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Characterizing left centralizers by their action on a polynomial

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Abstract. On algebras satisfying certain d-freeness condition we characterize left centralizers by their action on a fixed polynomial in noncommuting variables. The case of the polynomial X^n is studied in a greater detail.

1. Introduction

An additive map φ from a ring R into itself is called a *left centralizer* if $\varphi(xy) = \varphi(x)y$ for all $x, y \in R$. If R has a unity 1 then taking $x = 1$ we see that the left multiplications $y \mapsto ay$ are the only left centralizers on R. In the non-unital case there are often other examples.

In [9] Zalar showed that on a 2-torsionfree semiprime ring R every additive map φ satisfying $\varphi\left(x^2\right) = \varphi(x)x$, $x \in R$, is already a left centralizer. Molnár [7] obtained the same conclusion under the slightly milder assumption that $\varphi\left(x^3\right) = \varphi(x)x^2$, $x \in R$, in a semisimple H^*-algebra R. Molnar also mentioned that according to the formulation of his theorem this problem should be studied in a purely ring theoretical context. These results of Zalar and Molnár were motivated by the problem of representing quadratic forms by bilinear forms.

It is our aim in this paper to consider a considerably more general condition where an additive map acts as a left centralizer on an arbitrary

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multilinear polynomial in noncommuting indeterminates. In particular, we shall see that an additive map $\varphi : R \to R$ satisfying $\varphi(x^n) = \varphi(x)x^{n-1}$, $x \in R$, where $n \geq 2$ is an arbitrary fixed integer and R is a prime ring with $\mathrm{char}(R) = 0$ or $\mathrm{char}(R) \geq n$, is a left centralizer.

Our results are obtained as applications of the theory of *functional identities*. In particular we shall use some ideas from the paper of Beidar and Fong [3] where bijective additive maps preserving a fixed polynomial are characterized.

2. Preliminaries

The theory of functional identities considers set-theoretic maps on rings that satisfy some identical relations. When treating such relations one usually concludes that the form of the maps involved can be described, unless the ring is very special. We refer the reader to [5] for an introductory account on functional identities.

Let R be a ring and let S be a nonempty subset of R. The concept of *d-freeness* of S, introduced in [2], will play an important role in this paper (here d is a positive integer). We omit stating the exact definition since it will not be used in its full generality. Instead we just point out a special property satisfied by d-free sets: If $F_1, F_2, \ldots, F_d$ are arbitrary maps from $S^d = S \times S \times \ldots \times S$ to R and S is a d-free subset of R, then

$$\begin{gathered}\sum_{i=1}^{d} F_i(x_1 \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)x_i = 0 \quad \text{for all } x_1, \ldots, x_d \in S \\ \Longrightarrow F_1 = F_2 = \cdots = F_d = 0.\end{gathered} \tag{1}$$

In fact, in our main theorem (Theorem 3.2) we could replace the assumption on d-freeness by this condition which is actually weaker (cf. [6]).

Under some natural assumptions one can show that various subsets (such as ideals, Lie ideals, the sets of symmetric or skew elements in a ring with involution) of certain types of rings are d-free. We refer the reader to [1], [2] for results of this kind, which of course also show in which concrete situations Theorem 3.2 is applicable. Let us mention specifically only one result that we shall really need: A prime ring R is a d-free subset of its maximal right ring of quotients, unless R satisfies the standard polynomial identity of degree less than $2d$ [2, Theorem 2.4]. So, in particular such ring R satisfies (1) with $S = R$ and $F_i : R^d \to R$. Using this fact we can now easily establish the following result (which will be in this paper in fact used only in the PI case).

Lemma 2.1. *Let R be a prime ring and let $F : R \to R$ be an additive map. If there exists a positive integer n such that $F(x)x^n = 0$ for all $x \in R$, then $F = 0$.*

PROOF. Suppose first that R is not a PI ring. A complete linearization of $F(x)x^n = 0$ gives $\sum_{\pi \in \mathcal{S}_{n+1}} F(x_{\pi(1)})x_{\pi(2)} \dots x_{\pi(n+1)} = 0$ for all $x_1, \dots, x_{n+1} \in R$. Applying (1) successively it follows easily that $F = 0$.

