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# Approximation by some linear positive operators in polynomial weighted spaces

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**Abstract.** We consider certain linear positive operators $L_n$ in polynomial weighted spaces of functions of one variable and study approximation properties of these operators, including theorems on the degree of approximation.

## 1. Introduction

**1.1.** M. Becker in his paper [1] studied approximation problems for functions $f \in C_p$ and Szasz–Mirakyan operators

$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \qquad x \in R_0 := [0,+\infty), \qquad n \in N := \{1,2,\dots\}, \tag{1}$$

where $C_p$ with some fixed $p \in N_0 := \{0,1,2,\dots\}$ is a polynomial weighted space generated by the weight function

$$w_0(x) := 1, \quad w_p(x) := (1+x^p)^{-1}, \quad \text{if } p \geq 1, \tag{2}$$

i.e., $C_p$ is the set of all real-valued functions $f$, continuous on $R_0$ and such that $w_p f$ is uniformly continuous and bounded on $R_0$. The norm in $C_p$ is

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defined by the formula

$$\|f\|_p \equiv \|f(\cdot)\|_p := \sup_{x\in R_0} w_p(x)\,\|f(x)\|. \tag{3}$$

In [1] there were proved theorems on the degree of approximation of $f \in C_p$ by operators $S_n$ defined by (1). From these theorems it was deduced that

$$\lim_{n\to\infty} S_n(f;x) = f(x) \tag{4}$$

for every $f \in C_p$, $p \in N_0$ and $x \in R_0$. Moreover, the convergence (4) is uniform on every interval $[x_1, x_2]$, $x_2 > x_1 \geq 0$.

The Szasz–Mirakyan operators are important in approximation theory. They have been studied intensively, and their connections with different branches of analysis, such that as numerical analysis. Recently in many papers were introduced various modifications of operators $S_n$. Approximation properties of modified Szasz–Mirakyan operators

$$S_n^q(f;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!} f\left(\frac{k}{n+q}\right)$$

for $x \in R_0$, $n \in N$, $q > 0$, in exponential weighted spaces were examined in [3]. Their extensions can be found in, e.g., [4], [5].

The actual construction of the operator $S_n$ and $S_n^q$ requires estimation of infinite series which in a certain sense restricts their usefulness from the computational point of view. Thus the question arises, whether $S_n$, $S_n^q$ and their generalizations cannot be replaced by a finite sum. In connection with this question we introduce the operators (8) Moreover, we shall prove that the order of approximation of $f \in C_p$ by $L_n$ (defined by (8) is better that (25) and $(L_n(f))$ converges uniformly to $f$ on $R_0$. This together with the simple form of the operator makes the results, given in the present paper, more helpful e.g. in numerical methods.

**1.2.** In this paper we introduce certain linear positive operators and study their approximation properties. To this end, let $C_p$ be the space given above and let $f \in C_p^1 := \{f \in C_p : f' \in C_p\}$, where $f'$ is the first derivative of $f$. For $f \in C_p$ we define the modulus of continuity $\omega_1(f;\cdot)$ as usual ([2]) by the formula

$$\omega_1(f;C_p;t) := \sup_{0\leq h\leq t} \|\Delta_h f(\cdot)\|_p, \qquad t \in R_0, \tag{5}$$

where $\Delta_h f(x) := f(x+h) - f(x)$, for $x, h \in R_0$. From the above it follows that

$$\lim_{t\to 0+} \omega_1(f; C_p; t) = 0, \tag{6}$$

for every $f \in C_p$. Moreover, if $f \in C_p^1$, then there exists $M_1 = \text{const.} > 0$ such that

$$\omega_1(f; C_p; t) \leq M_1 \cdot t \quad \text{for } t \in R_0. \tag{7}$$

**1.3.** We introduce the following

*Definition.* Let $p \in N_0$ be a fixed number. For functions $f \in C_p$ we define the operators

$$\begin{aligned} L_n(f;x) := {} & \frac{1}{(1+(x+n^{-1})^2)^n} \\ & \times \sum_{k=0}^{n} \binom{n}{k} (x+n^{-1})^{2k} f\left(\frac{k}{n} \cdot \frac{1+(x+n^{-1})^2}{x+n^{-1}}\right), \end{aligned} \tag{8}$$

$x \in R_0$, $n \in N$. □

Similarly as $S_n$, the operator $L_n$ is linear and positive. In §2 we shall prove that $L_n$ is an operator from the space $C_p$ into $C_p$ for every fixed $p \in N_0$.

