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On t–norms dominating quasiarithmetic means

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Abstract. By solving several functional inequalities we characterize some t–norms dominating quasiarithmetic means.

In the framework of the theory of probabilistic metric spaces [7] several studies concerning geometrical properties of associative functions have been made in recent years. The aim of this paper is to analize some dominance relations between Archimedean t–norms and quasiarithmetic means. We begin with the following

Definition 1. A t–norm is a two-place function T from $[0,1]\times[0,1]$ into $[0,1]$ such that T is associative, commutative, nondecreasing in each place and with 1 as a unit. A t–norm T is Archimedean if it is continuous and $T(x,x)<x$ for all x in $(0,1)$.

Archimedean t–norms have been characterized in [5] where the following representation is given:

Theorem 1. *A t–norm T is Archimedean and satisfies $T(x,x)>0$ for all $0<x<1$ if and only if it admits the representation*

$$T(x,y)=t^{-1}(t(x)+t(y)) \tag{1}$$

where t (the additive generator of T) is a continuous strictly decreasing function from $[0,1]$ onto $[0,+\infty]$ such that $t(1)=0$ and $t(0)=+\infty$.

Definition 2. A quasiarithmetic mean on $[0,1]$ is a two place function of the form

$$M(x,y)=h^{-1}\left(\frac{h(x)+h(y)}{2}\right), \tag{2}$$

where $h:[0,1]\to[0,\operatorname{Max}(h(0),h(1))]$ is a continuous bijection. If $h(0)=0$ and $h(1)=1$ then M will be said to be of sum type (e.g.

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$h(x) = x, M(x,y) = \frac{x+y}{2}$). If $h(0) = +\infty$ and $h(1) = 0$, then M will be said to be of product type (e.g. $h(x) = -\log x, M(x,y) = \sqrt{xy}$).

Finally, following [7] we recall

Definition 3. Given two binary operations K, G in $[0,1]$, K dominates G if for all a, b, c, d in $[0,1]$ we have

$$K(G(a,b), G(c,d)) \geq G(K(a,c), K(b,d)).$$

Our chief concern here is to study when t–norms T dominate quasi-arithmetic means M either of sum or product type, i.e.,

$$T(M(a,b), M(c,d)) \geq M(T(a,c), T(b,d)), \tag{3}$$

for all a, b, c, d in $[0,1]$.

When M is of sum type then we can show the following

Theorem 2. *For any given quasiarithmetic mean M of sum type, Min is the only continuous t–norm dominating M.*

Proof. If a continuous t–norm T dominates M as given by (2) with $h(0) = 0$, $h(1) = 1$ and h is increasing then:

$$T\left(h^{-1}\left(\frac{h(a)+h(b)}{2}\right), h^{-1}\left(\frac{h(c)+h(d)}{2}\right)\right) \geq \geq h^{-1}\left(\frac{h(T(a,c))+h(T(b,d))}{2}\right). \tag{4}$$

If we define $H(x,y) = h(T(h^{-1}(x), h^{-1}(y)))$ then H is a continuous t–norm which by (4) necessarily is mid-point strictly concave, i.e.,

$$H\left(\frac{x+y}{2}, \frac{u+v}{2}\right) \geq \frac{H(x,u)+H(y,v)}{2}. \tag{5}$$

By the continuity of H condition (5) yields full concavity and therefore, since $H \leq \mathrm{Min}$,

$$x = \mathrm{Min}(x,x) \geq H(x,x) = H(x \cdot 1 + (1-x) \cdot 0,\ x \cdot 1 + (1-x) \cdot 0) \geq \geq xH(1,1) + (1-x)H(0,0) = x.$$

Then $H(x,x) = x$ and therefore $T(x,x) = x$ and deduce that $T = \mathrm{Min}$.

Now we turn our attention to the dominance in the case that M is of product type.

Theorem 3. *An Archimedean t–norm T dominates a quasiarithmetic mean M of product type if and only if T is strictly increasing in each place on $(0,1]\times(0,1]$ and has the form $T(x,y)=h^{-1}(F(h(x),h(y)))$ where F is a continuous convex associative operation on $\mathbb{R}^+$, non-decreasing in each place.*

Proof. If T is an Archimedean t–norm that dominates a quasiarithmetic mean M of product type and $T(x,x)=0$ for some $0<x<1$ then choosing $0<a<1$ such that $M(a,1)=x$, by (3) we have the contradiction

$$0=T(x,x)=T(M(a,1),M(a,1))\geq M(T(a,1),T(1,a))=M(a,a)=a.$$

By Theorem 1 T is representable in the form $T(x,y)=t^{-1}(t(x)+t(y))$. In this last case we will have that $T(x,y)=h^{-1}(F(h(x),h(y)))$ where

$$F(x,y)=f^{-1}(f(x)+f(y)), \tag{6}$$

is generated by $f=t\circ h^{-1}$. Then (3) is equivalent to the convexity of F.

