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On two realizability questions concerning strongly connected Moore automata

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Abstract. How is the simultaneous occurrence of some phenomena (namely: an automorphism, a partial quasi-isomorphism, indistinguishable state pairs of one or another type, as they are defined in [4]) possible in a strongly connected automaton? Two questions of this character are raised and solved in the paper.

1. Introduction

§ **1**. This paper is a continuation of the article [4] which is devoted to examining the causes of non-simplicity of strongly connected Moore automata. The concepts of partial quasi-isomorphism and three types of indistinguishable state pairs have been introduced in [4]; in addition, the notion of automorphism too plays an essential role in these considerations.

Among the papers (earlier than [4]) dealing with the behaviour and simplicity of Moore automata, we refer to the works [1], [2], [3]. We mention that the notions of endomorphism semigroup, automorphism group (and other structures) associated to an automaton are introduced in the article [7] of Peák in a detailed manner, together with the listing of a number of important open questions. The recent publication [5] continues [4] in another direction than the present paper.

Sixteen cases are imaginable, how the existence or lack of a nontrivial automorphism, a partial quasi-isomorphism, indistinguishable state pairs of type (II) or (III) can combine with each other. We show in Chapter II that seven cases are impossible and any of the remaining nine cases is realizable by strongly connected automata (even if automata without indistinguishable pairs of type (I) are considered only).

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If one takes three indistinguishable states, three state pairs can be formed from the triplet. We ask in Chapter III, how the types of the three pairs influence each other. We get that, out of ten cases, seven are possible.

The sufficiency parts of our theorems are proved by analyzing several examples. Sometimes we refer to automata contained already in the article [4] — common with I. Babcsányi —, and a further example is due to A. Nagy; we are indebted to our colleagues.

§ **2**. Let $\mathbf{A} = (A, X, Y, \delta, \lambda)$ be a finite Moore automaton where A, X, Y are the sets of states, input symbols, output symbols (respectively), $\delta : (A \times X) \to A$ is the transition function and $\lambda : A \to Y$ is the output function. The terminology to be introduced is the same as the terminology used in [2], [4].

$H_{a,b}$ is meant as the set of input words p satisfying $\delta(a,p) \neq \delta(b,p)$ where (a,b) is an unordered pair of states of $\mathbf{A}$. A state pair (a,b) is said to be of type (I) if $H_{a,b}$ is finite, of type (II) if $H_{a,b}$ contains every input word, of type (III) in the complementary case.

$\mathbf{A}$ is called a *strongly connected* automaton if to any ordered pair (a,b) (where $a \in A,\ b \in A$) there is an input word p such that $\delta(a,p) = b$.

A mapping α of the state set A into itself is called an *endomorphism* if

$$\alpha(\delta(a,x)) = \delta(\alpha(a),x), \quad \lambda(a) = \lambda(\alpha(a))$$

are fulfilled whenever a belongs to A and x belongs to X. If α is an endomorphism and a bijective mapping on A, then we say that α is an *automorphism.* It was shown by Oehmke that each endomorphism of a strongly connected automaton is an automorphism (see e.g. [4], Proposition 4).

A bijective mapping $\gamma : J \to K$ is called a *partial quasi-isomorphism* if J, K are disjoint subsets of the state set of an automaton such that the following three requirements are fulfilled:

(1) J is a strongly connected set and $|J| \geq 2$,

(2) we have

$$\delta(a,x) \in J \ \&\ \delta(\gamma(a),x) \in K \ \&\ \gamma(\delta(a,x)) = \delta(\gamma(a),x)$$

whenever $a \in J$, $x \in X$ such that $\delta(a,x) \neq \delta(\gamma(a),x)$, and

(3) $\lambda(a) = \lambda(\gamma(a))$ whenever $a \in J$.

Now we again study unordered state pairs (a,b). We say that (a,b) is a *proper* pair if $a \neq b$. The state pairs of type (II) and (III) are necessarily proper. A pair (a,b) is called *indistinguishable* if $\lambda(\delta(a,p)) = \lambda(\delta(b,p))$ is true for every input word p.

