and horizontal subspaces at u. If V is endowed with a regular connection, we introduce another bundle map $\beta := (\rho|_{H(\mathcal{T}V)})^{-1}$.

Let $\mathbf{T}$ be the torsion of ∇. The horizontal and mixed torsion tensors A and T are given by $A(\bar{X}, \bar{Y}) = \mathbf{T}(\beta\bar{X}, \beta\bar{Y})$, $T(\bar{X}, \bar{Y}) = \mathbf{T}(\gamma\bar{X}, \beta\bar{Y})$. If $\mathbf{R}$ is the curvature of ∇, then the horizontal, mixed and vertical curvature tensors R, P and S are given by $R(\bar{X}, \bar{Y})\bar{Z} = \mathbf{R}(\beta\bar{X}, \beta\bar{Y})\bar{Z}$, $P(\bar{X}, \bar{Y})\bar{Z} = \mathbf{R}(\gamma\bar{X}, \beta\bar{Y})\bar{Z}$, $S(\bar{X}, \bar{Y})\bar{Z} = \mathbf{R}(\gamma\bar{X}, \gamma\bar{Y})\bar{Z}$.

Let (V, L) be a Finsler manifold, where L is the fundamental function. Let g be the Finsler metric associated with L and ∇ be the Cartan connection determined by the metric g. The angular metric tensor h is defined by $h = g - \ell \otimes \ell$, where the π-form ℓ is given by $\ell(\bar{X}) = L^{-1}g(\bar{X}, \vartheta)$.

The tensor T induces a π-tensor field of type $(0, 3)$, denoted again by T, defined by $T(\bar{X}, \bar{Y}, \bar{Z}) = g(T(\bar{X}, \bar{Y}), \bar{Z})$ and also induces a π-form

C defined by $C(\bar{X}) :=$ trace of the map $\bar{Y} \longmapsto T(\bar{X}, \bar{Y})$. The horizontal and vertical Ricci tensors Ric^h and Ric^v are defined respectively by $\mathrm{Ric}^h(\bar{X}, \bar{Y}) :=$ trace of the map $\bar{Z} \longmapsto R(\bar{X}, \bar{Z})\bar{Y}$ and $\mathrm{Ric}^v(\bar{X}, \bar{Y}) :=$ trace of the map $\bar{Z} \longmapsto S(\bar{X}, \bar{Z})\bar{Y}$. The horizontal and vertical Ricci maps $\mathrm{Ric}^h{}_o$ and $\mathrm{Ric}^v{}_o$ are defined respectively by $\mathrm{Ric}^h(\bar{X}, \bar{Y}) = g(\mathrm{Ric}^h{}_o(\bar{X}), \bar{Y})$ and $\mathrm{Ric}^v(\bar{X}, \bar{Y}) = g(\mathrm{Ric}^v{}_o(\bar{X}), \bar{Y})$. The horizontal and vertical Ricci scalar curvatures Sc^h and Sc^v are defined respectively by $\mathrm{Sc}^h :=$ trace of the map $\bar{X} \longmapsto \mathrm{Ric}^h{}_o(\bar{X})$ and $\mathrm{Sc}^v :=$ trace of the map $\bar{X} \longmapsto \mathrm{Ric}^v{}_o(\bar{X})$.

Let $u \in \mathcal{T}V$ and let π be a 2-plane in $\pi^{-1}(TV)$ spanned by two orthogonal unit π-vectors $\bar{X}$, $\bar{Y}$. The sectional curvature of π in V is defined by

$$\kappa(\pi) = g(R(\bar{X}, \bar{Y})\bar{X}, \bar{Y}).$$

Definition 1. A Finsler manifold (V, L) endowed with the Cartan connection ∇ is said to be:

(a) a Landsberg manifold, if $P(\bar{X}, \bar{Y})\vartheta = 0$ for all $\bar{X}, \bar{Y} \in \mathfrak{X}(\pi(V))$;

(b) a Berwald manifold, if $\nabla_{\beta\bar{X}} T = 0$ for all $\bar{X} \in \mathfrak{X}(\pi(V))$;

(c) a locally Minkowski manifold, if it is a Berwald manifold of vanishing horizontal curvature.

Definition 2. A Finsler manifold (V, L) endowed with the Cartan connection ∇ is said to be:

(a) a semi-C-reducible manifold if dim $V > 2$ and if the mixed torsion T of ∇ has the form $T(\bar{X}, \bar{Y}, \bar{Z}) = \frac{\sigma}{(n+1)} \mathfrak{S}_{\bar{X},\bar{Y},\bar{Z}}(h \otimes C)(\bar{X}, \bar{Y}, \bar{Z}) + {}+\frac{\tau}{c^2}\, C(\bar{X})C(\bar{Y})C(\bar{Z})$, where $c^2 := g(C, C) \neq 0$, σ and τ are scalars such that $\sigma + \tau = 1$ whereas $\mathfrak{S}_{\bar{X},\bar{Y},\bar{Z}}$ denotes the cyclic sum over $\bar{X}, \bar{Y}, \bar{Z}$;

(b) an $S4$-like manifold if dim $V > 4$ and if the vertical curvature S of ∇ has the form

$$\begin{aligned} S(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = {} & \frac{\mathrm{Sc}^v}{(n-2)(n-3)}\{h(\bar{X}, \bar{W})h(\bar{Y}, \bar{Z}) - h(\bar{X}, \bar{Z})h(\bar{Y}, \bar{W})\} \\ & + \frac{1}{(n-3)}\{h(\bar{Y}, \bar{W})\,\mathrm{Ric}^v(\bar{X}, \bar{Z}) - h(\bar{X}, \bar{W})\,\mathrm{Ric}^v(\bar{Y}, \bar{Z}) \\ & + h(\bar{X}, \bar{Z})\,\mathrm{Ric}^v(\bar{Y}, \bar{W}) - h(\bar{Y}, \bar{Z})\,\mathrm{Ric}^v(\bar{X}, \bar{W})\}. \end{aligned}$$

Definition 3. A Finsler manifold (V, L) endowed with the Cartan connection ∇ is said to be:

(a) a partially isotropic manifold if there exists a scalar κ such that the horizontal curvature R of ∇ has the form

$$R(\bar{X}, \bar{Y})\bar{Z} = \kappa[g(\bar{X}, \bar{Z})\bar{Y} - g(\bar{Y}, \bar{Z})\bar{X}] \quad \text{for all } \bar{X}, \bar{Y}, \bar{Z} \in \mathfrak{X}(\pi(V));$$

(b) a manifold of scalar curvature if there exists a function $K : \mathcal{T}V \longrightarrow \mathbb{R}$ such that

$$R(\vartheta, \bar{X}, \vartheta, \bar{Y}) = KL^2 h(\bar{X}, \bar{Y}) \quad \text{for all } \bar{X}, \bar{Y} \in \mathfrak{X}(\pi(V));$$

(c) a manifold of constant curvature if the function K in (b) is constant.

For a systematic study of the introduced special Finsler manifolds we refer to [3], [7], [8], [12] and [14].