For $t \in R_0$ and $n \in N$

$$(1+t^2)^n = \sum_{k=0}^{n} \binom{n}{k} t^{2k}. \tag{9}$$

From (8) and (9) we derive the following formulas

$$L_n(1;x) = 1,$$

$$L_n(t;x) = x + n^{-1},$$

$$L_n(t^2;x) = (x+n^{-1})^2 \left[1 + \frac{n^{-1}}{(x+n^{-1})^2}\right],$$

$$L_n(t^3;x) = (x+n^{-1})^3 \left[1 + \frac{3n^{-1} - n^{-2}}{(x+n^{-1})^2} + \frac{n^{-2}}{(x+n^{-1})^4}\right],$$

$$L_n(t^4;x) = (x+n^{-1})^4 \left[1 + \frac{6n^{-1}-4n^{-2}+n^{-3}}{(x+n^{-1})^2} \right.$$
$$\left. + \frac{7n^{-2}-4n^{-3}}{(x+n^{-1})^4} + \frac{n^{-3}}{(x+n^{-1})^6}\right], \tag{10}$$

for all $n \in N$ and $x \in R_0$.

## 2. Main results

**2.1.** From formulas (8), (9) and $L_n(t^k;x)$, $1 \le k \le 4$, given in the above we obtain

**Lemma 1.** *For all $x \in R_0$ and $n \in N$ we have*

$$L_n(t-x;x) = n^{-1},$$

$$L_n((t-x)^2;x) = n^{-2} + n^{-1},$$

$$L_n((t-x)^3;x) = n^{-2}\left(3 - x + \frac{1}{x+n^{-1}}\right),$$

$$L_n((t-x)^4;x) = n^{-2}\left(7 - \frac{4x}{x+n^{-1}} + \frac{n^{-1}}{(x+n^{-1})^2} \right.$$
$$\left. + n^{-1}(x^2-4x+2) + n^{-2}(2x-3) + n^{-3}\right). \qquad \square$$

Next we shall prove

**Lemma 2.** *Let $s \in N$ be a fixed number. Then there exist coefficients $\alpha_{s,j}(n)$, depending only on $s$, $j$, $n$ and bounded with respect to $n$ such that*

$$L_n(t^s;x) = (x+n^{-1})^s \sum_{j=1}^{s} \frac{\alpha_{s,j}(n)}{(x+n^{-1})^{2(j-1)}} \tag{11}$$

*for all $n \in N$ and $x \in R_0$. Moreover, $\alpha_{s,1}(n) = 1$, $\alpha_{s,s}(n) = n^{1-s}$ and $\alpha_{s,j}(n) = O(1/n^{j-1})$ for $j = 1, 2 \ldots, s$.*

Proof. We shall use mathematical induction for $s$. The formula (11) for $1 \leq s \leq 4$ is given in above. Let (11) hold for $f(x) = x^j$, $1 \leq j \leq s$, with fixed $s \in N$. We shall prove (11) for $f(x) = x^{s+1}$. From (8) and (9) it follows that