Consider six assertions defined for an automaton $\mathbf{A}$:

(b–1) $\mathbf{A}$ has an indistinguishable proper state pair of type (I).

(b–2) $\mathbf{A}$ has an indistinguishable state pair of type (II).

(b–3) $\mathbf{A}$ has an indistinguishable state pair of type (III).

(c–1)
$$(\forall x)[\delta(a,x)=\delta(b,x)]\ \&\ \lambda(a)=\lambda(b)$$
holds for some proper state pair (a,b) of **A** (where $x \in X$).

(c–2) **A** has a nontrivial automorphism.

(c–3) **A** has a partial quasi-isomorphism.

The following statement has been proved in [4] (as Proposition 2):

Proposition A. *The assertions* (b–1) *and* (c–1) *are equivalent.*

II. Automorphisms, partial quasi-isomorphisms and indistinguishability

§ **3**. In Chapter II we consider strongly connected finite Moore automata **A**. Our aim is to study how the four assertions (b–2), (b–3), (c–2), (c–3) can be combined with each other if **A** does not fulfil (b–1). As usual, we write e.g. $\overline{\text{(b–2)}}$ for the negation of (b–2), and it is expressed by $\widetilde{\text{(b–2)}}$ that we leave indetermined whether (b–2) is negated or not.

Consider a conjunction of form

$$\widetilde{\text{(b–2)}}\&\widetilde{\text{(b–3)}}\&\widetilde{\text{(c–2)}}\&\widetilde{\text{(c–3)}}. \tag{3.1}$$

We call (3.1) *realizable* if there exists a strongly connected automaton **A** such that

A satisfies the assertions being non-negated (3.1),

A does not satisfy the assertions being negated in (3.1), and

A does not satisfy (b–1).

Later all the sixteen conjunctions of form (3.1) will be specified, see (3.2) and (4.1).

The next two results have been verified in [4] (see Propositions 5, 12):

Proposition B. (c–2) *implies* (b–2).

Proposition C. (c–3) *implies* (b–1)$\vee$(b–2)$\vee$(b–3).

We show an improvement of Proposition C:

Proposition 1. (c–3) *implies* (b–1)$\vee$(b–3).

Proof. Suppose that an automaton **A** satisfies (c–3), i.e., **A** has a partial quasi-isomorphism γ. The definition domain J of γ is properly included in A. Since **A** is strongly connected, there exists a state $a(\in J)$ and an $x(\in X)$ such that $\delta(a,x) \notin J$. The definition of partial quasi-isomorphism implies $\delta(a,x)=\delta(\gamma(a),x)$. Thus the type of the pair $a, \gamma(a)$ is either (I) or (III). This pair is indistinguishable, hence (b–1) or (b–3) is valid for **A**.

An easy consequence of Propositions B and 1 is:

Corollary 1. *The seven conjunctions*

$$
(3.2)\qquad
\left\{
\begin{array}{l}
\overline{\text{(b–2)}} \;\&\; \overline{\text{(b–3)}} \;\&\; \overline{\text{(c–2)}} \;\&\; \text{(c–3)},\\
\overline{\text{(b–2)}} \;\&\; \overline{\text{(b–3)}} \;\&\; \text{(c–2)} \;\&\; \overline{\text{(c–3)}},\\
\overline{\text{(b–2)}} \;\&\; \overline{\text{(b–3)}} \;\&\; \text{(c–2)} \;\&\; \text{(c–3)},\\
\overline{\text{(b–2)}} \;\&\; \text{(b–3)} \;\&\; \text{(c–2)} \;\&\; \overline{\text{(c–3)}},\\
\overline{\text{(b–2)}} \;\&\; \text{(b–3)} \;\&\; \text{(c–2)} \;\&\; \text{(c–3)},\\
\text{(b–2)} \;\&\; \overline{\text{(b–3)}} \;\&\; \overline{\text{(c–2)}} \;\&\; \text{(c–3)},\\
\text{(b–2)} \;\&\; \overline{\text{(b–3)}} \;\&\; \text{(c–2)} \;\&\; \text{(c–3)}
\end{array}
\right.
$$

are not realizable.

§ **4. Theorem 1.** *A conjunction of form* (3.1) *is realizable if and only if it does not occur in* (3.2).