Let $(V, {}^*L)$ be an n-dimensional Finsler manifold and (M, L) an m-dimensional Finsler submanifold of $(V, {}^*L)$, where $n = m + p$, $p \geq 1$. Entities of $(V, {}^*L)$ will be marked by an asterisk "*". If ${}^*g$ and g are respectively the Finsler metrics associated to ${}^*L$ and L then $g = {}^*g|_{\pi^{-1}(TM)}$. Let $\mathcal{C}(\mathcal{N})$ be the set of all differentiable sections of the normal vector bundle $\mathcal{N}$. Elements of $\mathcal{C}(\mathcal{N})$ will be called π-normal vector fields.

The Cartan connection ${}^*\nabla$ in $\pi^{-1}(TV)$ and the induced connection ∇ in $\pi^{-1}(TM)$ are related by the Gauss formula [6]

$${}^*\nabla_X \bar{Y} = \nabla_X \bar{Y} + \tilde{H}(X, \bar{Y}); \quad X \in \mathfrak{X}(\mathcal{T}M),\ \bar{Y} \in \mathfrak{X}(\pi(M)), \tag{1}$$

where $\tilde{H}$ is the second fundamental form for the given immersion. $\tilde{H}$ gives rise to the so called horizontal and vertical second fundamental forms H and Q defined in [11]. The normal curvature vector N and the normal curvature N_o are defined respectively by $N(\bar{X}) = H(\bar{X}, \vartheta)$ and $N_o = N(\vartheta)$. The normal connection $\nabla^{\perp}$ in the normal vector bundle $\mathcal{N}$ is related to ${}^*\nabla$ by the Weingarten formula

$${}^*\nabla_X \xi = -\tilde{B}_\xi X + \nabla^{\perp}{}_X \xi; \quad \xi \in \mathcal{C}(\mathcal{N}) \tag{2}$$

where $\tilde{B}_\xi$ is the Weingarten operator associated to ξ. We have

$${}^*g(\tilde{H}(X, \bar{Y}), \xi) = g(\tilde{B}_\xi X, \bar{Y}); \quad X \in \mathfrak{X}(\mathcal{T}M),\ \bar{Y} \in \mathfrak{X}(\pi(M)). \tag{3}$$

Notice that the normal connection $\nabla^{\perp}$ is $g^{\perp}$-metric, where $g^{\perp} = {}^*g|_{\mathcal{N}}$. The Weingarten operator $\tilde{B}_\xi$ gives rise to the so called horizontal and vertical Weingarten operators B_ξ and W_ξ defined respectively by $B_\xi = \tilde{B}_\xi \circ \beta$ and $W_\xi = \tilde{B}_\xi \circ \gamma$. Using equation (1) and the fact that $\beta\bar{X} = {}^*\beta\bar{X} + \gamma N(\bar{X})$, we deduce that

$$\left.\begin{aligned} {}^*T(\bar{X},\bar{Y}) &= T(\bar{X},\bar{Y}) + Q(\bar{X},\bar{Y}),\\ {}^*T(N(\bar{X}),\bar{Y}) - {}^*T(N(\bar{Y}),\bar{X}) &= A(\bar{X},\bar{Y}) + H(\bar{X},\bar{Y}) - H(\bar{Y},\bar{X}).\end{aligned}\right\} \quad (4)$$

Definition 4. For a Finsler submanifold (M, L) of a Finsler manifold, the normal connection $\nabla^{\perp}$ of M is said to be h-flat, hv-flat, v-flat, if $R^N = 0$, $P^N = 0$ and $S^N = 0$, respectively.

Definition 5. An m-dimensional Finsler submanifold (M,L) of a Finsler manifold is said to be:

(a) totally geodesic, if $N_o = 0$;

(b) totally h-umbilic, if $H = g \otimes \mu$, where $\mu := \frac{1}{m}$ trace H is the horizontal mean curvature;

(c) totally v-umbilic, if $Q = L^{-1}\ h \otimes \nu$, where $\nu := \frac{1}{m}$ trace Q is the vertical mean curvature;

(d) h-minimal, if $\mu = 0$;

(e) v-minimal, if $\nu = 0$.

A detailed study of Finsler submanifolds can be found in references [11], [13], [15] and [16].

2. Totally h-umbilic Finsler submanifolds

In this section we investigate some properties of totally h-umbilic submanifolds of a Finsler manifold.

Proposition 1. *For a totally h-umbilic Finsler submanifold M of a Finsler manifold V, we have $LA = \ell \wedge {}^*T(N_o, .)$.*

Proof. If the embedded manifold M is totally h-umbilic, then $H = g \otimes \mu$ and $N = L^{-1}\ell \otimes N_o$. Consequently equation (4) takes the form

$$LA(\bar{X},\bar{Y}) = \ell(\bar{X}){}^*T(N_o,\bar{Y}) - \ell(\bar{Y}){}^*T(N_o,\bar{X}), \quad \text{for all } \bar{X},\bar{Y} \in \mathfrak{X}(\pi(M)),$$

and the result then follows. $\square$

Theorem 1. *A necessary and sufficient condition for the coincidence of the induced and intrinsic connections of a totally h-umbilic Finsler submanifolds is that either the normal curvature N_o vanishes or ${}^*T(N_o,.)$ vanishes.*

Proof. It should firstly be noticed that the induced and intrinsic connections of M coincide if and only if the horizontal torsion tensor A vanishes [2]. By Proposition 1 we have $LA = \ell \wedge {}^*T(N_o,.)$. If either $N_o = 0$ or ${}^*T(N_o,.) = 0$, then the horizontal torsion tensor A vanishes.

Conversely, assume that the horizontal torsion tensor A vanishes. Proposition 1 implies that $\ell \wedge {}^*T(N_o,.) = 0$. Then there exists a function $f \in \mathfrak{F}(\mathcal{T}V)$ such that

$$ {}^*T(N_o,\bar{X}) = f\ell(\bar{X}) \quad \forall\ \bar{X} \in \mathfrak{X}(\pi(M)). \tag{5} $$

Setting $\bar{X} = \vartheta$ and noting that $\ell(\vartheta) = L$, we get ${}^*T(N_o,\vartheta) = 0 = fL$ so that $f = 0$. Consequently, equation (5) reduces to ${}^*T(N_o,.) = 0$. But ${}^*T(N_o,.) = 0$ if either $N_o = 0$ or ${}^*T(N_o,.) = 0$. $\square$

We recall from [11] the following lemma which will be used in the sequel.