So assume that R is PI. As it is well-known, then R has a nonzero center [8]. Let c be a nonzero central element. Since c is not a zero divisor, $F(c)c^n = 0$ implies $F(c) = 0$. Accordingly, for any $x \in R$ we have $0 = F(x+c)(x+c)^n = F(x)(x+c)^n$, and hence also $F(x)(x+c)^n x^{n-1} = 0$. Since $F(x)x^n = 0$ this identity reduces to $F(x)c^n x^{n-1} = 0$, and therefore $F(x)x^{n-1} = 0$ for every $x \in R$. Repeating this argument we get $F = 0$. $\square$

3. The results

It is more convenient to state our first results in the setting of algebras over commutative rings. So let Φ be a commutative ring with unity, and let A be an algebra over Φ. Further, let $n \geq 2$ and let

$$p(X_1, X_2, \dots, X_n) = \sum_{\pi \in \mathcal{S}_n} \alpha_\pi X_{\pi(1)} X_{\pi(2)} \dots X_{\pi(n)}, \quad \alpha_\pi \in \Phi,\ \alpha_e = 1$$

be a fixed multilinear polynomial in noncommuting indeterminates X_i over Φ. Further, let $\mathcal{L}$ be a subset of A closed under p, i.e. $p(\overline{x}_n) \in \mathcal{L}$ for all $x_1, \dots, x_n \in \mathcal{L}$, where $\overline{x}_n = (x_1, x_2, \dots, x_n)$. We shall consider a map $\varphi : \mathcal{L} \to A$ satisfying

$$\varphi\left(p\left(\overline{x}_n\right)\right) = \sum_{\pi \in \mathcal{S}_n} \alpha_\pi \varphi\left(x_{\pi(1)}\right) x_{\pi(2)} \dots x_{\pi(n)} \tag{2}$$

for all $x_1, \dots, x_n \in \mathcal{L}$. Of course, every left centralizer satisfies (2). Our goal is to show that under certain assumptions these are in fact the only maps with this property. In the first and crucial step of the proof we derive a functional identity from (2). It should be mentioned that the idea of considering the expression $[p(\overline{x}_n), p(\overline{y}_n)]$ in its proof is taken from [3].

Lemma 3.1. *Let $\mathcal{L}$ be a Lie subalgebra of A closed under p. If $\varphi : \mathcal{L} \to A$ is an additive map satisfying (2) then*

$$\sum_{\pi \in \mathcal{S}_n} \sum_{\sigma \in \mathcal{S}_n} \alpha_\pi \alpha_\sigma F\left(x_{\pi(1)}, y_{\sigma(1)}\right) \left[x_{\pi(2)} \dots x_{\pi(n)}, y_{\sigma(2)} \dots y_{\sigma(n)}\right] = 0 \tag{3}$$

for all $\overline{x}_n, \overline{y}_n \in \mathcal{L}^n$, where $F(x, y) = \varphi([x, y]) - \varphi(x) y + \varphi(y) x$.

Proof. Note that for any $a \in A$ and $\overline{x}_n \in \mathcal{L}^n$ we have

$$[p(\overline{x}_n), a] = \sum_{i=1}^{n} p(x_1, \dots, x_{i-1}, [x_i, a], x_{i+1}, \dots, x_n).$$

Accordingly, by (2) we see that when expanding

$$\varphi([p(\overline{x}_n), a]) = \sum_{i=1}^{n} \varphi\left(p(x_1, \dots, x_{i-1}, [x_i, a], x_{i+1}, \dots, x_n)\right)$$

we obtain $n \cdot n!$ summands, where those corresponding to $\pi \in \mathcal{S}_n$ are

$$\begin{aligned} &\alpha_\pi\big(\varphi([x_{\pi(1)}, a]) x_{\pi(2)} \dots x_{\pi(n)} + \varphi(x_{\pi(1)})[x_{\pi(2)}, a] x_{\pi(3)} \dots x_{\pi(n)} \\ &\qquad + \dots + \varphi(x_{\pi(1)}) x_{\pi(2)} \dots x_{\pi(n-1)}[x_{\pi(n)}, a]\big) \\ &= \alpha_\pi\big(\varphi([x_{\pi(1)}, a]) x_{\pi(2)} \dots x_{\pi(n)} + \varphi(x_{\pi(1)})[x_{\pi(2)} \dots x_{\pi(n)}, a]\big). \end{aligned}$$