The necessity part of Theorem 1 was already stated as Corollary 1. For proving the sufficiency, we are going to show that there exist nine automata $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_9$ (such that (b–1) is false for them) realizing the conjuctions

$$
(4.1)\qquad
\left\{
\begin{array}{l}
\overline{\text{(b–2)}} \;\&\; \overline{\text{(b–3)}} \;\&\; \overline{\text{(c–2)}} \;\&\; \overline{\text{(c–3)}},\\
\text{(b–2)} \;\&\; \overline{\text{(b–3)}} \;\&\; \text{(c–2)} \;\&\; \overline{\text{(c–3)}},\\
\overline{\text{(b–2)}} \;\&\; \text{(b–3)} \;\&\; \overline{\text{(c–2)}} \;\&\; \text{(c–3)},\\
\text{(b–2)} \;\&\; \text{(b–3)} \;\&\; \overline{\text{(c–2)}} \;\&\; \overline{\text{(c–3)}},\\
\overline{\text{(b–2)}} \;\&\; \text{(b–3)} \;\&\; \overline{\text{(c–2)}} \;\&\; \overline{\text{(c–3)}},\\
\text{(b–2)} \;\&\; \overline{\text{(b–3)}} \;\&\; \overline{\text{(c–2)}} \;\&\; \overline{\text{(c–3)}},\\
\text{(b–2)} \;\&\; \text{(b–3)} \;\&\; \text{(c–2)} \;\&\; \overline{\text{(c–3)}},\\
\text{(b–2)} \;\&\; \text{(b–3)} \;\&\; \overline{\text{(c–2)}} \;\&\; \text{(c–3)},\\
\text{(b–2)} \;\&\; \text{(b–3)} \;\&\; \text{(c–2)} \;\&\; \text{(c–3)},
\end{array}
\right.
$$

respectively.

Example 1. $\mathbf{A}_1 = (A, X, Y, \delta, \lambda)$ is the trivial automaton: $A = \{1\}$, $X = \{x\}$, $Y = \{y\}$, $\delta(1, x) = 1$, $\lambda(1) = y$. We note that every simple automaton (among these, every automaton with a bijective λ) realizes the first conjunction of (4.1).

We refer to § 3 of the previous paper [4] concerning automata that realize the three following conjuctions in (4.1). Let $\mathbf{A}_2$ $\mathbf{A}_3$ $\mathbf{A}_4$ be the automata constructed in [4] as Examples 2, 3, 7, respectively. ($|X| = 2$ for any of them. The value of $|A|$ is 6, 6, 5; the value of $|Y|$ is 2, 3, 2, resp.)

Example 2. Put $A = \{1,2,3\}$, $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$; let δ, λ be defined by *Table 1* (see *Fig. 1*). This automaton $\mathbf{A}_5$ has neither a nontrivial automorphism nor a partial quasi-isomorphism. The type of the single indistinguishable pair (2, 3) is (III).

Figure 1.

i	$\delta(i, x_1)$	$\delta(i, x_2)$	$\lambda(i)$
1	2	3	y_1
2	1	3	y_2
3	1	2	y_2

Table 1.

Example 3. Put $A = \{1,2,3,4,5,6\}$, $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$; let δ, λ be defined by *Table 2* (see *Fig. 2*). This automaton $\mathbf{A}_6$ — constructed by A. Nagy — again without automorphisms (except the identical one) and partial quasi-isomorphisms. The classes of $\pi_{\max}$ are $\{1,2,3\}$ and $\{4,5,6\}$. It can be seen easily that all the six indistinguishable state pairs are of type (II).

Example 4. Put $A = \{1,2,3,4,5,6\}$, $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$; let δ, λ be defined by *Table 3* (see *Fig. 3*). The mapping

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 6 & 5 \end{pmatrix}$$

Figure 2.

i	$\delta(i,x_1)$	$\delta(i,x_2)$	$\lambda(i)$
1	5	1	y_1
2	4	3	y_1
3	6	2	y_1
4	4	1	y_2
5	5	2	y_2
6	6	3	y_2

Table 2.

is an automorphism of this automaton $\mathbf{A}_7$. There is no partial quasi-isomorphism. The classes of $\pi_{\max}$ are $\{1,2,3,4\}$ and $\{5,6\}$. It follows from Proposition 5 of [4] that the indistinguishable pairs (1, 4), (2, 3), (5, 6) belong to the type (II). A discussion shows that each of the remaining four indistinguishable pairs is of type (III).