Lemma 1. *For every $X, Y \in \mathfrak{X}(\mathcal{T}M)$ and $\bar{Z} \in \mathfrak{X}(\pi(M))$, we have*

$$ \begin{aligned} {}^*\mathbf{R}(X,Y)\bar{Z} = \mathbf{R}(X,Y)\bar{Z} &+ \tilde{B}_{\tilde{H}(Y,\bar{Z})}X - \tilde{B}_{\tilde{H}(X,\bar{Z})}Y + (\bar{\nabla}_Y H)(\rho X,\bar{Z}) \\ &- (\bar{\nabla}_X H)(\rho Y,\bar{Z}) + (\bar{\nabla}_Y Q)(K(X),\bar{Z}) - (\bar{\nabla}_X Q)(K(Y),\bar{Z}) \\ &- H(\mathbf{T}(X,Y),\bar{Z}) + Q(\mathbf{R}(X,Y)\vartheta,\bar{Z}). \end{aligned} $$

A generalization of Proposition 3.1 of [4] is given by

Theorem 2. *A totally h-umbilic submanifold M of a Finsler manifold V of scalar curvature is of scalar curvature.*

Proof. Applying Lemma 1 for $X = \beta\bar{X}$, $Y = \beta\bar{Y}$, we get

$$ \begin{aligned} &{}^*R(\bar{X},\bar{Y})\bar{Z} + {}^*P(N(\bar{X}),\bar{Y})\bar{Z} - {}^*P(N(\bar{Y}),\bar{X})\bar{Z} + {}^*S(N(\bar{X}),N(\bar{Y}))\bar{Z} \\ &\quad = R(\bar{X},\bar{Y})\bar{Z} + B_{H(\bar{Y},\bar{Z})}\bar{X} - B_{H(\bar{X},\bar{Z})}\bar{Y} + (\bar{\nabla}_{\beta\bar{Y}}H)(\bar{X},\bar{Z}) \\ &\quad\quad -(\bar{\nabla}_{\beta\bar{X}}H)(\bar{Y},\bar{Z}) - H(A(\bar{X},\bar{Y}),\bar{Z}) + Q(R(\bar{X},\bar{Y})\vartheta,\bar{Z}). \end{aligned} \tag{6} $$

Setting $\bar{X}=\bar{Z}=\vartheta$ in (6), using the properties of the curvature tensors ${}^*P$, ${}^*S$ and taking equation (3) into account, we deduce that

$$
\begin{aligned}
{}^*R(\vartheta,\bar{Y},\vartheta,\bar{W})+{}^*P(N_o,\bar{Y},\vartheta,\bar{W}) &= R(\vartheta,\bar{Y},\vartheta,\bar{W})+{}^*g(H(\vartheta,\bar{W}),N(\bar{Y}))\\
&\quad -{}^*g(N_o,H(\bar{Y},\bar{W})). \qquad (7)
\end{aligned}
$$

As the embedded manifold M is totally h-umbilic and ${}^*\nabla_{{}^*\beta\vartheta}{}^*T(L^2\mu,\bar{Y})$ is a normal π-vector field, equation (7) leads to

$$R(\vartheta,\bar{Y},\vartheta,\bar{W})={}^*R(\vartheta,\bar{Y},\vartheta,\bar{W})+\|\mu\|^2L^2h(\bar{Y},\bar{W}).$$

Since the ambient manifold V is of scalar curvature ${}^*K$, the horizontal curvature ${}^*R$ satisfies the condition ${}^*R(\vartheta,\bar{Y},\vartheta,\bar{W})={}^*K{}^*L^{2*}h(\bar{Y},\bar{W})$. Consequently, since $h={}^*h|_{\pi^{-1}(TM)}$ and $L={}^*L|_{\mathcal{T}M}$, we deduce that

$$R(\vartheta,\bar{Y},\vartheta,\bar{W})=L^2(K+\|\mu\|^2)h(\bar{Y},\bar{W}),$$

where we have put $K={}^*K|_{\mathcal{T}M}$. Hence M is of scalar curvature $K+\|\mu\|^2$. □

Corollary 1. *A totally h-umbilic submanifold of a Finsler manifold of constant curvature is of constant curvature if and only if μ is constant.*

Theorem 3. *A totally geodesic Finsler submanifold M of a Finsler manifold V of scalar curvature is of scalar curvature.*

Proof. Using the properties of the curvature tensors ${}^*R$, ${}^*P$, equation (6) takes the form

$$
\begin{aligned}
&{}^*R(\bar{X},\bar{Y},\bar{Z},\bar{W})+{}^*P(N(\bar{X}),\bar{Y},\bar{Z},\bar{W})-{}^*P(N(\bar{Y}),\bar{X},\bar{Z},\bar{W})\\
&\qquad +{}^*S(N(\bar{X}),N(\bar{Y}),\bar{Z},\bar{W})\\
&= R(\bar{X},\bar{Y},\bar{Z},\bar{W})+{}^*g(H(\bar{X},\bar{W}),H(\bar{Y},\bar{Z}))-{}^*g(H(\bar{Y},\bar{W}),H(\bar{X},\bar{Z})).
\end{aligned}
$$

The above equation for $\bar{X}=\bar{Z}=\vartheta$ leads to

$$
\begin{aligned}
&{}^*R(\vartheta,\bar{Y},\vartheta,\bar{W})+{}^*P(N_o,\bar{Y},\vartheta,\bar{W})\\
&= R(\vartheta,\bar{Y},\vartheta,\bar{W})+{}^*g(H(\vartheta,\bar{W}),N(\bar{Y}))-{}^*g(N_o,H(\bar{Y},\bar{W})).
\end{aligned}
$$

We recall [11] that $N_o=0$ if and only if $N=0$. Assume that M is totally geodesic ($N_o=0$). Then $N=0$, and the last equation reduces to

${}^*R(\vartheta,\bar{Y},\vartheta,\bar{W}) = R(\vartheta,\bar{Y},\vartheta,\bar{W})$. As the ambient manifold V is of scalar curvature, we conclude that $R(\vartheta,\bar{Y},\vartheta,\bar{W}) = KL^2h(\bar{Y},\bar{W})$, which means that M is of scalar curvature. This completes the proof. □

Theorem 4. *A totally h-umbilic submanifold M of a partially isotropic Landsberg manifold V is partially isotropic if and only if $\nabla^{\perp}{}_{\beta\bar{X}}\mu = 0$ for all $\bar{X} \in \mathfrak{X}(\pi(M))$.*

Proof. Since the ambient manifold V is a Landsberg manifold (${}^*P = 0$) and since the embedded manifold M is totally h-umbilic, equation (6) takes the form

$$\begin{aligned}{}^*R(\bar{X},\bar{Y})\bar{Z} =& R(\bar{X},\bar{Y})\bar{Z} + B_{H(\bar{Y},\bar{Z})}\bar{X} - B_{H(\bar{X},\bar{Z})}\bar{Y} + (\bar{\nabla}_{\beta\bar{Y}}H)(\bar{X},\bar{Z}) \\ &- (\bar{\nabla}_{\beta\bar{X}}H)(\bar{Y},\bar{Z}) - H(A(\bar{X},\bar{Y}),\bar{Z}) + Q(R(\bar{X},\bar{Y})\vartheta,\bar{Z}).\end{aligned} \tag{8}$$