$$\begin{aligned}
L_n(t^{s+1};x) &= \frac{x+n^{-1}}{n(1+(x+n^{-1})^2)^{n-1}} \\
&\quad \times \sum_{j=0}^{n-1} \binom{n}{j} (n-j)(x+n^{-1})^{2j} \left( \frac{j+1}{n} \cdot \frac{1+(x+n^{-1})^2}{x+n^{-1}} \right)^s \\
&= \frac{x+n^{-1}}{(1+(x+n^{-1})^2)^{n-1}} \\
&\quad \times \sum_{j=0}^{n} \binom{n}{j} (x+n^{-1})^{2j} \left( \frac{j+1}{n} \cdot \frac{1+(x+n^{-1})^2}{x+n^{-1}} \right)^s \\
&\quad - \frac{x+n^{-1}}{n(1+(x+n^{-1})^2)^{n-1}} \\
&\quad \times \sum_{j=0}^{n} \binom{n}{j} (x+n^{-1})^{2j} j \left( \frac{j+1}{n} \cdot \frac{1+(x+n^{-1})^2}{x+n^{-1}} \right)^s \\
&= \frac{x+n^{-1}}{(1+(x+n^{-1})^2)^{n-1}} \\
&\quad \times \sum_{j=0}^{n} \binom{n}{j} (x+n^{-1})^{2j} \left( \frac{1+(x+n^{-1})^2}{x+n^{-1}} \right)^s n^{-s} \sum_{\mu=0}^{s} \binom{s}{\mu} j^{\mu} \\
&\quad - \frac{x+n^{-1}}{n(1+(x+n^{-1})^2)^{n-1}} \\
&\quad \times \sum_{j=0}^{n} \binom{n}{j} (x+n^{-1})^{2j} \left( \frac{1+(x+n^{-1})^2}{x+n^{-1}} \right)^s n^{-s} \sum_{\mu=0}^{s} \binom{s}{\mu} j^{\mu+1} \\
&= (x+n^{-1})(1+(x+n^{-1})^2) \\
&\quad \times \sum_{\mu=0}^{s} \binom{s}{\mu} \left( \frac{1+(x+n^{-1})^2}{x+n^{-1}} \right)^{s-\mu} n^{\mu-s} L_n(t^{\mu};x)
\end{aligned}$$

$$- (x+n^{-1})^2 L_n(t^{s+1};x) - (x+n^{-1})^2$$
$$\times \sum_{\mu=0}^{s-1} \binom{s}{\mu} \left(\frac{1+(x+n^{-1})^2}{x+n^{-1}}\right)^{s-\mu} n^{\mu-s} L_n(t^{\mu+1};x).$$

Consequently

$$L_n(t^{s+1};x) = (x+n^{-1}) \sum_{\mu=0}^{s} \binom{s}{\mu} \left(\frac{1+(x+n^{-1})^2}{x+n^{-1}}\right)^{s-\mu} n^{\mu-s} L_n(t^{\mu};x)$$
$$- (x+n^{-1}) \sum_{\mu=0}^{s-1} \binom{s}{\mu} \left(\frac{1+(x+n^{-1})^2}{x+n^{-1}}\right)^{s-\mu-1} n^{\mu-s} L_n(t^{\mu+1};x).$$

From these we obtain

$$L_n(t^{s+1};x) = (x+n^{-1}) \left(\frac{1+(x+n^{-1})^2}{x+n^{-1}}\right)^{s} n^{-s}$$
$$+ (x+n^{-1}) \sum_{\mu=1}^{s} \binom{s}{\mu} \left(\frac{1+(x+n^{-1})^2}{x+n^{-1}}\right)^{s-\mu} n^{\mu-s} L_n(t^{\mu};x)$$
$$- (x+n^{-1}) \sum_{\mu=1}^{s} \binom{s}{\mu-1} \left(\frac{1+(x+n^{-1})^2}{x+n^{-1}}\right)^{s-\mu} n^{\mu-s-1} L_n(t^{\mu};x).$$