Example 5. Put $A=\{1,2,3,4,5,6,7\}$, $X=\{x_1,x_2,x_3\}$, $Y=\{y_1,y_2\}$; let δ,λ be defined by *Table 4* (see *Fig. 4*). This automaton $\mathbf{A}_8$ has no other automorphism than the trivial one, and

$$\begin{pmatrix} 2 & 6 \\ 3 & 7 \end{pmatrix}$$

is a partial quasi-isomorphism. The classes of $\pi_{\max}$ are $\{1,2,3,6,7\}$ and

Figure 3.

i	$\delta(i, x_1)$	$\delta(i, x_2)$	$\delta(i, x_3)$	$\lambda(i)$
1	5	2	5	y_1
2	5	1	6	y_1
3	6	4	5	y_1
4	6	3	6	y_1
5	6	6	1	y_2
6	5	5	4	y_2

Table 3.

$\{4, 5\}$. The type of (1, 2), (1, 3), (1, 6), (1, 7), (4, 5) is (II); the remaining six indistinguishable pairs are of type (III).

Example 6. Put $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$; let δ, λ be defined by *Table 5* (see *Fig. 5*). The mapping

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 7 & 8 & 9 & 10 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

is an automorphism of this automaton $\mathbf{A}_9$, and

$$\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$$

is a partial quasi-isomorphism in it. The classes of $\pi_{\max}$ are $\{1, 6\}$, $\{2, 5, 7, 10\}$, $\{3, 4, 8, 9\}$. Among the thirteen indistinguishable pairs, (2, 5), (3, 4), (7, 10), (8, 9) are of type (III), the type of the remaining nine pairs is (II).

Figure 4.

i	$\delta(i, x_1)$	$\delta(i, x_2)$	$\delta(i, x_3)$	$\lambda(i)$
1	1	1	4	y_1
2	2	6	5	y_1
3	3	7	5	y_1
4	5	1	1	y_2
5	4	2	3	y_2
6	6	2	5	y_1
7	7	3	5	y_1

Table 4.

III. Indistinguishable pairs in state triplets

§ **5**. Consider a triplet states a, b, c of an automaton. We can form three pairs (a, b), (b, c), (a, c) from these states, and each pair belongs to one of the types (I), (II), (III). (The ordering of pairs and triplets is indifferent in these considerations.) To any triplet of states we assign in

Figure 5.

this manner one of the ten possible triplets of types:

$$
(5.1) \qquad \begin{cases} \text{(I, I, I), (I, I, II), (I, I, III), (I, II, II),} \\ \text{(I, II, III), (I, III, III), (II, II, II),} \\ \text{(II, II, III), (II, III, III), (III, III, III).} \end{cases}
$$

Let $\mathfrak{T}$ be one of the triplets occurring in (5.1). We say that $\mathfrak{T}$ is *realizable* if there exist a strongly connected Moore automaton **A** and three states a, b, c in it such that

(1) a, b, c are pairwise different and indistinguishable, and

(2) the type triplet, assigned to a, b, c, equals $\mathfrak{T}$.

Lemma 1. *If a, b, c are arbitrary states of an automaton, then* $H_{a,b} \cup H_{b,c} \supseteq H_{a,c}$.

Proof. Suppose that an input word p belongs neither to $H_{a,b}$ nor to $H_{b,c}$. This means $\delta(a,p) = \delta(b,p)$ and $\delta(b,p) = \delta(c,p)$. Consequently $\delta(a,p) = \delta(c,p)$, hence $p \notin H_{a,c}$.