If V is partially isotropic, then ${}^*R(\bar{X},\bar{Y})\bar{Z} = \kappa[{}^*g(\bar{X},\bar{Z})\bar{Y} - {}^*g(\bar{Y},\bar{Z})\bar{X}]$ for all $\bar{X},\bar{Y},\bar{Z} \in \mathfrak{X}(\pi(V))$. Hence we have ${}^*R(\bar{X},\bar{Y})\bar{Z} \in \mathfrak{X}(\pi(M))$ for all $\bar{X},\bar{Y},\bar{Z} \in \mathfrak{X}(\pi(M))$. Therefore, for every $\bar{W} \in \mathfrak{X}(\pi(M))$, equation (8) reduces to

$$R(\bar{X},\bar{Y},\bar{Z},\bar{W}) = (\kappa + \|\mu\|^2)\big[g(\bar{Y},\bar{W})g(\bar{X},\bar{Z}) - g(\bar{X},\bar{W})g(\bar{Y},\bar{Z})\big]. \tag{9}$$

Now, by equation (9) M is partially isotropic if and only if μ is constant, and hence if and only if $\nabla^{\perp}{}_X\mu = 0$ for every $X \in \mathfrak{X}(\mathcal{T}M)$. As $\nabla^{\perp}{}_{\gamma\overline{X}}\mu = 0$ by Theorem 2 of [13], M is partially isotropic if and only if $\nabla^{\perp}{}_{\beta\bar{X}}\mu$ vanishes. □

The following results follow from Theorem 4.

Corollary 2. *A minimal totally h-umbilic submanifold of a partially isotropic Landsberg manifold is partially isotropic.*

Corollary 3. *A totally geodesic totally h-umbilic submanifold of a partially isotropic Landsberg manifold is partially isotropic.*

Corollary 4. *A totally h-umbilic submanifold of a locally Minkowski manifold is partially isotropic if and only if $\nabla^{\perp}{}_{\beta\bar{X}}\mu = 0$ for all $\bar{X} \in \mathfrak{X}(\pi(M))$.*

Theorem 5. *For a totally h-umbilic submanifold M of a partially isotropic Landsberg manifold V we have*

(a) *the horizontal scalar curvature* Sc^h *is nonnegative if and only if* κ *is nonnegative;*

(b) $\nabla_{\gamma\bar{X}}\,\mathrm{Sc}^h = 0$ *for all* $\bar{X} \in \mathfrak{X}(\pi(M))$*; i.e.,* Sc^h *does not vary in the vertical direction;*

(c) $\nabla_{\beta\bar{X}}\,\mathrm{Sc}^h = 0$ *for all* $\bar{X} \in \mathfrak{X}(\pi(M))$ *if and only if* M *is h-minimal,* μ *is constant or* ${\nabla^{\perp}}_{\beta\bar{X}}\mu = 0$.

Proof. By equation (9), we have $\mathrm{Ric}^h(\bar{X},\bar{Y}) = (m-1)(\kappa + \|\mu\|^2)$ $g(\bar{X},\bar{Y})$. By the nondegeneracy of g, we deduce from this relation that ${\mathrm{Ric}^h}_0 = (m-1)(\kappa + \|\mu\|^2)I$, consequently

$$\mathrm{Sc}^h = m(m-1)(\kappa + \|\mu\|^2). \tag{10}$$

Now, (a) follows from (10), (b) follows from (10) and the fact that ${\nabla^{\perp}}_{\gamma\bar{X}}\mu = 0$ for every $\bar{X} \in \mathfrak{X}(\pi(M))$. Finally by equation (10) we can easily show that $\nabla_{\beta\bar{X}}\,\mathrm{Sc}^h = 2m(m-1){}^*g({\nabla^{\perp}}_{\beta\bar{X}}\mu,\mu)$. Hence, $\nabla_{\beta\bar{X}}\,\mathrm{Sc}^h = 0$ if and only if $\mu = 0$, $\mu =$ constant or ${\nabla^{\perp}}_{\beta\bar{X}}\mu = 0$. $\square$

The following two corollaries follow from Theorem 5.

Corollary 5. *For a totally h-umbilic submanifold* M *of a locally Minkowski manifold* V*, we have*

(a) *the scalar curvature* Sc^h *is nonnegative and* $\mathrm{Sc}^h = 0$ *if and only if* M *is h-minimal,*

(b) Sc^h *does not vary in the vertical direction,*

(c) $\nabla_{\beta\bar{X}}\,\mathrm{Sc}^h = 0$ *for all* $\bar{X} \in \mathfrak{X}(\pi(M))$ *if and only if* M *is h-minimal,* μ *is constant or* ${\nabla^{\perp}}_{\beta\bar{X}}\mu = 0$.

Corollary 6. *For a minimal totally h-umbilic submanifold of a locally Minkowski manifold, the curvature tensor* R *vanishes.*

Let $\mathbf{R}^N$ be the curvature transformation of the normal connection $\nabla^{\perp}$. Using equations (1) and (2), we get

Lemma 2. *For every* $X, Y \in \mathfrak{X}(\mathcal{T}M)$ *and for every* $\xi \in \mathcal{C}(\mathcal{N})$*, we have*

$$\begin{aligned}{}^*\mathbf{R}(X,Y)\xi = \mathbf{R}^{\mathbf{N}}(X,Y)\xi + \tilde{H}(X,\tilde{B}_\xi Y) - \tilde{H}(Y,\tilde{B}_\xi X) + \tilde{B}_{\nabla^{\perp}_Y\xi}X \\ - \tilde{B}_{\nabla^{\perp}_X\xi}Y + \mathbf{T}_{\tilde{B}_\xi}(X,Y),\end{aligned}$$

where $\mathbf{T}_{\tilde{B}_\xi}(X,Y)$ *is given by* $\mathbf{T}_{\tilde{B}_\xi}(X,Y) = \nabla_X \tilde{B}_\xi Y - \nabla_Y \tilde{B}_\xi X - \tilde{B}_\xi[X,Y]$.

Theorem 6. *If M is a totally h-umbilic Finsler submanifold of a locally Minkowski manifold V, then $\nabla^\perp$ is h-flat.*

Proof. Applying Lemma 2 for $X = \beta\bar{X}, Y = \beta\bar{Y}$ and taking equation (3) into account, we conclude that

$$\begin{aligned} {}^*R(\bar{X},\bar{Y},\xi,\eta) + {}^*P(N(\bar{X}),\bar{Y},\xi,\eta) - {}^*P(N(\bar{Y}),\bar{X},\xi,\eta) \\ + {}^*S(N(\bar{X}),N(\bar{Y}),\xi,\eta) = R^N(\bar{X},\bar{Y},\xi,\eta) \\ + g(B_\eta\bar{X},B_\xi\bar{Y}) - g(B_\eta\bar{Y},B_\xi\bar{X}). \end{aligned} \tag{11}$$

If V is locally Minkowski (${}^*R = {}^*P = 0$) and if M is totally h-umbilic ($B_\xi = {}^*g(\xi,\mu)I$ and $N = L\ell \otimes \mu$), it follows that

$${}^*S(N(\bar{X}),N(\bar{Y}),\xi,\eta) = 0 \quad \text{and} \quad g(B_\eta\bar{X},B_\xi\bar{Y}) = g(B_\eta\bar{Y},B_\xi\bar{X}).$$

Consequently, by substituting into (11), we have $R^N(\bar{X},\bar{Y},\xi,\eta) = 0$. This completes the proof. □

3. Submanifolds of a Landsberg manifold

In this section, the ambient manifold $(V, {}^*L)$ will be a Landsberg manifold.