Therefore

$$\begin{aligned} \varphi([p(\overline{x}_n), a]) = &\sum_{\pi \in \mathcal{S}_n} \alpha_\pi \varphi\left(\left[x_{\pi(1)}, a\right]\right) x_{\pi(2)} \dots x_{\pi(n)} \\ &+ \sum_{\pi \in \mathcal{S}_n} \alpha_\pi \varphi\left(x_{\pi(1)}\right) \left[x_{\pi(2)} \dots x_{\pi(n)}, a\right]. \end{aligned}$$

In particular

$$\varphi\left(\left[p\left(\overline{x}_n\right), p\left(\overline{y}_n\right)\right]\right) = \sum_{\pi\in\mathcal{S}_n} \alpha_\pi \varphi\left(\left[x_{\pi(1)}, p\left(\overline{y}_n\right)\right]\right) x_{\pi(2)} \dots x_{\pi(n)} + \sum_{\pi\in\mathcal{S}_n} \alpha_\pi \varphi\left(x_{\pi(1)}\right)\left[x_{\pi(2)} \dots x_{\pi(n)}, p\left(\overline{y}_n\right)\right] \tag{4}$$

and

$$\begin{aligned}\varphi\left(\left[x_{\pi(1)}, p\left(\overline{y}_n\right)\right]\right) &= -\varphi([p(\overline{y}_n), x_{\pi(1)}]) \\ &= \sum_{\sigma\in\mathcal{S}_n} \alpha_\sigma \varphi\left(\left[x_{\pi(1)}, y_{\sigma(1)}\right]\right) y_{\sigma(2)} \dots y_{\sigma(n)} \\ &\quad + \sum_{\sigma\in\mathcal{S}_n} \alpha_\sigma \varphi\left(y_{\sigma(1)}\right)\left[x_{\pi(1)}, y_{\sigma(2)} \dots y_{\sigma(n)}\right]\end{aligned}$$

for all $\overline{x}_n, \overline{y}_n \in \mathcal{L}^n$. Using this together with

$$\left[x_{\pi(2)} \dots x_{\pi(n)}, p\left(\overline{y}_n\right)\right] = \sum_{\sigma\in\mathcal{S}_n} \alpha_\sigma \left[x_{\pi(2)} \dots x_{\pi(n)}, y_{\sigma(1)} y_{\sigma(2)} \dots y_{\sigma(n)}\right]$$

in (4) we obtain

$$\begin{aligned}&\varphi\left(\left[p\left(\overline{x}_n\right), p\left(\overline{y}_n\right)\right]\right) \\ &= \sum_{\pi\in\mathcal{S}_n} \sum_{\sigma\in\mathcal{S}_n} \alpha_\pi \alpha_\sigma \varphi\left(\left[x_{\pi(1)}, y_{\sigma(1)}\right]\right) y_{\sigma(2)} \dots y_{\sigma(n)} x_{\pi(2)} \dots x_{\pi(n)} \\ &\quad + \sum_{\pi\in\mathcal{S}_n} \sum_{\sigma\in\mathcal{S}_n} \alpha_\pi \alpha_\sigma \varphi\left(y_{\sigma(1)}\right)\left[x_{\pi(1)}, y_{\sigma(2)} \dots y_{\sigma(n)}\right] x_{\pi(2)} \dots x_{\pi(n)} \\ &\quad + \sum_{\pi\in\mathcal{S}_n} \sum_{\sigma\in\mathcal{S}_n} \alpha_\pi \alpha_\sigma \varphi\left(x_{\pi(1)}\right)\left[x_{\pi(2)} \dots x_{\pi(n)}, y_{\sigma(1)} y_{\sigma(2)} \dots y_{\sigma(n)}\right]\end{aligned} \tag{5}$$

for all $\overline{x}_n, \overline{y}_n \in \mathcal{L}^n$. Since $[p(\overline{x}_n), p(\overline{y}_n)] = -[p(\overline{y}_n), p(\overline{x}_n)]$, we see from (5) (where for convenience we replace the roles of denotations π and σ) that on the other hand