In view of the previous result we will turn now our attention to the characterization of convex associative functions on $\mathbb{R}^+$ of the form (6) with $f(0)=0$ and f strictly increasing. While, in general, it is not possible to find from the convexity of F an equivalent condition for f involving two variables, if some regularity of f is assumed then it is possible to find a suitable characterization.

Theorem 4. *Let F be a binary operation on $\mathbb{R}^+$ representable in the form (6) where the additive generator $f:[0,+\infty]\to[0,+\infty]$ is such that f' and f'' exist and are continuous functions on $(0,+\infty)$ with $f'(x),f''(x)\neq 0$ for all $x>0$. Then F is convex if and only if there exist a continuous non-drecreasing and superadditive function $S:(0,+\infty)\to(0,+\infty)$ such that*

$$f^{-1}(x)=\frac{1}{k}\int_0^x \exp\left[-\int_1^u \frac{dt}{S(t)}\right]du,$$

where k is an arbitrary positive constant.

Proof. By the differentiability conditions assumed on f we have that F is twice differentiable on $(0,+\infty)\times(0,+\infty)$ and

$$\frac{\partial^2 F}{\partial x^2}(x,y)=\frac{f''(x)}{f'(F(x,y))}-\frac{f'(x)^2 f''(F(x,y))}{f'(F(x,y))^3}, \tag{7}$$

$$\frac{\partial^2 F}{\partial y^2}(x,y)=\frac{f''(y)}{f'(F(x,y))}-\frac{f'(y)^2 f''(F(x,y))}{f'(F(x,y))^3}, \tag{8}$$

$$\frac{\partial^2 F}{\partial x\partial y}(x,y)=-\frac{f'(x)f'(y)f''(F(x,y))}{f'(F(x,y))^3}. \tag{9}$$

If F is convex then at any point the Hessian matrix of F will be positive semidefinite, i.e., we will have, for all $x, y > 0$

$$(10)\qquad \frac{\partial^2 F}{\partial x^2}(x,y) > 0 \quad \text{and} \quad \frac{\partial^2 F}{\partial x^2}(x,y)\frac{\partial^2 F}{\partial y^2}(x,y) - \left(\frac{\partial^2 F}{\partial x \partial y}(x,y)\right)^2 \geq 0$$

In view of (7), (8) and (9) we obtain from (10) that the function f will necessarily satisfy the inequalities

$$(11)\qquad \frac{f''(x)}{f'(x)^2} \geq \frac{f''(F(x,y))}{f'(F(x,y))^2}$$

and

$$(12)\qquad \frac{f''(x)f''(y)}{f'(x)^2 f'(y)^2} \geq \left(\frac{f''(x)}{f'(x)^2} + \frac{f''(y)}{f'(y)^2}\right)\frac{f''(F(x,y))}{f'(F(x,y))^2}.$$

If we introduce the variables $u = f(x)$, $v = f(y)$ and the function $g : (0, +\infty) \to \mathbb{R}$ given by

$$g(z) = \frac{f''(f^{-1}(z))}{f'(f^{-1}(z))^2},$$

then (11) and (12) become

$$(13)\qquad g(u) \geq g(u+v)$$

and

$$(14)\qquad g(u)g(v) \geq (g(u) + g(v))g(u+v),$$

for all u, v in $(0, +\infty)$. Since F is convex, in particular we have

$$F\left(\frac{x+y}{2}, \frac{x+y}{2}\right) < \frac{F(x,y) + F(y,x)}{2} = F(x,y)$$

and therefore f is convex. Thus by the assumption $f''(x) \neq 0$ on $(0, +\infty)$ we deduce $f''(x) > 0$ and $g(x) > 0$ for all $x > 0$. Consequently by (13) and (14) we have that g is non-increasing and

$$\frac{1}{g(u+v)} \geq \frac{1}{g(u)} + \frac{1}{g(v)},$$

i.e., $S(x) = \frac{1}{g(x)}$ is superadditive and non-decreasing. Moreover by the definition of g we obtain by simple integration:

$$(15)\qquad \int_1^x \frac{dt}{S(t)} = \int_1^x \frac{f''(f^{-1}(t))}{f'(f^{-1}(t))^2}dt = \log \frac{f'(f^{-1}(x))}{f'(f^{-1}(1))},$$

and if we call $k = f'(f^{-1}(1)) > 0$, (15) may be rewritten in the form:

$$\frac{1}{f'(f^{-1}(x))} = \frac{1}{k}\exp\left[-\int_1^x \frac{dt}{S(t)}\right],$$

i.e.,

$$f^{-1}(x) = \int_0^x \frac{du}{f'(f^{-1}(u))} = \frac{1}{k}\int_0^x \exp\left[-\int_1^u \frac{dt}{S(t)}\right] du. \tag{16}$$

Conversely if f^{-1} is given by (16) with $S(x) > 0$, S non-decreasing and superadditive, we obtain immediately that $(f^{-1})'(x) = \frac{1}{k}\exp\left[-\int_1^x \frac{dt}{S(t)}\right]$ and consequently $f'(x) \neq 0$ in $(0, +\infty)$. Therefore $f'(f^{-1}(x)) = k\exp\left[\int_1^x \frac{dt}{S(t)}\right]$ yields $f''(x) \neq 0$ for all x in $(0, +\infty)$. Since $S(x) > 0$ and $\frac{1}{S(x)} = \frac{f''(f^{-1}(x))}{f'(f^{-1}(x))^2}$ we necessarily have $f''(x) > 0$ on $(0, +\infty)$. Finally, our assumptions on S imply that $g(x) = 1/S(x)$ will satisfy (13) and (14). Thus (10) holds and F is convex.

Example 1. If we consider $S(x) = \frac{c}{c-1}x$, with $c > 1$, that verifies the hypothesis of theorem 4, we obtain the convex binary operation $F(x, y) = (x^c + y^c)^{\frac{1}{c}}$. Such functions have Gauss curvature zero and were characterized in [3].

Thus we have

Theorem 5. *Let T be a continuous t–norm generated by $t : [0, 1] \to [0, +\infty]$ and M a quasiarithmetic mean of product type generated by h. Assume that t', t'', h', h'' exist and are continuous on $(0, 1)$ with $t'(x)$, $h'(x) \neq 0$ for all x in $(0, 1)$. Then T dominates M if and only if:*

$$t^{-1}(x) = h^{-1}\left[\frac{1}{k}\int_0^x \exp\left[-\int_1^u \frac{dt}{S(t)}\right] du\right]$$

where $S : (0, +\infty) \to (0, +\infty)$ is continuous, non-decreasing, superadditive and $k > 0$

Corollary 1. *If $M(x, y) = \sqrt{xy}$, i.e., $h(x) = -\log x$, then a t–norm T such that t', t'' exist, are continuous and $t'(x) \neq 0$ for all x in $(0, 1)$, verifies*

$$T(\sqrt{ab}, \sqrt{cd}) > \sqrt{T(a, c), T(b, d)}$$

if and only if

$$t^{-1}(x) = \exp -\left[\frac{1}{k}\int_0^x \exp\left[-\int_1^u \frac{dt}{S(t)}\right] du\right]$$

where $S : (0, +\infty) \to (0, +\infty)$ is continuous, non-decreasing, superadditive and $k > 0$.

Remark. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$f(x) = \begin{cases} \log \dfrac{2}{2-x}, & \text{if } 0 \le x \le 1, \\ \frac{x^2}{2} + \log 2 - \frac{1}{2}, & \text{if } x \ge 1. \end{cases}$$

Then $f(0) = 0$ and f is strictly increasing and strictly convex. Nevertheless the function $F(x,y) = f^{-1}(f(x) + f(y))$ cannot be convex because, e.g., a straigthforward computation shows that the Hessian matrix of F in the point $(\frac{\log 2}{2}, \frac{\log 2}{2})$ is not positive semidefinite. Thus the strict convexity of the generator does not imply (even with smooth properties) the strict convexity of F.

To end we must mention that inequalities related to t–norms dominated by quasiarithmetic means have been studied previously. It is remarkable, for example that, as shown in [2] smooth convex t–norms do not exist. This situation clearly contrasts with the case $\mathbb{R}^+$ studied here in Theorem 3.

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