Publ. Math. Debrecen
64/3-4 (2004), 319–328

Decomposable subspaces of Banach spaces

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Abstract. We introduce and study the notion of hereditarily A-indecomposable Banach space for A a space ideal. The case $\mathsf{A} = \mathsf{F}$, the finite dimensional spaces, corresponds to the hereditarily indecomposable spaces. We show that several properties of the case $\mathsf{A} = \mathsf{F}$ extend to some other space ideals.

1. Introduction

W. T. Gowers and B. Maurey [10] constructed the first example of a *hereditarily indecomposable Banach space* (HI space, for short). The main property of the HI spaces is that they do not contain unconditional basic sequences. So they provide a counterexample to an old question in Banach space theory. Moreover, a result of Weis [18] characterizes the HI spaces as those spaces X such that for every Banach space Y any operator in $L(X,Y)$ is upper semi-Fredholm or strictly singular.

In this paper we consider a natural notion of hereditarily A-indecomposable Banach space associated to a space ideal A: a Banach space X is hereditarily A-indecomposable (HAI space, for short) if no (closed) subspace of X can be written as the direct sum of two subspaces which are not in A. In the case $\mathsf{A} = \mathsf{F}$, the finite dimensional spaces, we obtain the

Mathematics Subject Classification: 46B03, 47A53, 47B10.
Key words and phrases: hereditarily indecomposable Banach spaces, space ideals.
Research of the first author supported in part by DGI (Spain), Proyecto BFM2001-1147.
Research of the second author supported in part by Gobierno de Canarias (Spain) PI2001/039.

HI spaces. We show that the notion of HAI space is nontrivial for A the reflexive spaces, the weakly sequentially complete spaces, the spaces with the Mazur property and for some other space ideals.

Note that X HI means that any two infinite dimensional subspaces M and N of X are very close, in the sense that $\text{dist}(S_M, S_N) = 0$, where S_M is the unit sphere of M. Similarly, X HAI means that $\text{dist}(S_M, S_N) = 0$ when M and N do not belong to A.

We show that the HAI spaces do not contain unconditional basic sequences of some kinds related to A. Moreover, we show that X is HAI if and only if $L(X,Y) = \mathsf{A}\Phi_+(X,Y) \cup \mathsf{A}SS(X,Y)$, for any space Y. Here $\mathsf{A}\Phi_+$ and $\mathsf{A}SS$ are classes of operators which were introduced in [8], [9]. These classes generalize the upper semi-Fredholm operators and the strictly singular operators, respectively.

We also consider space ideals satisfying $\mathsf{A} = \mathsf{A}_{ii}$. This is a condition defined in terms of incomparability. It means that A coincides with the class of all Banach spaces X such that every infinite dimensional subspace of X contains an infinite dimensional subspace isomorphic to a subspace of a space in A.

In the case $X \in \mathsf{A} = \mathsf{A}_{ii}$ we prove that the union $L(X,Y) = \mathsf{A}\Phi_+(X,Y) \cup \mathsf{A}SS(X,Y)$ is disjoint, and that for each $T \in L(X,Y)$, $in_{\mathsf{A}}(TJ_M) = in_{\mathsf{A}}(T)$ and $sj_{\mathsf{A}}(TJ_M) = sj_{\mathsf{A}}(T)$ for all the subspaces M of X which are not in A, where $in_{\mathsf{A}}(T) > 0$ and $sj_{\mathsf{A}}(T) = 0$ characterize $T \in \mathsf{A}\Phi_+$ and $T \in \mathsf{A}SS$, respectively.

Notation and terminology. Along the paper, X, Y and Z are Banach spaces, and we denote by X^* and X^{**} the dual space and the second dual of X, respectively. A subspace is always a closed linear subspace. Given a subspace M of X, we denote by J_M the inclusion of M into X, and by Q_M the quotient map from X onto X/M.

We denote by $L(X,Y)$ the set of all (continuous linear) operators from X into Y. An operator $T \in L(X,Y)$ is strictly singular if from TJ_M an isomorphism, we obtain that M has finite dimension; T is upper semi-Fredholm operator if it has finite dimensional null space and closed range.

A space ideal A is a class of Banach spaces which contains the finite dimensional spaces and is stable under passing to isomorphic spaces, finite products or complemented subspaces. We refer to [14] for information on operator ideals and space ideals.