Proposition 2. *Let a, b, c be three states of an automaton. If the pairs (a,b), (b,c) belong to the type (I), then (a,c) too is of type (I).*

i	$\delta(i,x_1)$	$\delta(i,x_2)$	$\delta(i,x_3)$	$\lambda(i)$
1	2	5	6	y_1
2	3	3	2	y_2
3	2	1	3	y_1
4	5	1	4	y_1
5	4	4	5	y_2
6	7	10	1	y_1
7	8	8	7	y_2
8	7	6	8	y_1
9	10	6	9	y_1
10	9	9	10	y_2

Table 5.

PROOF. $H_{a,b}$ and $H_{b,c}$ are finite by the assumption. Thus $|H_{a,b} \cup H_{b,c}| < \infty$ and (by Lemma 1) $|H_{a,c}| < \infty$.

Proposition 3. *Let a, b, c be as in Proposition 2. If (a, b) belongs to the type* (I) *and (b, c) belongs to the type* (III), *then the type of (a, c) is not* (II).

PROOF. The supposition implies that $|H_{a,b}| < \infty$ and $H_{b,c}$ is properly included in the set $F(X)$ consisting of all input words. It is known that $|F(X) - H_{b,c}| = \infty$ (because $px \notin H_{b,c}$ if $p \notin H_{b,c}$; cf. [4], § 1). Therefore

$$F(X) \supset H_{a,b} \cup H_{b,c} \supseteq H_{a,c}$$

where the second inclusion follows from Lemma 1.

§ **6. Theorem 2.** *Let $\mathfrak{T}$ be a triplet occurring in (5.1). $\mathfrak{T}$ is realizable if and only if $\mathfrak{T}$ differs from* (I, I, II), (I, I, III), (I, II, III).

The necessity part of Theorem 2 follows immediately from Propositions 2 and 3. We verify the sufficiency statement of the theorem by observing some automata in which the seven types in question really exist.

Let first the examples $\mathbf{A}_6$, $\mathbf{A}_7$, $\mathbf{A}_8$, exposed in § 4, be studied. Both indistinguishable state triplets in $\mathbf{A}_6$ show the realizability of (II, II, II). The triplet 1, 2, 3 in $\mathbf{A}_7$ guarantees that (II, III, III) is realizable. If we consider the triplets 1, 2, 3 and 2, 3, 6 in $\mathbf{A}_8$, we get examples for the realizability of the types (II, II, III) and (III, III, III), respectively. (We note that a triplet realizing (II, II, III) can be found also in an automaton having five states only; see the states 3, 4, 5 in $\mathbf{A}_4$, i.e., Example 7 of [4].)

For verifying the realizability of the type (I, II, II), recall Example 5 in § 3 of [4], and consider the states 1, 2, 4 of this automaton (fulfilling $|A| = 5$, $|X| = 3$, $|Y| = 2$).

Finally we have to prove that the types (I, I, I) and (I, III, III) are realizable.

Example 7. Put $A = \{1,\ 2,\ 3,\ 4\}$, $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$; let δ, λ be defined by *Table 6* (see *Fig. 6*). The classes of $\pi_{\max}$ are $\{1,\ 2,\ 3\}$ and $\{4\}$. The three-element class realizes (I, I, I).

Figure 6.

i	$\delta(i, x_1)$	$\delta(i, x_2)$	$\delta(i, x_3)$	$\lambda(i)$
1	4	4	4	y_1
2	4	4	4	y_1
3	4	4	4	y_1
4	1	2	3	y_2

Table 6.

Figure 7.

i	$\delta(i,x_1)$	$\delta(i,x_2)$	$\delta(i,x_3)$	$\delta(i,x_4)$	$\lambda(i)$
1	1	5	5	5	y_1
2	1	5	5	5	y_1
3	4	5	5	5	y_1
4	3	5	5	5	y_1
5	1	2	3	4	y_2

Table 7.

Example 8. Put $A = \{1,2,3,4,5\}$, $X = \{x_1,x_2,x_3,x_4\}$, $Y = \{y_1,y_2\}$; let δ, λ be defined by *Table 7* (see *Fig. 7*). The classes of $\pi_{\max}$ are $\{1,\ 2,\ 3,\ 4\}$ and $\{5\}$. The state triplets 1, 2, 3 and 1, 2, 4 realize (I, III, III). (The type of 1, 3, 4 and 2, 3, 4 is (III, III, III).)

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