$$\begin{aligned}&\varphi\left(\left[p\left(\overline{x}_n\right), p\left(\overline{y}_n\right)\right]\right) \\ &= \sum_{\pi\in\mathcal{S}_n} \sum_{\sigma\in\mathcal{S}_n} \alpha_\pi \alpha_\sigma \varphi\left(\left[x_{\pi(1)}, y_{\sigma(1)}\right]\right) x_{\pi(2)} \dots x_{\pi(n)} y_{\sigma(2)} \dots y_{\sigma(n)}\end{aligned}$$

$$+\sum_{\pi\in\mathcal{S}_n}\sum_{\sigma\in\mathcal{S}_n}\alpha_\pi\alpha_\sigma\varphi\left(x_{\pi(1)}\right)\left[x_{\pi(2)}\dots x_{\pi(n)},y_{\sigma(1)}\right]y_{\sigma(2)}y_{\sigma(3)}\cdots y_{\sigma(n)}$$

$$+\sum_{\pi\in\mathcal{S}_n}\sum_{\sigma\in\mathcal{S}_n}\alpha_\pi\alpha_\sigma\varphi\left(y_{\sigma(1)}\right)\left[x_{\pi(1)}\dots x_{\pi(n)},y_{\sigma(2)}\cdots y_{\sigma(n)}\right] \tag{6}$$

for all $\overline{x}_n,\overline{y}_n\in\mathcal{L}^n$. Comparing (5) and (6) we obtain the conclusion of the lemma. □

Theorem 3.2. *Let $\mathcal{L}$ be a $2n$-free Lie subalgebra of an algebra A closed under p. If $\varphi:\mathcal{L}\to A$ is an additive map satisfying* (2) *then $\varphi([x,y])=\varphi(x)\,y-\varphi(y)\,x$ for all $x,y\in\mathcal{L}$. Moreover, if $\mathcal{L}=A$ is a 2-torsionfree algebra, then φ is a left centralizer.*

Proof. Writing x_{n+i} instead of y_i in (3) we can express this identity as $\sum_{i=1}^{2n}F_i(x_1\dots,x_{i-1},x_{i+1},\dots,x_{2n})x_i=0$ where, for instance,

$$F_{2n}(x_1,\dots,x_{2n-1})=F(x_1,x_{n+1})x_2\dots x_nx_{n+2}\dots x_{2n-1}+\dots$$

Therefore (cf. (1)) it follows that $F_1=\dots=F_{2n}=0$. Each identity $F_i=0$ can be expressed in a way that (1) can be applied again. Repeating this argument we finally arrive at $F(x,y)=0$ for all $x,y\in\mathcal{L}$, i.e. $\varphi([x,y])=\varphi(x)\,y-\varphi(y)\,x$ for all $x,y\in\mathcal{L}$.

Now suppose that $\mathcal{L}=A$ is 2-torsionfree (i.e. $2a=0$ implies $a=0$ in A). Set $x\circ y=xy+yx$ and note that $[x,y\circ z]+[y,z\circ x]+[z,x\circ y]=0$. Therefore, by what we just proved it follows that

$$\begin{aligned}0&=\varphi\left([x,y\circ z]+[y,z\circ x]+[z,x\circ y]\right)\\&=\varphi(x)\,(y\circ z)-\varphi(y\circ z)\,x+\varphi(y)\,(z\circ x)\\&\quad-\varphi(z\circ x)\,y+\varphi(z)\,(x\circ y)-\varphi(x\circ y)\,z\\&=G(x,y)z+G(y,z)x+G(z,x)y,\end{aligned}$$

where $G(x,y)=\varphi(x)\,y+\varphi(y)\,x-\varphi(x\circ y)$ for all $x,y,z\in A$. Since $2n>3$ it follows that $G(x,y)=0$ for all $x,y\in A$. Therefore

$$2\varphi(xy)=\varphi([x,y]+x\circ y)=\varphi([x,y])+\varphi(x\circ y)=2\varphi(x)\,y$$

for all $x,y\in A$, proving that φ is a left centralizer. □

Theorem 3.3. *Let R be a prime ring and let $\varphi : R \to R$ be an additive map satisfying $\varphi(x^n) = \varphi(x)\, x^{n-1}$ for all $x \in R$, where $n \geq 2$ is a fixed integer. If* $\operatorname{char}(R) = 0$ *or* $\operatorname{char}(R) \geq n$*, then φ is a left centralizer.*