2. Hereditarily indecomposable Banach spaces

Let A be a space ideal in the sense of PIETSCH [14]. For each Banach space X we consider

$$S_{\mathsf{A}}(X) := \{M \subset X : M \text{ is a subspace of } X \text{ and } M \notin \mathsf{A}\}.$$

Definition 1. A Banach space X is said to be A-indecomposable if there are no subspaces M and N in S_{A} such that $X = M \oplus N$. Equivalently, if $M, N \in S_{\mathsf{A}}(X)$ and $X = M + N$, then $M \cap N \neq \{0\}$.

The space X is said to be *hereditarily* A*-indecomposable* (HAI) if every subspace M of X is A-indecomposable.

Remark 1. Every space in A is A-indecomposable. Moreover, if $\mathsf{A}_1 \subset \mathsf{A}_2$, then the (hereditarily) A_1-indecomposable spaces are (hereditarily) A_2-indecomposable.

Remark 2. Let $\mathsf{N}\ell_1$ be the ideal of the spaces that contain no copies of ℓ_1.

This case is trivial because the spaces in $\mathsf{N}\ell_1$ are the only hereditarily $\mathsf{N}\ell_1$-indecomposable spaces. Indeed, if X contains a subspace isomorphic to ℓ_1, this subspace can be written as the direct sum of two subspaces isomorphic to ℓ_1. The same happens for $\mathsf{N}\ell_p$, $1 \leq p < \infty$.

A nontrivial example has to include a A-indecomposable space X which is not in A. Observe that this space X cannot be isomorphic to $X \times X$.

Example 1. Let $\mathsf{A} = \mathsf{F}$, the finite dimensional spaces. The hereditarily F-indecomposable spaces are precisely the hereditarily indecomposable spaces.

The existence of infinite dimensional hereditarily indecomposable spaces has been a long-standing open problem in Banach space theory. Finally, GOWERS and MAUREY have constructed an example that we denote X_{GM} [10].

Example 2. Let $\mathsf{A} = \mathsf{R}$ be the reflexive spaces.

James' space J is a hereditarily R-indecomposable, non-reflexive space. The reason is that $\dim(J^{**}/J) = 1$.

Example 3. Let $\mathsf{A} = \mathsf{WSC}$ be the weakly sequentially complete spaces.

A Banach space X is weakly sequentially complete if and only if $B_1(X) = X$, where $B_1(X)$ is the subspace of all $F \in X^{**}$ which are the weak*-limit of some weakly Cauchy sequence in X. Observe that $B_1(M) = B_1(X) \cap M^{\perp\perp}$ for every subspace M of X.

The space J is a hereditarily WSC-indecomposable space which is not weakly sequentially complete. The reason is that $\dim(B_1(J)/J) = 1$.

Example 4. Let $\mathsf{A} = \mathsf{M}$ be the spaces with the Mazur property [19], [11].

Recall that X is in M when $S_1(X) = X$, where $S_1(X)$ is the subspace of all weak*-sequentially continuous elements of X^{**}. Observe that $S_1(M) = S_1(X) \cap M^{\perp\perp}$ for every subspace M of X.

The space $C[0, \omega_1]$ is a hereditarily M-indecomposable space which has not the Mazur property. Again, the reason is that $\dim(S_1(C[0, \omega_1])/C[0, \omega_1]) = 1$. See [17] or [5, Proposition 3.6.b] for a direct proof.

Let $Q : X^{**} \longrightarrow X^{**}/X$ denote the quotient map. Given a closed subspace M of X, we can identify M^{**}/M with $Q(M^{\perp\perp})$. Thus,

$$\mathsf{A}^{co} := \{X : X^{**}/X \in \mathsf{A}\}$$

is a space ideal. For the properties of Q we refer to [20].

Observe that for every separable space Z there exists a separable space X so that X^{**}/X is isomorphic to Z [12].

Example 5. Let A be one of the space ideals F, R or WSC. Let X be a Banach space such that X^{**}/X is isomorphic to X_{GM}, J or J, respectively.

The space X is a hereditarily A^{co}-indecomposable space which is not in A^{co}.

Remark 3. The HI spaces contain no unconditional basic sequence [10].