Proposition 2. *If M is a Landsberg submanifold of a Landsberg manifold V, then the induced and intrinsic connections of M coincide.*

Proof. Applying Lemma 1 for $X = \gamma\bar{X}$, $Y = \beta\bar{Y}$, we get

$$\begin{aligned} {}^*P(\bar{X},\bar{Y})\bar{Z} + {}^*S(\bar{X},N(\bar{Y}))\bar{Z} = P(\bar{X},\bar{Y})\bar{Z} + W_{H(\bar{Y},\bar{Z})}\bar{X} \\ - B_{Q(\bar{X},\bar{Z})}\bar{Y} - (\bar{\nabla}_{\gamma\bar{X}}H)(\bar{Y},\bar{Z}) + (\bar{\nabla}_{\beta\bar{Y}}Q)(\bar{X},\bar{Z}) \\ - H(T(\bar{X},\bar{Y}),\bar{Z}) + Q(P(\bar{X},\bar{Y})\vartheta,\bar{Z}). \end{aligned} \tag{12}$$

Since both the ambient and the embedded manifolds are Landsberg, then equation (12) takes the form

$$\begin{aligned} {}^*S(\bar{X},N(\bar{Y}))\bar{Z} = W_{H(\bar{Y},\bar{Z})}\bar{X} - B_{Q(\bar{X},\bar{Z})}\bar{Y} - (\bar{\nabla}_{\gamma\bar{X}}H)(\bar{Y},\bar{Z}) \\ + (\bar{\nabla}_{\beta\bar{Y}}Q)(\bar{X},\bar{Z}) - H(T(\bar{X},\bar{Y}),\bar{Z}). \end{aligned}$$

Setting $\bar{Y} = \bar{Z} = \vartheta$, we get $W_{N_o}\bar{X} = 0$. This together with (3) gives ${}^*g(Q(\bar{X}, \bar{Y}), N_o) = 0$ for every $\bar{Y} \in \mathfrak{X}(\pi(M))$. Now the result follows from Theorem 1 of [11]. $\square$

Lemma 3. *For a submanifold M of a Landsberg manifold V, we have $P^N = 0$ if and only if ${}^*S(\bar{X}, N(\bar{Y}), \xi, \eta) = g(W_\eta\bar{X}, B_\xi\bar{Y}) - g(B_\eta\bar{Y}, W_\xi\bar{X})$, for every $\xi, \eta \in \mathcal{C}(\mathcal{N})$.*

Proof. Applying Lemma 2 for $X = \gamma\bar{X}$, $Y = \beta\bar{Y}$ and using equation (3) we have, for every $\xi, \eta \in \mathcal{C}(\mathcal{N})$,

$$\begin{aligned} {}^*P(\bar{X}, \bar{Y}, \xi, \eta) + {}^*S(\bar{X}, N(\bar{Y}), \xi, \eta) = P^N(\bar{X}, \bar{Y}, \xi, \eta) &+ g(W_\eta\bar{X}, B_\xi\bar{Y}) \\ &- g(B_\eta\bar{Y}, W_\xi\bar{X}). \end{aligned} \tag{13}$$

As the ambient manifold V is Landsberg, the result follows from equation (13). $\square$

Using Lemma 3, we obtain

Theorem 7. *For a submanifold of a Landsberg manifold, $\nabla^\perp$ is hv-flat if and only if ${}^*S(\bar{X}, N(\bar{Y}), \xi, \eta) = g(W_\eta\bar{X}, B_\xi\bar{Y}) - g(B_\eta\bar{Y}, W_\xi\bar{X})$, for every $\xi, \eta \in \mathcal{C}(\mathcal{N})$.*

In [11] we have shown that for a submanifold M of a Landsberg manifold V, if $N_o = 0$ then M is also a Landsberg manifold and $H = 0$. Combining this fact with Theorem 7, we have

Corollary 7. *If M is a totally geodesic submanifold of a Landsberg manifold V, then $\nabla^\perp$ is hv-flat.*

Theorem 8. *A minimal totally v-umbilic Finsler submanifold M of a Berwald manifold V is a Berwald manifold.*

Proof. Using equations (1), (2) and (4), we have, for every $\bar{X}, \bar{Y}, \bar{Z} \in \mathfrak{X}(\pi(M))$,

$$\begin{aligned} ({}^*\nabla_{{}^*\beta\bar{X}}{}^*T)(\bar{Y}, \bar{Z}) = (\nabla_{\beta\bar{X}}T)(\bar{Y}, \bar{Z}) &- B_{Q(\bar{Y}, \bar{Z})}\bar{X} + (\bar{\nabla}_{\beta\bar{X}}Q)(\bar{Y}, \bar{Z}) \\ &- ({}^*\nabla_{\gamma N(X)}{}^*T)(\bar{Y}, \bar{Z}) + H(\bar{X}, T(\bar{Y}, \bar{Z})) \\ &- {}^*T(H(\bar{X}, \bar{Y}), \bar{Z}) - {}^*T(\bar{Y}, H(\bar{X}, \bar{Z})). \end{aligned}$$

Since the ambient manifold V is Berwald (${}^*\nabla_{{}^*\beta\bar{X}}{}^*T = 0$) and since ${}^*\nabla_{\gamma N(X)}{}^*T$, ${}^*T(H(\bar{X},\bar{Y}),\bar{Z})$ are π-normal, the above equation gives $(\nabla_{\beta\bar{X}}T)(\bar{Y},\bar{Z}) = B_{Q(\bar{Y},\bar{Z})}\bar{X}$. If M is totally v-umbilic and minimal, then $B_{Q(\bar{Y},\bar{Z})}\bar{X} = 0$ and the result follows. □

Theorem 8 shows that a totally geodesic submanifold of a Berwald manifold is a Berwald manifold if and only if $B_{Q(\bar{Y},\bar{Z})}\bar{X} = 0$.

4. Submanifolds of semi-C-reducible manifolds

In this section, the ambient Finsler manifold $(V, {}^*L)$ will be a semi-C-reducible manifold.

The following result generalizes Theorem 2.1 of [10].