By our assumption we get

$$L_n(t^{s+1};x) = (x+n^{-1}) \left(\frac{1+(x+n^{-1})^2}{x+n^{-1}}\right)^{s} n^{-s}$$
$$+ (x+n^{-1}) \sum_{\mu=1}^{s} \left\{\binom{s}{\mu} n^{\mu-s} - \binom{s}{\mu-1} n^{\mu-s-1}\right\} \left(\frac{1+(x+n^{-1})^2}{x+n^{-1}}\right)^{s-\mu}$$
$$\times (x+n^{-1})^{\mu} \sum_{j=1}^{\mu} \frac{\alpha_{\mu,j}(n)}{(x+n^{-1})^{2(j-1)}} = (x+n^{-1})^{s+1} \Bigg\{ \left(\frac{1+(x+n^{-1})^2}{(x+n^{-1})^2}\right)^{s} n^{-s}$$
$$+ \sum_{\mu=1}^{s} \sum_{j=1}^{\mu} \left\{\binom{s}{\mu} n^{\mu-s} - \binom{s}{\mu-1} n^{\mu-s-1}\right\}$$
$$\times \left(\frac{1+(x+n^{-1})^2}{(x+n^{-1})^2}\right)^{s-\mu} \frac{\alpha_{\mu,j}(n)}{(x+n^{-1})^{2(j-1)}} \Bigg\}$$

$$= (x+n^{-1})^{s+1}\Bigg\{ \left(\frac{1+(x+n^{-1})^2}{(x+n^{-1})^2}\right)^s n^{-s} + \sum_{\mu=1}^{s}\sum_{j=1}^{\mu}\sum_{k=0}^{s-\mu}\binom{s-\mu}{k}$$

$$\times (x+n^{-1})^{2(\mu-s-j+k+1)}\left\{\binom{s}{\mu}n^{\mu-s}-\binom{s}{\mu-1}n^{\mu-s-1}\right\}\alpha_{\mu,j}(n)\Bigg\}.$$

Since $\alpha_{s,1}(n)=1$, $\alpha_{s,s}(n)=n^{1-s}$ and $\alpha_{s,j}(n)=O(1/n^{j-1})$ for $j=1,2\ldots,s$, we have for $\mu=1,2,\ldots,s$

$$\left\{\binom{s}{\mu}n^{\mu-s}-\binom{s}{\mu-1}n^{\mu-s-1}\right\}\alpha_{\mu,j}(n)=O(1/n^{j+s-\mu-1}),\quad j=2,3,\ldots,\mu,$$

$$\sum_{\mu=1}^{s}\left\{\binom{s}{\mu}n^{\mu-s}-\binom{s}{\mu-1}n^{\mu-s-1}\right\}\alpha_{\mu,1}(n)=1-n^{-s}.$$

From the above and by elementary calculations we can write

$$\sum_{\mu=1}^{s}\sum_{j=1}^{\mu}\sum_{k=0}^{s-\mu}\binom{s-\mu}{k}(x+n^{-1})^{2(\mu-s-j+k+1)}$$

$$\times\left\{\binom{s}{\mu}n^{\mu-s}-\binom{s}{\mu-1}n^{\mu-s-1}\right\}\alpha_{\mu,j}(n)=1-n^{-s}+\sum_{\mu=2}^{s}\frac{\beta_{s,\mu}(n)}{(x+n^{-1})^{2(\mu-1)}},$$

where $\beta_{s,\mu}(n)$ are coefficients depending only on $s$, $\mu$, $n$ and bounded with respect to $n$ and $\beta_{s,\mu}(n)=O(1/n^{\mu-1})$ for $\mu=2,\ldots,s$. Consequently we have

$$L_n(t^{s+1};x)=(x+n^{-1})^{s+1}\Bigg\{\sum_{\mu=0}^{s}\frac{n^{-s}}{(x+n^{-1})^{2(s-\mu)}}$$

$$+1-n^{-s}+\sum_{\mu=2}^{s}\frac{\beta_{s,\mu}(n)}{(x+n^{-1})^{2(\mu-1)}}\Bigg\}$$

$$=(x+n^{-1})^{s+1}\Bigg\{1+\sum_{\mu=2}^{s}\frac{n^{-s}}{(x+n^{-1})^{2(s-\mu+1)}}$$

$$+\sum_{\mu=2}^{s}\frac{\beta_{s,\mu}(n)}{(x+n^{-1})^{2(\mu-1)}}+\frac{n^{-s}}{(x+n^{-1})^{2s}}\Bigg\}$$

$$= (x+n^{-1})^{s+1} \sum_{\mu=1}^{s+1} \frac{\alpha_{s+1,\mu}(n)}{(x+n^{-1})^{2\mu-2}}$$

and $\alpha_{s+1,1}(n) = 1$, $\alpha_{s+1,s+1}(n) = n^{-s}$, $\alpha_{s+1,j}(n) = O(1/n^{j-1})$ for $j = 1, 2, \ldots, s+1$, which proves (11) for $f(x) = x^{s+1}$. Hence the proof of (11) is completed. □