In the case $\mathsf{A} = \mathsf{R}$, the reflexive spaces, every unconditional basic sequence in a HRI space X generates a reflexive subspace; indeed, if an unconditional basic sequence in X generates a nonreflexive subspace, then

X contains an isomorphic copy of c_0 or ℓ_1 [13, 1.c.13]. Since c_0 and ℓ_1 are isomorphic copy of $c_0 \times c_0$ and $\ell_1 \times \ell_1$ respectively, X is not a HRI space.

Similarly, if X is hereditarily WSC-indecomposable, then every unconditional basic sequence in X generates a weakly sequentially complete subspace [13, 1.c.13].

In general, if $(x_n)_{n\in\mathbb{N}}$ is an unconditional basic sequence of X which is HAI and $\mathbb{N} = I \cup J$ is a partition of $\mathbb{N}$ into two infinite subsets, then the subspace generated by $(x_n)_{n\in I}$ or the subspace generated by $(x_n)_{n\in J}$ belongs to A.

The following characterizations of the hereditarily A-indecomposable spaces will be useful later.

Proposition 1. *Let* A *be a space ideal. For a Banach space* X *the following assertions are equivalent:*

1. X *is* HAI.
2. *If* $M, N \in S_{\mathsf{A}}(X)$, *then* $Q_M J_N$ *is not an isomorphism.*
3. *If* $M, N \in S_{\mathsf{A}}(X)$, *then* $\operatorname{dist}(S_M, S_N) = 0$.

Proof. Since the kernel of $Q_M J_N$ is $M \cap N$ and its range is $(M + N)/M$, (1) and (2) are clearly equivalent. The equivalence between (1) and (3) for infinite dimensional subspaces M and N is well known [6, Exercise 5.15]. □

3. Operators on HAI spaces

Recall that the injection modulus of an operator $T \in L(X, Y)$ is defined by

$$j(T) := \inf\{\|Tx\| : x \in X, \|x\| = 1\}.$$

From the norm and from j we derive two operational quantities that allows us to define two classes of operators.

Definition 2. Let A be a space ideal. Suppose that $S_{\mathsf{A}}(X) \neq \emptyset$ and let Y be a Banach space. For each operator $T \in L(X, Y)$ we define the following quantities:

$$sj_{\mathsf{A}}(T) := \sup\{j(TJ_M) : M \in S_{\mathsf{A}}(X)\},$$

$$in_{\mathsf{A}}(T) := \inf\{\|TJ_M\| : M \in S_{\mathsf{A}}(X)\}.$$

Definition 3. For $S_{\mathsf{A}}(X) \neq \emptyset$ we define

1. $\mathsf{A}SS(X,Y) := \{T \in L(X,Y) : sj_{\mathsf{A}}(T) = 0\}$
2. $\mathsf{A}\Phi_+(X,Y) := \{T \in L(X,Y) : in_{\mathsf{A}}(T) > 0\}$

For $S_{\mathsf{A}}(X) = \emptyset$ we define $\mathsf{A}SS(X,Y) = \mathsf{A}\Phi_+(X,Y) = L(X,Y)$.

In the case $\mathsf{A} = \mathsf{F}$, the finite dimensional spaces, the quantities in_{F} and sj_{F} were introduced in [16]. In this case $\mathsf{F}\Phi_+ = \Phi_+$, the upper semi-Fredholm operators, and $\mathsf{F}SS = SS$, the strictly singular operators.

In the general case the quantities in_{A} and sj_{A} and the associated classes of operators were introduced in [8], [9].

Theorem 1. *Let* A *be a space ideal. For a Banach space* X *the following assertions are equivalent:*

1. X *is* HAI
2. *For every space* Y *and every* $T \in L(X,Y)$, $sj_{\mathsf{A}}(T) \leq in_{\mathsf{A}}(T)$
3. *For every space* Y, $L(X,Y) = \mathsf{A}\Phi_+(X,Y) \cup \mathsf{A}SS(X,Y)$
4. *For every* $M \in S_{\mathsf{A}}(X)$, *the quotient map* Q_M *belongs to* $\mathsf{A}SS$

Proof. (1) $\Longrightarrow$ (2) Assume that X is HAI and let $M, N \in S_{\mathsf{A}}(X)$. From Proposition 1 we obtain $\mathrm{dist}(S_M, S_N) = 0$; that is, given $\varepsilon > 0$, there exist $u \in S_M$ and $v \in S_N$ such that $\|u - v\| < \varepsilon$. Then

$$\|Tu\| - \|Tv\| \leq \|T(u-v)\| \leq \varepsilon\|T\|.$$

Consequently

$$j(TJ_M) \leq \|Tu\| \leq \|Tv\| + \varepsilon\|T\| \leq \|TJ_N\| + \varepsilon\|T\|.$$

Since $\varepsilon > 0$ is arbitrary we obtain $j(TJ_M) \leq \|TJ_N\|$, for every $M, N \in S_{\mathsf{A}}(X)$, hence $sj_{\mathsf{A}}(T) \leq in_{\mathsf{A}}(T)$.