Theorem 9. *A submanifold of a semi-C-reducible manifold is semi-C-reducible.*

Proof. From the definition of a semi-C-reducible manifold, we have

$$\begin{aligned}{}^*T(\bar{X},\bar{Y},\bar{Z}) = \frac{{}^*\sigma}{(n+1)}\big\{&{}^*h(\bar{X},\bar{Y}){}^*C(\bar{Z}) + {}^*h(\bar{Y},\bar{Z}){}^*C(\bar{X}) \\ &+ {}^*h(\bar{X},\bar{Z}){}^*C(\bar{Y})\big\} + \frac{{}^*\tau}{{}^*c^2}\ {}^*C(\bar{X}){}^*C(\bar{Y}){}^*C(\bar{Z}),\end{aligned} \tag{14}$$

for every $\bar{X},\bar{Y},\bar{Z} \in \mathfrak{X}(\pi(V))$, where the scalars ${}^*\sigma$ and ${}^*\tau$ satisfy the condition ${}^*\sigma + {}^*\tau = 1$ and the function ${}^*c^2$ is given by ${}^*c^2 = {}^*g({}^*C, {}^*C) \neq 0$. As $h = {}^*h|_{\pi^{-1}(TM)}$ we deduce, from equation (14), that

$$C = \varrho{}^*C|_{\pi^{-1}(TM)} \quad \text{and} \quad c^2 = \varrho^{2*}c^2 \quad \text{where} \quad \varrho = \frac{m+1}{n+1}{}^*\sigma + \frac{{}^*\tau}{{}^*c^2}\ c^2.$$

Consequently equations (4) and (14) imply, for every $\bar{X},\bar{Y},\bar{Z} \in \mathfrak{X}(\pi(M))$, that

$$\begin{aligned}T(\bar{X},\bar{Y},\bar{Z}) = &\frac{{}^*\sigma}{\varrho(n+1)}\{h(\bar{X},\bar{Y})C(\bar{Z}) + h(\bar{Y},\bar{Z})C(\bar{X}) + h(\bar{X},\bar{Z})C(\bar{Y})\} \\ &+ \frac{{}^*\tau}{\varrho^3\ {}^*c^2}\ C(\bar{X})C(\bar{Y})C(\bar{Z}).\end{aligned}$$

By the nondegeneracy of the metric g, we obtain

$$T(\bar{X},\bar{Y}) = \frac{{}^*\sigma}{\varrho(n+1)}\big\{h(\bar{X},\bar{Y})\bar{b} + \varphi(\bar{Y})C(\bar{X}) + \varphi(\bar{X})C(\bar{Y})\big\} + \frac{{}^*\tau}{\varrho^3 \ {*c^2}}\ C(\bar{X})C(\bar{Y})\bar{b},$$

where $\bar{b}$ is the π-vector field associated with the π-form C under the duality defined by g and φ is the π-tensor field defined by $\varphi = I - L^{-1}\ell\otimes\vartheta$. Taking the trace of both sides of the above equation and noting that $h(\bar{X},\bar{b}) = C(\bar{X})$, $C(\varphi(\bar{X})) = C(\bar{X})$, we deduce that $(1-\frac{(m+1)^*\sigma}{(n+1)\varrho}-\frac{c^{2*}\tau}{\varrho^{3*}c^2})C = 0$. Since the embedded manifold is Finsler, $C \neq 0$. Thus $(1-\frac{(m+1)^*\sigma}{(n+1)\varrho}-\frac{c^{2*}\tau}{\varrho^{3*}c^2})C = 0$ if and only if $\frac{(m+1)^*\sigma}{(n+1)\varrho}+\frac{c^{2*}\tau}{\varrho^{3*}c^2} = 1$. If we put $\sigma = \frac{(m+1)^*\sigma}{(n+1)\varrho}$ and $\tau = \frac{c^{2*}\tau}{\varrho^{3*}c^2}$, then $\sigma+\tau = 1$ and the π-tensor field T takes the form

$$T(\bar{X},\bar{Y},\bar{Z}) = \frac{\sigma}{m+1}\big\{h(\bar{X},\bar{Y})C(\bar{Z}) + h(\bar{Y},\bar{Z})C(\bar{X}) + h(\bar{X},\bar{Z})C(\bar{Y})\big\} + \frac{\tau}{c^2}\ C(\bar{X})C(\bar{Y})C(\bar{Z}).$$

This means that M is a semi-C-reducible submanifold. □

Proposition 3. *For a submanifold M of a semi-C-reducible manifold V, the vertical second fundamental form Q takes the form $Q = (\frac{{}^*\sigma}{n+1}\ h + \frac{{}^*\tau}{\varrho^* c^2}\ C\otimes C)\otimes{}^*\bar{b}$, where ${}^*\bar{b}$ is the π-vector field associated with the π-form ${}^*C$ under the duality defined by ${}^*g$.*

Proof. Using equation (4), it follows that ${}^*g(Q(\bar{X},\bar{Y}),\xi) = {}^*g({}^*T(\bar{X},\bar{Y}),\xi)$, for all $\bar{X},\bar{Y}\in\mathfrak{X}(\pi(M))$, $\xi\in\mathcal{C}(\mathcal{N})$. By (14) this equation reduces to

$${}^*g(Q(\bar{X},\bar{Y}),\xi) = \left(\frac{{}^*\sigma}{n+1}\ h(\bar{X},\bar{Y}) + \frac{{}^*\tau}{\varrho^{2*}c^2}\ C(\bar{X})C(\bar{Y})\right){}^*C(\xi),$$

since ${}^*h(\bar{X},\xi) = 0$ and $h = {}^*h|_{\pi^{-1}(TM)}$. Let ${}^*\bar{b}$ be the π-vector field associated with the π-form ${}^*C$ under the duality defined by the metric ${}^*g$. Then the result follows from the nondegeneracy of the metric ${}^*g$. □

Corollary 8. *If ${}^*\bar{b}$ is tangential to a submanifold M of a semi-C-reducible manifold V, then the vertical second fundamental form Q vanishes.*

Proposition 4. *The horizontal torsion tensor A of a submanifold M of a semi-C-reducible manifold V can be given by*

$$A(\bar{X},\bar{Y}) = {}^*g({}^*\bar{b}, N(\bar{X}))F(\bar{Y}) - {}^*g({}^*\bar{b}, N(\bar{Y}))F(\bar{X}),$$

where F is the π-tensor field defined by $F = (\frac{^*\sigma}{n+1}\varphi + \frac{^*\tau}{\varrho^2 {}^*c^2}\ C \otimes \bar{b})$.

Proof. By equation (4), we have, for every $\bar{X},\bar{Y} \in \mathfrak{X}(\pi(M))$,

$$g(A(\bar{X},\bar{Y}),\bar{Z}) = {}^*g({}^*T(N(\bar{X}),\bar{Y}),\bar{Z}) - {}^*g({}^*T(N(\bar{Y}),\bar{X}),\bar{Z}).$$

From the symmetry of the torsion tensor ${}^*T$, it follows that

$$g(A(\bar{X},\bar{Y}),\bar{Z}) = {}^*g(Q(\bar{Y},\bar{Z}),N(\bar{X})) - {}^*g(Q(\bar{X},\bar{Z}),N(\bar{Y})).$$

Then, by Proposition 3, we have

$$g(A(\bar{X},\bar{Y}),\bar{Z}) = {}^*g({}^*\bar{b}, N(\bar{X}))\left(\frac{^*\sigma}{n+1}\ h + \frac{^*\tau}{\varrho^2 {}^*c^2}\ C\otimes C\right)(\bar{Y},\bar{Z})$$
$$- {}^*g({}^*\bar{b}, N(\bar{Y}))\left(\frac{^*\sigma}{n+1}\ h + \frac{^*\tau}{\varrho^2 {}^*c^2}\ C\otimes C\right)(\bar{X},\bar{Z}).$$

The result follows from the nondegeneracy of the Finsler metric g. □

Now we present a generalization of Theorem 1 of [9].