Proof. The complete linearization of $\varphi(x^n) = \varphi(x)\, x^{n-1}$ gives us

$$\varphi\Big(\sum_{\pi \in \mathcal{S}_n} x_{\pi(1)} x_{\pi(2)} \dots x_{\pi(n)}\Big) = \sum_{\pi \in \mathcal{S}_n} \varphi\left(x_{\pi(1)}\right) x_{\pi(2)} \dots x_{\pi(n)} \tag{7}$$

for all $\overline{x}_n \in R^n$. Therefore, we can apply Theorem 3.2 (for the case when $\alpha_\pi = 1$ for each $\pi \in \mathcal{S}_n$ and $\Phi = \mathbb{Z}$) and conclude that φ is indeed a left centralizer unless R is a PI ring (satisfying the standard identity of degree less than $4n$, cf. Section 2).

So we may now assume that R is a PI ring and so it contains a nonzero central element c. Picking any $x \in R$ and setting $x_1 = \dots = x_{n-1} = cx$ and $x_n = x$ in (7) we obtain

$$\begin{aligned}\varphi\left(n! c^{n-1} x^n\right) = (n-1)\,(n-1)!\varphi(cx)\, c^{n-2} x^{n-1}\\ + (n-1)!\varphi(x)\, c^{n-1} x^{n-1}.\end{aligned}$$

On the other hand, setting $x_1 = \dots = x_{n-1} = c$ and $x_n = x^n$ in (7) we get

$$\begin{aligned}\varphi\left(n! c^{n-1} x^n\right) &= (n-1)\,(n-1)!\varphi(c)\, c^{n-2} x^n + (n-1)!\varphi(x^n)\, c^{n-1}\\ &= (n-1)\,(n-1)!\varphi(c)\, c^{n-2} x^n + (n-1)!\varphi(x)\, x^{n-1} c^{n-1}.\end{aligned}$$

Comparing these two identities we obtain

$$(n-1)\,(n-1)!\,(\varphi(cx) - \varphi(c)\, x)\, c^{n-2} x^{n-1} = 0,$$

which clearly yields $(\varphi(cx) - \varphi(c)\, x)\, x^{n-1} = 0$. By Lemma 2.1 it follows that $\varphi(cx) = \varphi(c)\, x$ for all $x \in R$. Now setting $x_1 = \dots = x_{n-1} = x$ and $x_n = c$ in (7) we get

$$\begin{aligned}(n-1)\,(n-1)!\varphi(x)\, c x^{n-2} + (n-1)!\varphi(c)\, x^{n-1} &= \varphi\left(n! c x^{n-1}\right)\\ &= n!\varphi(c)\, x^{n-1}\end{aligned}$$

and hence

$$(n-1)\,(n-1)!\,(\varphi(x)\, c - \varphi(c)\, x)\, x^{n-2} = 0$$

for all $x \in R$. Applying Lemma 2.1 again we see that $\varphi(x)c = \varphi(c)x$ for all $x \in R$. Accordingly, $\varphi(xy)c = \varphi(c)xy = \varphi(x)cy = \varphi(x)yc$ which implies that φ is a left centralizer. □

We remark that every left centralizer on a prime ring R is of the form $x \mapsto qx$ for some q in the right Martindale ring of quotients of R [4, Proposition 2.2.1 (iv)].

We conclude the paper by an example showing that some restrictions concerning char(R) are really necessary in Theorem 3.3.

Example 3.4. Let R be a Galois field of order $n = p^s$ where p is prime and $s \geq 2$. Then $x^{n-1} = 1$ for every $x \neq 0$ in R, and so any additive map φ satisfies $\varphi(x^n) = \varphi(x)x^{n-1}$ for all $x \in R$. However, not every additive map on R is a left centralizer, i.e. is of the form $x \mapsto ax$ with $a \in R$. Namely, there are only n left centralizers on R, while the number of additive (i.e. $\mathbb{Z}_p$-linear) maps on R is equal to n^s since R can be viewed as an s-dimensional vector space over $\mathbb{Z}_p$.

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