**Lemma 3.** *Let $p \in N_0$ be a fixed number. Then there exists a positive constant $M_2 \equiv M_2(p)$, depending only on the parameter $p$ such that*

$$\|L_n(1/w_p(t);\,\cdot)\|_p \le M_2, \quad n \in N. \tag{12}$$

*Moreover, for every $f \in C_p$ we have*

$$\|L_n(f;\,\cdot)\|_p \le M_2 \|f\|_p, \quad n \in N. \tag{13}$$

*Formula* (8) *and inequality* (13) *show that $L_n$, $n \in N$, is a positive linear operator from the space $C_p$ into $C_p$, for every $p \in N_0$.*

Proof. The inequality (12) is obvious for $p = 0$ by (2), (3) and (10). Let $p \in N$. By (2) and (8)–(11) we have

$$\begin{aligned} w_p(x) L_n(1/w_p(t); x) &= w_p(x) \left\{1 + L_n(t^p; x)\right\} \\ &= \frac{1}{1+x^p} + \frac{(x+n^{-1})^p}{1+x^p} \sum_{j=1}^{p} \frac{\alpha_{p,j}(n)}{(x+n^{-1})^{2j-2}}. \end{aligned}$$

For $x \in [1, +\infty)$, we get using Lemma 2

$$w_p(x) L_n(1/w_p(t); x) \le 1 + \sum_{k=0}^{p} \binom{p}{k} \frac{x^{p-k}}{1+x^p} \sum_{j=1}^{p} \alpha_{p,j}(n) \le M_2(p).$$

Let $x \in [0, 1)$ and

$$g(x) := (x+n^{-1})^{p+2-2j}. \tag{14}$$

We remark that $g$ on $[0, 1)$ is an increasing function for $1 \le j < (p+2)/2$ and a decreasing function for $(p+2)/2 < j \le p$. From this we immediately obtain

$$\frac{\alpha_{p,j}(n)}{(x+n^{-1})^{2j-2-p}} \le \frac{\alpha_{p,j}(n)}{(1+n^{-1})^{2j-2-p}} \le \alpha_{p,j}(n), \quad 1 \le j < (p+2)/2,$$

$$\frac{\alpha_{p,j}(n)}{(x+n^{-1})^{2j-2-p}} \le \frac{\alpha_{p,j}(n)}{n^{-2j+2+p}} \le \frac{n^{j-1}\alpha_{p,j}(n)}{n^{-j+1+p}}, \qquad (p+2)/2 < j \le p.$$

Applying Lemma 2, we get

$$w_p(x)L_n(1/w_p(t);x) \le 1 + \sum_{j=1}^{p} \frac{\alpha_{p,j}(n)}{(x+n^{-1})^{2j-2-p}} \le M_2(p)$$

for $x \in [0,1)$, $n \in N$, where $M_2(p)$ is a positive constant depending only upon $p$. Therefore, the proof of inequality (12) is completed.

The formulas (8)–(9) and (2) imply

$$\|L_n(f(t);\,\cdot)\|_p \le \|f\|_p \|L_n(1/w_p(t);\,\cdot)\|_p, \quad n \in N,$$

for every $f \in C_p$. Applying (12), we obtain (13). $\square$

**Lemma 4.** *Let $p \in N_0$ be a fixed number. Then there exists a positive constant $M_3 \equiv M_3(p)$ such that*

$$\left\| L_n\left(\frac{(t-\,\cdot)^2}{w_p(t)};\,\cdot\right)\right\|_p \le \frac{M_3}{n} \quad \textit{for all} \;\; n \in N. \tag{15}$$

Proof. The formulas given in Lemma 1 and (2), (3) imply (15) for $p=0$.