Theorem 10. *For a submanifold M of a semi-C-reducible manifold V, the induced and intrinsic connections of M coincide if and only if M is totally geodesic, ${}^*\bar{b} = 0$ or ${}^*\bar{b}$ is tangential to M.*

Proof. If M is either totally geodesic, ${}^*\bar{b} = 0$ or ${}^*\bar{b}$ is tangential to M, Proposition 4 implies that A vanishes. This means that the induced connection ∇ of M is g-Cartan [11].

Conversely, assume that the horizontal torsion tensor A vanishes. It follows, from Proposition 4 again, that ${}^*g({}^*\bar{b}, N(\bar{X}))F(\bar{Y}) = {}^*g({}^*\bar{b}, N(\bar{Y}))\times F(\bar{X})$. Setting $\bar{Y} = \vartheta$ in this equality, the left-hand side vanishes identically. Hence, ${}^*g({}^*\bar{b}, N_o)F = 0$. But ${}^*g({}^*\bar{b}, N_o)F = 0$ if and only if ${}^*g({}^*\bar{b}, N_o) = 0$ or $F = 0$. If $F = 0$, then $\varphi = 0$, since the rank of the matrix representing φ is greater than one. But $\varphi = 0$ implies that $I = L^{-1}\ell\otimes\vartheta$ which means that M is a submanifold of dimension one, contradicting Theorem 9. Therefore, ${}^*g({}^*\bar{b}, N_o)F = 0$ if and only if $N_o = 0$, ${}^*\bar{b} = 0$ or ${}^*\bar{b}$ is tangential to M. This completes the proof. □

In [14] we have shown that for a semi-C-reducible manifold V the relation

$$\begin{aligned}{}^*S(\bar{X}_1,\bar{X}_2,\bar{X}_3,\bar{X}_4) &= {}^*h(\bar{X}_1,\bar{X}_4){}^*G(\bar{X}_2,\bar{X}_3) - {}^*h(\bar{X}_2,\bar{X}_4){}^*G(\bar{X}_1,\bar{X}_3)\\ &+ {}^*h(\bar{X}_2,\bar{X}_3){}^*G(\bar{X}_1,\bar{X}_4) - {}^*h(\bar{X}_1,\bar{X}_3){}^*G(\bar{X}_2,\bar{X}_4),\end{aligned} \tag{15}$$

holds, for all $\bar{X}_1,\bar{X}_2,\bar{X}_3,\bar{X}_4 \in \mathfrak{X}(\pi(V))$, where ${}^*G$ is the π-tensor field defined by

$${}^*G = \frac{1}{2}\left(\frac{{}^*\sigma{}^*c}{n+1}\right)^2 {}^*h + \left(\left(\frac{{}^*\sigma}{n+1}\right)^2 + \frac{{}^*\sigma{}^*\tau}{n+1}\right){}^*C \otimes {}^*C.$$

Using Lemmas 1, 2 and taking into account the fact that the vertical Wiengarten operator is self-adjoint, one can deduce

Lemma 4. *For every $\bar{X}_1,\bar{X}_2,\bar{X}_3,\bar{X}_4 \in \mathfrak{X}(\pi(M))$, $\xi_1,\xi_2 \in \mathcal{C}(\mathcal{N})$, we have*

(a) $$\begin{aligned}{}^*S(\bar{X}_1,\bar{X}_2,\bar{X}_3,\bar{X}_4) = S(\bar{X}_1,\bar{X}_2,\bar{X}_3,\bar{X}_4) &+ {}^*g(Q(\bar{X}_1,\bar{X}_4),Q(\bar{X}_2,\bar{X}_3))\\ &- {}^*g(Q(\bar{X}_1,\bar{X}_3),Q(\bar{X}_2,\bar{X}_4)),\end{aligned}$$

(b) $${}^*S(\bar{X}_1,\bar{X}_2,\xi_1,\xi_2) = S^N(\bar{X}_1,\bar{X}_2,\xi_1,\xi_2) + g([W_{\xi_1},W_{\xi_2}]\bar{X}_1,\bar{X}_2).$$

Theorem 11. *If M is a totally v-umbilic submanifold of a semi-C-reducible manifold V, then $\nabla^\perp$ is v-flat.*

Proof. Since M is totally v-umbilic, we have $W_\xi = L^{-1}{}^*g(\nu,\xi)\varphi$ for all $\xi \in \mathcal{C}(\mathcal{N})$. Consequently, it follows from Lemma 4(b) that

$${}^*S(\bar{X}_1,\bar{X}_2,\xi_1,\xi_2) = S^N(\bar{X}_1,\bar{X}_2,\xi_1,\xi_2)$$

$$\forall\ \bar{X}_1,\bar{X}_2 \in \mathfrak{X}(\pi(M)),\ \xi_1,\xi_2 \in \mathcal{C}(\mathcal{N}).$$

On the other hand, since the ambient Finsler manifold V is semi-C-reducible, we obtain from (15) that

$$\begin{aligned}{}^*S(\bar{X}_1,\bar{X}_2,\xi_1,\xi_2) &= {}^*h(\bar{X}_1,\xi_2)\ {}^*G(\bar{X}_2,\xi_1) - {}^*h(\bar{X}_2,\xi_2)\ G^*(\bar{X}_1,\xi_1)\\ &+ {}^*h(\bar{X}_2,\xi_1)\ {}^*G(\bar{X}_1,\xi_2) - {}^*h(\bar{X}_1,\xi_1)\ {}^*G(\bar{X}_2,\xi_2).\end{aligned}$$

Since ${}^*h(\bar{X}_i,\xi_i) = 0$; $i = 1,2$; it follows that ${}^*S(\bar{X}_1,\bar{X}_2,\xi_1,\xi_2) = 0$ and consequently $S^N = 0$. Hence $\nabla^\perp$ is v-flat. $\square$

5. Submanifolds of $S4$-like manifolds

In this section, the ambient Finsler manifold $(V, {}^*L)$ will be an $S4$-like manifold.