(2) $\Longrightarrow$ (3) If $T \notin \mathsf{A}\Phi_+$, then $in_{\mathsf{A}}(T) = 0$, hence $sj_{\mathsf{A}}(T) = 0$ and we have $T \in \mathsf{A}SS$.

(3) $\Longrightarrow$ (4) Let $M \in S_{\mathsf{A}}(X)$. As $Q_M \notin \mathsf{A}\Phi_+$ we have $Q_M \in \mathsf{A}SS$.

(4) $\Longrightarrow$ (1) Let $M, N \in S_{\mathsf{A}}(X)$. Since $Q_M \in \mathsf{A}SS$, $Q_M J_N$ is not an isomorphism. By Proposition 1 we have that X is HAI. □

We say that two Banach spaces X and Y are *totally incomparable* [15] if no infinite dimensional subspace of X is isomorphic to a subspace of Y. Given a class $\mathcal{C}$ of Banach spaces, the class of incomparability $\mathcal{C}_i$ was defined in [3] as follows:

$$\mathcal{C}_i := \{X : X \text{ is totally incomparable with every } Y \in \mathcal{C}\}.$$

The class $\mathcal{C}_i$ is a space ideal. Moreover it is not difficult to see that $X \in \mathcal{C}_{ii}$ if and only if X has no infinite dimensional subspace in $\mathcal{C}_i$, and that $\mathcal{C}_{iii} = \mathcal{C}_i$.

In the case $\mathsf{A} = \mathsf{A}_{ii}$ the class $\mathsf{A}\Phi_+$ was studied in [8] and the class $\mathsf{A}SS$ was studied in [4], [9], [2]; see also [1, Section 4.2].

Theorem 2. *Let $\mathsf{A} = \mathsf{A}_{ii}$ be a space ideal. Suppose that $S_\mathsf{A}(X) \neq \emptyset$. Then*

$$\mathsf{A}\Phi_+(X,Y) \cap \mathsf{A}SS(X,Y) = \emptyset$$

for every Banach space Y. Moreover, if X is a HAI space, then the union

$$L(X,Y) = \mathsf{A}\Phi_+(X,Y) \cup \mathsf{A}SS(X,Y)$$

is disjoint.

Proof. Let $M \in S_\mathsf{A}(X)$. Since $\mathsf{A} = \mathsf{A}_{ii}$, there exists an infinite dimensional subspace N of M such that $N \in \mathsf{A}_i$. Since $S_\mathsf{A}(N) = S_\mathsf{F}(N)$, $sj_\mathsf{A}(TJ_N) = sj_\mathsf{F}(TJ_N)$.

Let $T \in L(X,Y)$. Suppose that $T \in \mathsf{A}SS$; i.e., $sj_\mathsf{A}(T) = 0$. For every $M \in S_\mathsf{A}(X)$, we take the subspace N introduced in the previous paragraph. Then $sj_\mathsf{A}(TJ_N) = sj_\mathsf{F}(TJ_N) = 0$, so TJ_N is a strictly singular operator. Thus, for every $\varepsilon > 0$, there exists an infinite dimensional subspace P of N such that $\|TJ_P\| < \varepsilon$ [13, 2.c.4]. Since $P \notin \mathsf{A}$, we have $in_\mathsf{A}(T) = 0$, hence $T \notin \mathsf{A}\Phi_+$. □

Proposition 2. *Let $\mathsf{A} = \mathsf{A}_{ii}$ be a space ideal. Suppose that X is a HAI space and $S_\mathsf{A}(X) \neq \emptyset$. Let $T \in L(X,Y)$. Then for every $M \in S_\mathsf{A}(X)$,*

$$in_\mathsf{A}(TJ_M) = in_\mathsf{A}(T) \quad \text{and} \quad sj_\mathsf{A}(TJ_M) = sj_\mathsf{A}(T).$$

Thus $in_\mathsf{A}(TJ_M)$ and $sj_\mathsf{A}(TJ_M)$ are constant for $M \in S_\mathsf{A}(X)$.