By (2) and (10) we have

$$L_n\left((t-x)^2/w_p(t);x\right) = L_n\left((t-x)^2;x\right) + L_n\left(t^p(t-x)^2;x\right),$$

for $p,n \in N$. If $p=1$, then by the equality we get

$$\begin{aligned} L_n\left((t-x)^2/w_1(t);x\right) &= L_n\left((t-x)^2;x\right) + L_n\left(t(t-x)^2;x\right) \\ &= L_n\left((t-x)^3;x\right) + (1+x)L_n\left((t-x)^2;x\right), \end{aligned}$$

which by (2), (3) and Lemma 1 yields (15) for $p=1$.

Let $p \ge 2$. Applying Lemma 2, we get

$$w_p(x)L_n(t^p(t-x)^2;x) = w_p(x)\left\{L_n(t^{p+2};x) - 2xL_n(t^{p+1};x) + x^2L_n(t^p;x)\right\}$$

$$= w_p(x)\left\{(x+n^{-1})^{p+2}\sum_{j=1}^{p+2}\frac{\alpha_{p+2,j}(n)}{(x+n^{-1})^{2(j-1)}} - 2x(x+n^{-1})^{p+1}\right.$$

$$\left.\times\sum_{j=1}^{p+1}\frac{\alpha_{p+1,j}(n)}{(x+n^{-1})^{2(j-1)}} + x^2(x+n^{-1})^p\sum_{j=1}^{p}\frac{\alpha_{p,j}(n)}{(x+n^{-1})^{2(j-1)}}\right\}$$

$$= w_p(x)(x+n^{-1})^p\left\{n^{-2} + (x+n^{-1})^2\sum_{j=2}^{p+2}\frac{\alpha_{p+2,j}(n)}{(x+n^{-1})^{2(j-1)}}\right.$$

$$\left. + 2x(x+n^{-1})\sum_{j=2}^{p+1}\frac{\alpha_{p+1,j}(n)}{(x+n^{-1})^{2(j-1)}} + x^2\sum_{j=2}^{p}\frac{\alpha_{p,j}(n)}{(x+n^{-1})^{2(j-1)}}\right\},$$

which by (2) and Lemma 2 implies for $x \in [1, +\infty)$

$$w_p(x)L_n(t^p(t-x)^2;x) \le n^{-1}\frac{(1+x)^p}{1+x^p}\left\{1 + \sum_{j=2}^{p+2}\frac{n\alpha_{p+2,j}(n)}{(x+n^{-1})^{2(j-2)}}\right.$$

$$\left. + 2\sum_{j=2}^{p+1}\frac{n\alpha_{p+1,j}(n)}{(x+n^{-1})^{2(j-2)}} + \sum_{j=2}^{p}\frac{n\alpha_{p,j}(n)}{(x+n^{-1})^{2(j-2)}}\right\} \le \frac{M_4(p)}{n}, \quad n \in N.$$

Let $x \in [0, 1)$. Applying Lemma 2 and arguing as in the proof of Lemma 3, we easily obtain

$$w_p(x)L_n\left(t^p(t-x)^2;x\right) \le \frac{M_4(p)}{n}, \quad n \in N.$$

Thus the proof is completed. □

**2.2.** Now we shall give approximation theorems for $L_n$.

**Theorem 1.** *Let $p \in N_0$ be a fixed number. Then there exists a positive constant $M_5 \equiv M_5(p)$ such that for every $f \in C_p^1$ we have*

$$\|L_n(f;\cdot) - f(\cdot)\|_p \le \frac{M_5}{\sqrt{n}}\|f'\|_p, \quad n \in N. \tag{16}$$