Using Lemma 4(a) and taking into account the fact that $h = {}^*h|_{\pi^{-1}(TM)}$, we have

Proposition 5. *If the ambient Finsler manifold is an $S4$-like manifold, then the π-tensor field S of the embedded manifold M is given by*

$$\begin{aligned}
&S(\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4) \\
&\quad = \frac{{}^*\operatorname{Sc}^v}{(n-2)(n-3)} \big[h(\bar{X}_1, \bar{X}_4)h(\bar{X}_2, \bar{X}_3) - h(\bar{X}_1, \bar{X}_3)h(\bar{X}_2, \bar{X}_4)\big] \\
&\qquad + \frac{1}{(n-3)} \big[h(\bar{X}_2, \bar{X}_4))^*\operatorname{Ric}^v(\bar{X}_1, \bar{X}_3) + h(\bar{X}_1, \bar{X}_3)^*\operatorname{Ric}^v(\bar{X}_2, \bar{X}_4) \\
&\qquad - h(\bar{X}_1, \bar{X}_4)^*\operatorname{Ric}^v(\bar{X}_2, \bar{X}_3) - h(\bar{X}_2, \bar{X}_3)^*\operatorname{Ric}^v(\bar{X}_1, \bar{X}_4)\big] \\
&\qquad + {}^*g(Q(\bar{X}_1, \bar{X}_3), Q(\bar{X}_2, \bar{X}_4)) - {}^*g(Q(\bar{X}_1, \bar{X}_4), Q(\bar{X}_2, \bar{X}_3)),
\end{aligned}$$

for all $\bar{X}_i \in \mathfrak{X}(\pi(M))$; $1 \le i \le 4$.

Lemma 5. *For a totally v-umbilic submanifold M of an $S4$-like manifold V we have*

(a) $$\begin{aligned}\operatorname{Ric}^v(\bar{X}_1, \bar{X}_2) = {}&\left(\frac{n-m}{(n-2)(n-3)}{}^*\operatorname{Sc}^v + (m-2)L^{-2}\|\nu\|^2\right) h(\bar{X}_1, \bar{X}_2) \\ &+ \frac{m-3}{n-3}{}^*\operatorname{Ric}^v(\bar{X}_1, \bar{X}_2),\end{aligned}$$

(b) $$\operatorname{Sc}^v = (m-2)\left(\frac{2n-m-3}{(n-2)(n-3)}{}^*\operatorname{Sc}^v + (m-1)L^{-2}\|\nu\|^2\right),$$

for every $\bar{X}_1, \bar{X}_2 \in \mathfrak{X}(\pi(M))$.

Proof. (a) Since M is a totally v-umbilic submanifold, it follows from Proposition 5 that

$$\begin{aligned}S(\bar{X}_1, \bar{X}_2)\bar{X}_3 = {}&\left(\frac{{}^*\operatorname{Sc}^v}{(n-2)(n-3)} - L^{-2}\|\nu\|^2\right) \\ &\times \{h(\bar{X}_2, \bar{X}_3)\varphi(\bar{X}_1) - h(\bar{X}_1, \bar{X}_3)\varphi(\bar{X}_2)\}\end{aligned}$$

$$+\frac{1}{(n-3)}\{h(\bar{X}_1,\bar{X}_3)^*\operatorname{Ric}^v{}_o(\bar{X}_2)-\ h(\bar{X}_2,\bar{X}_3)^*\operatorname{Ric}^v{}_o(\bar{X}_1)$$

$$+{}^*\operatorname{Ric}^v(\bar{X}_1,\bar{X}_3)\varphi(\bar{X}_2)-{}^*\operatorname{Ric}^v(\bar{X}_2,\bar{X}_3)\varphi(\bar{X}_1)\},$$

for every $\bar{X}_1,\bar{X}_2,\bar{X}_3\in\mathfrak{X}(\pi(M))$.

Since

$$h(\varphi(\bar{X}_1),\bar{X}_2)=h(\bar{X}_1,\bar{X}_2),\quad {}^*h(\bar{X}_2,{}^*\operatorname{Ric}^v{}_o(\bar{X}_1))={}^*\operatorname{Ric}^v(\bar{X}_2,\bar{X}_1),$$

$$^*\operatorname{Ric}^v(\bar{X}_2,\varphi(\bar{X}_1))={}^*\operatorname{Ric}^v(\bar{X}_2,\bar{X}_1)\text{ and }{}^*\operatorname{Ric}^v(\bar{X}_1,\bar{X}_2)={}^*\operatorname{Ric}^v(\bar{X}_2,\bar{X}_1),$$

we conclude that

$$\operatorname{Ric}^v(\bar{X}_1,\bar{X}_2)=\left(\frac{n-m}{(n-2)(n-3)}{}^*\operatorname{Sc}^v+(m-2)L^{-2}\|\nu\|^2\right)h(\bar{X}_1,\bar{X}_2)$$

$$+\frac{m-3}{n-3}{}^*\operatorname{Ric}^v(\bar{X}_1,\bar{X}_2).$$

(b) By the nondegeneracy of the metric ${}^*g$, we deduce from (a) that

$$\operatorname{Ric}^v{}_o(\bar{X}_1)=\left(\frac{n-m}{(n-2)(n-3)}{}^*\operatorname{Sc}^v+(m-2)L^{-2}\|\nu\|^2\right)\varphi(\bar{X}_1)$$

$$+\frac{m-3}{n-3}\ {}^*\operatorname{Ric}^v{}_o(\bar{X}_1).$$

The result then follows by taking the trace of both sides of the above relation. □

Combining Lemma 5 and Proposition 5, we infer

Theorem 12. *A totally v-umbilic submanifold of an $S4$-like manifold is $S4$-like.*

Theorem 13. *If M is a totally v-umbilic submanifold of an $S4$-like manifold V, then $\nabla^\perp$ is v-flat.*

Proof. Since M is a totally v-umbilic submanifold, Lemma 4(b) implies that

$$^*S(\bar{X}_1,\bar{X}_2,\xi_1,\xi_2)=S^N(\bar{X}_1,\bar{X}_2,\xi_1,\xi_2),$$

$$\text{for all } \bar{X}_1, \bar{X}_2 \in \mathfrak{X}(\pi(M)),\ \xi_1, \xi_2 \in \mathcal{C}(\mathcal{N}).$$

On the other hand, since the ambient Finsler manifold is $S4$-like, we get

$$\begin{aligned}
&{}^*S(\bar{X}_1, \bar{X}_2, \xi_1, \xi_2) \\
&\quad = \frac{{}^*\operatorname{Sc}^v}{(n-2)(n-3)}\{{}^*h(\bar{X}_1,\xi_2){}^*h(\bar{X}_2,\xi_1) - {}^*h(\bar{X}_1,\xi_1){}^*h(\bar{X}_2,\xi_2)\} \\
&\qquad + \frac{1}{(n-3)}\{{}^*h(\bar{X}_2,\xi_2){}^*\operatorname{Ric}^v(\bar{X}_1,\xi_1) - {}^*h(\bar{X}_1,\xi_2){}^*\operatorname{Ric}^v(\bar{X}_2,\xi_1) \\
&\qquad + {}^*h(\bar{X}_1,\xi_1){}^*\operatorname{Ric}^v(\bar{X}_2,\xi_2) - {}^*h(\bar{X}_2,\xi_1){}^*\operatorname{Ric}^v(\bar{X}_1,\xi_2)\}.
\end{aligned}$$

Since ${}^*h(\bar{X}_i, \xi_i) = 0$; $i = 1, 2$; it follows that ${}^*S(\bar{X}_1, \bar{X}_2, \xi_1, \xi_2) = 0$ and, consequently $S^N = 0$. Hence $\nabla^\perp$ is v-flat. $\square$

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ALY A. TAMIM
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
CAIRO UNIVERSITY
GIZA
EGYPT

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