Proof. Let $M \in S_{\mathsf{A}}(X)$. Basically we follow the proof of [7, Lemma 3]. As in [7, Lemma 1] we can prove that for each $P \in S_{\mathsf{A}}(X)$, there exist $U \in S_{\mathsf{A}}(P)$ and a strictly singular operator $S : U \longrightarrow X$ such that

$$J_U + S : x \in U \longrightarrow x + Sx \in M$$

defines an isomorphism onto $N := (J_U + S)U \in S_{\mathsf{A}}(M)$. Note that the hypothesis $\mathsf{A} = \mathsf{A}_{ii}$ allows us to choose $U \in \mathsf{A}_i$.

Obviously, $in_{\mathsf{A}}(TJ_M) \geq in_{\mathsf{A}}(T)$ and $sj_{\mathsf{A}}(TJ_M) \leq sj_{\mathsf{A}}(T)$. Thus it is enough to prove that for each $P \in S_{\mathsf{A}}(X)$ and each $\varepsilon > 0$

$$in_{\mathsf{A}}(TJ_M) \leq \|TJ_P\| + \varepsilon \quad \text{and} \quad j(TJ_P) - \varepsilon \leq sj_{\mathsf{A}}(TJ_M).$$

In order to show the first inequality, note that, for any $\varepsilon' > 0$, we can choose U so that $\|S\| < \varepsilon'$ and $\|(J_U + S)^{-1}\| < 1 + \varepsilon'$. Since $TJ_N = T(J_U + S)(J_U + S)^{-1}$, we obtain

$$\begin{aligned} in_{\mathsf{A}}(TJ_M) &\leq \|TJ_N\| \leq \|TJ_U + TS\| \, \|(J_U + S)^{-1}\| \\ &\leq (\|TJ_U\| + \varepsilon'\|T\|)(1 + \varepsilon') \leq \|TJ_P\| + \varepsilon'(2 + \varepsilon')\|T\|. \end{aligned}$$

For the second inequality, we choose U verifying $\|S\| < \varepsilon'$ and $\|J_U + S\|^{-1} = j((J_U + S)^{-1}) \geq 1 - \varepsilon'$. As $TJ_N = T(J_U + S)(J_U + S)^{-1}$, we have

$$\begin{aligned} sj_{\mathsf{A}}(TJ_M) &\geq j(TJ_N) \geq j(TJ_U + TS) \, j((J_U + S)^{-1}) \\ &\geq (j(TJ_U) - \|TS\|)(1 - \varepsilon') \geq j(TJ_P) - \varepsilon'(2 - \varepsilon')\|T\|. \quad \square \end{aligned}$$

Remark 4. In the case $\mathsf{A} = \mathsf{A}_{ii}$, the components $\mathsf{A}\Phi_+(X, Y)$ are open. Moreover, the class $\mathsf{A}\Phi_+$ is stable under taking products: $T \in \mathsf{A}\Phi_+(X, Y)$ and $S \in \mathsf{A}\Phi_+(Y, Z)$ imply $ST \in \mathsf{A}\Phi_+(X, Z)$. Analogously, in this case, the class $\mathsf{A}SS$ has closed components and it is an operator ideal [8], [9]

Proposition 3. *Let $\mathsf{A} = \mathsf{A}_{ii}$ be a space ideal and let X be a* HAI *space. For every subspace M of X, either $M \in \mathsf{A}$ or M contains a subspace $N \notin \mathsf{A}$ which is a HI space.*

Proof. Let X be a HAI space and let M be a subspace of X. Suppose that $M \notin \mathsf{A}$. Then M contains an infinite dimensional subspace $N \in \mathsf{A}_i$. Let us see that N is HI. If $N = N_1 \oplus N_2$, then $N_1 \in \mathsf{A}$ or $N_2 \in \mathsf{A}$. Since $N_1, N_2 \in \mathsf{A}_i$, we obtain that N_1 or N_2 is finite dimensional. Thus N is HI. $\square$

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(Received November 8, 2002; revised August 5, 2003)