Proof. Let $x \in R_0$ be a fixed point. Then for $f \in C_p^1$ we have

$$f(t) - f(x) = \int_x^t f'(u)du, \quad t \in R_0.$$

From this and by (8) and (10) we get

$$L_n(f(t); x) - f(x) = L_n\left(\int_x^t f'(u)du; x\right), \quad n \in N.$$

But by (2) and (3) we have

$$\left|\int_x^t f'(u)du\right| \leq \|f'\|_p \left(\frac{1}{w_p(t)} + \frac{1}{w_p(x)}\right)|t-x|, \quad t \in R_0,$$

which implies

$$\begin{aligned} w_p(x)&|L_n(f;x) - f(x)| \\ &\leq \|f'\|_p \left\{L_n(|t-x|;x) + w_p(x)L_n\left(\frac{|t-x|}{w_p(t)};x\right)\right\} \end{aligned} \tag{17}$$

for $n \in N$. By the Hölder inequality and by (10) and Lemmas 1, 3, 4 it follows that

$$L_n(|t-x|;x) \leq \left\{L_n\left((t-x)^2;x\right) L_n(1;x)\right\}^{1/2} \leq \sqrt{\frac{2}{n}},$$

$$\begin{aligned} w_p(x)&L_n\left(\frac{|t-x|}{w_p(t)};x\right) \\ &\leq w_p(x)\left\{L_n\left(\frac{(t-x)^2}{w_p(t)};x\right)\right\}^{1/2}\left\{L_n\left(\frac{1}{w_p(t)};x\right)\right\}^{1/2} \leq \frac{M_7(p)}{\sqrt{n}} \end{aligned}$$

for $n \in N$. From this and by (17) we immediately obtain (16). $\square$

**Theorem 2.** *Let $p \in N_0$ be a fixed number. Then there exists $M_8 \equiv M_8(p)$ such that for every $f \in C_p$ and $n \in N$ we have*

$$\|L_n(f;\cdot) - f(\cdot)\|_p \leq M_8\omega_1\left(f; C_p; \frac{1}{\sqrt{n}}\right). \tag{18}$$

Proof. We use the Steklov function $f_h$ of $f \in C_p$

$$f_h(x) := \frac{1}{h}\int_0^h f(x+t)dt, \quad x \in R_0,\ h > 0. \tag{19}$$

From (19) we get

$$f_h(x) - f(x) = \frac{1}{h}\int_0^h \Delta_t f(x)dt,$$

$$f_h'(x) = \frac{1}{h}\Delta_h f(x), \quad x \in R_0,\ h > 0,$$

which imply

$$\|f_h - f\|_p \le \omega_1\left(f; C_p; h\right), \tag{20}$$

$$\|f_h'\|_p \le h^{-1}\omega\left(f; C_p; h\right), \tag{21}$$

for $h > 0$. From this we deduce that $f_h \in C_p^1$ if $f \in C_p$ and $h > 0$. Hence we can write

$$w_p(x)|L_n(f;x) - f(x)| \le w_p(x)\big\{|L_n(f - f_h; x)| + |L_n(f_h; x) - f_h(x)| + |f_h(x) - f(x)|\big\} := A_1(x) + A_2(x) + A_3(x),$$

for $n \in N$, $h > 0$ and $x \in R_0$. From (13) and (20) we get

$$\|A_1\|_p \le M_2\|f_h - f\|_p \le M_2\omega_1\left(f; C_p; h\right),$$

$$\|A_3\|_p \le \omega_1\left(f; C_p; h\right).$$

By Theorem 1 and (21) it follows that

$$\|A_2\|_p \le \frac{M_5}{\sqrt{n}}\|f_h'\|_p \le \frac{M_5}{\sqrt{n}h}\omega_1\left(f; C_p; h\right).$$

Consequently

$$\|L_n(f;\,\cdot) - f(\,\cdot\,)\|_p \le \left(1 + M_2 + \frac{M_5}{\sqrt{n}h}\right)\omega_1(f; C_p; h).$$

Now, for fixed $n \in N$, setting $h = \frac{1}{\sqrt{n}}$, we obtain

$$\|L_n(f;\,\cdot) - f(\,\cdot\,)\|_p \le M_8(p)\omega_1\left(f; C_p; \frac{1}{\sqrt{n}}\right)$$

and we complete the proof. $\square$

From Theorem 1 and Theorem 2 we derive following two corollaries:

**Corollary 1.** *For $f \in C_p$, $p \in N_0$, we have*

$$\lim_{n\to\infty} \|L_n(f;\,\cdot) - f(\,\cdot\,)\|_p = 0. \qquad \square$$

**Corollary 2.** *If $f \in C_p^1$, $p \in N_0$, then*

$$\|L_n(f;\,\cdot) - f(\,\cdot\,)\|_p = O(1/\sqrt{n}\,). \qquad \square$$

**2.3.** Finally, we shall give the Voronovskaya type theorem for $L_n$.

**Theorem 3.** *Let $f \in C_p^2 := \left\{f \in C_p : f^{'}, f^{''} \in C_p\right\}$. Then*

$$\lim_{n\to\infty} n\left\{L_n\left(f;x\right) - f(x)\right\} = f'(x) + \frac{1}{2}f''(x) \tag{22}$$

*for every $x > 0$.*

Proof. Let $x > 0$ be a fixed point. Then by the Taylor formula we have

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \varepsilon(t;x)(t-x)^2$$

for $t \in R_0$, where $\varepsilon(t) \equiv \varepsilon(t;x)$ is a function belonging to $C_p$ and $\varepsilon(x) = 0$. Hence by (8) and (10) we get

$$\begin{aligned} L_n(f;x) = f(x) + f'(x)L_n(t-x;x) + \frac{1}{2}f''(x)L_n((t-x)^2;x) \\ + L_n(\varepsilon(t)(t-x)^2;x), \quad n \in N, \end{aligned} \tag{23}$$

which by Lemma 1 yields

$$\begin{aligned} &\lim_{n\to\infty} n\left\{L_n\left(f;x\right) - f(x)\right\} \\ &\qquad = f'(x) + \frac{1}{2}f''(x) + \lim_{n\to\infty} nL_n(\varepsilon(t)(t-x)^2;x). \end{aligned} \tag{24}$$

By the Hölder inequality we have

$$|L_n(\varepsilon(t)(t-x)^2;x)| \le \left\{L_n\left(\varepsilon^2(t);x\right)\right\}^{1/2}\left\{L_n\left((t-x)^4;x\right)\right\}^{1/2}.$$

The properties of $\varepsilon$ and Corollary 1 imply that

$$\lim_{n\to\infty} L_n\left(\varepsilon^2(t);x\right) = \varepsilon^2(x) = 0.$$

From this and by Lemma 1 we get

$$\lim_{n\to\infty} nL_n(\varepsilon(t)(t-x)^2;x)=0$$

and (22) follows from (24). $\square$

*Remark.* In [1] it was proved that if $f\in C_p$, $p\in N_0$, then for the Szasz–Mirakyan operators $S_n$ (defined by (1)) one has the following inequality

$$w_p(x)|S_n(f;x)-f(x)|\le M_9\omega_2\left(f;C_p;\sqrt{\frac{x}{n}}\right),\quad x\in R_0,\ n\in N_0,$$

where $M_9=\text{const.}>0$ and $\omega_2\,(f;\,\cdot)$ is the modulus of smoothness defined by the formula

$$\omega_2(f;C_p;t):=\sup_{0\le h\le t}\|\Delta_h^2 f(\,\cdot\,)\|_p,\qquad t\in R_0,$$

where $\Delta_h^2 f(x):=f(x)-2f(x+h)+f(x+2h)$. In particular, if $f\in C_p^1$, $p\in N_0$, then

$$w_p(x)|S_n(f;x)-f(x)|\le M_{10}\sqrt{\frac{x}{n}},\tag{25}$$

for $x\in R_0$ and $n\in N$ $(M_{10}=\text{const.}>0)$.

Theorem 1, Theorem 2 and Corollary 2 in our paper show that the operators $L_n$, $n\in N$, give better degree of approximation of functions $f\in C_p$ and $f\in C_p^1$ than $S_n$.

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