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Some results on primitive words, palindromes and polyslender languages

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Dedicated to Professor Manfred Kudlek, on the occasion of his 60th birthday

Abstract. We show that the language of all primitive palindromes is not context-free. In addition, a characterization of slender and polyslender palindromic context-free languages is given. Some related results and problems are also discussed.

1. Introduction

Combinatorial properties of words play an important role in mathematics and theoretical computer science (see, e.g., [5], [10], [15], [32], etc.).

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In this paper we study the language of all palindromic primitive words over a nontrivial alphabet.

Let us fix a (nonempty, finite) alphabet X, having at least two letters. A *primitive word* (over X, or actually over an arbitrary alphabet) is a nonempty word not of the form w^m for any nonempty word w and integer $m \geq 2$. The set of all primitive words over X will be denoted by $Q(X)$, or simply by Q if X is understood. Q has received special interest: Q and $X^+ \setminus Q$ play an important role in the algebraic theory of codes and formal languages (see M. Lothaire [23] and H. J. Shyr [32]).

In [4] and [5] the authors proved that Q is not deterministic context-free. It was also shown that Q is not unambiguous context-free (J.-P. Allouche [1], H. Petersen [25]), moreover, that it is not linear context-free and not bounded (S. Horváth [13]). Furthermore, in [14] and [16], decidability and related questions concerning Q were studied. Returning to the relation of Q to the Chomsky language classes, it is easy to see that Q is deterministic context-sensitive. The last-in-first-out nature of a pushdown store does not provide the means to remember one substring and then check it for equality or nonequality against another substring. The information needed first for such a check is bound to reside at the bottom of the store. Therefore, we strongly believe that the following conjecture is valid.

Conjecture (P. Dömösi, S. Horváth, M. Ito, 1991 [4])**.** *Q is not context-free.*

We formulated this conjecture also in [5] and in all of our later papers concerning Q. We tried to prove this conjecture by applying different types of strong pumping lemmas (see [6], [7]), but it has turned out that Q satisfies all of the considered pumping properties, and even a strengthened, new interchange property (see [13]). In [13] it was (easily) observed that the relative density of nonprimitive words exponentially quickly tends to zero as the length of words tends to infinity, and it was remarked that intuitively this fact is the reason, why Q is closed even under quite strong combinatorial manipulations (since "almost all words are primitive", therefore there is very little chance to "go out from Q" by means of such manipulations). Another possibility is to consider an appropriate regular language R and prove that $Q \cap R$ is not context-free. However, contrary to our expectation, $Q \cap (ab^*)^n$ has proved to be context-free for exponents n, belonging to wider and wider (infinite) classes of positive integers (see [6], [7], [20]). Therefore, the main problem has remained open. In this paper we study these problems within the class of palindromic languages. Moreover, we have some investigations in the family of slender and polyslender palindromic languages, too.

2. Preliminaries

For any word uvw, we say that v is a *subword* of uvw. Let w be a word. We put $w^0 = \lambda$, and $w^n = w^{n-1}w$ for $n >> 0$.Thus w^k $(k \geq 0)$ is the *k-th power* (or, in short, a *power*) of w. As is customary, we put $w^* = \{w^n : n \geq 0\}$ and $w^+ = \{w^n : n > 0\}$. A word is called *primitive* iff it cannot be written in the form of a power with exponent greater than 1. (Thus the empty word λ is nonprimitive.) We will denote by $Q(X)$ the set of all primitive words over X, or simply by Q if X is clear. For a word $p = x_1 \dots x_n \in X^+$, where $x_i \in X$ $(i = 1, \dots, n)$, we denote its reverse $x_n \dots x_1$ by p^R. A word p is a *palindrome* iff $p = p^R$. (Of course, λ is a palindrome, too.) The set of all palindromes over X will be denoted by $\mathrm{Pal}(X)$, or simply by Pal if X is understood. Furthermore, we call a language *palindromic* iff every word in L is a palindrome (i.e., iff $L \subseteq \mathrm{Pal}$). A nonempty language $L \subseteq X^*$ is said to be *self-embedding* iff it can be generated by a context-free grammar $G = (V, X, S, P)$ such that $S \to ASB \in P$ for some $A, B \in (V \cup X)^*$, moreover, for an appropriate pair w, z with $wz \in X^+$, $A \overset{*}{\Rightarrow} w$ and $B \overset{*}{\Rightarrow} z$. Following, e.g., H. J. Shyr [32], we call a language $L \subseteq X^*$ *dense* (in X^*) iff for every word $w \in X^*$, $L \cap X^*wX^* \neq \emptyset$. We shall use the following well-known property of regular languages.

Proposition 2.1 (See, e.g., [27]). *A language $L \subseteq X^*$ is regular if and only if $X^* \setminus L$ is regular.* □

We shall also use the following pumping lemmas for context-free languages, the first of which (Theorem 2.2) is traditionally called *Bar–Hillel's lemma* in the literature.

Theorem 2.2 (Y. Bar–Hiller, M. Perles, E. Shamir [2], see also in [12], [27] or [28]). *For any context-free grammar G, one can effectively compute a constant $p \geq 1$ from G, such that, for any $z \in L(G)$, if $|z| > p$, then z can be factorized into $z = uvwxy$ so that,*

(1) $|vx| > 0$,

(2) $|vwx| \leq p$, *and*

(3) $uv^iwx^iy \in L(G)$ *for all* $i \geq 0$. □

Theorem 2.3 (P. Dömösi, M. Ito, M. Katsure, C. L. Nehaniv [8]). *For any context-free grammar G, one can effectively compute a constant $c \geq 2$ from G, such that, for any $z \in L(G)$ and any $e > 0$, if $|z| \geq ce$, and in z, e positions are excluded, then z can be factorized into $z = uvwxy$ so that,*

(1) $|vx| > 0$,

(2*) vx *contains no excluded position, and*

(3) $uv^iwx^iy \in L(G)$ *for all* $i \geq 0$. □

By comparing the conditions (2) and (2*) we can observe, that the latter result, unlike the former one (Bar–Hillel's lemma, Theorem 2.2), does not contain any upper bound on the length of vwx. Therefore it is not an extension of Bar–Hillel's lemma.

Consider the set Q of all primitive words over an alphabet X. The next theorem is widely known as *Borwein's lemma*.

Theorem 2.4 (D. Borwein, see [32], p. 8). *If $a \in X$, $pq \in X^+ \setminus a^+$ and $paq \in X^+ \setminus Q$ then $pq \in Q$.* □

We shall also use the following results.

Theorem 2.5 (S. Horváth, J. Karhumäki, J. Kleinj [15]). *A regular language $L \subseteq X^*$ is palindromic if and only if it is the union of finitely many languages of the form*

$$L_p = \{p\},\ L_{q,r,s} = qr(sr)^*q^R, \quad (p, q, r, s \in X^*),$$

where p, r and s are palindromes. □

Theorem 2.6 (S. Horváth, J. Karhumäki, J. Kleijn [15])**.** *A context-free language $L \subseteq X^*$ is palindromic if and only if it is of the form*

$$L = \bigcup_{a \in X \cup \{\lambda\}} \{pap^R : p \in L(a)\},$$

where the $L(a) \subseteq X^*$ $(a \in X \cup \{\lambda\})$ are regular languages (uniquely determined by L). □

3. Slenderness and Polyslenderness

Denote by $|H|$ the *cardinality* of H for any set H and let N be denote the set of *nonnegative integers*. A language L is said to be *length bounded* (or more precisely, density bounded) by a function $f : N \to N$ if we have $|\{w \in L : |w| = n\}| \leq f(n)$. Note that every language $L \subseteq X^*$ is length bounded by $f(n) = |X|^n$. A language that is length bounded by a polynomial of degree k is termed *k-polyslender. Slender languages* coincide with 0-polyslender languages. A language is called *polyslender* iff it is k-polyslender for some k. A language L is called *k-bounded* iff $L \subseteq {w_1}^* \cdots {w_k}^*$ holds for some words $w_1, \ldots, w_k \in X^+$.

Theorem 3.1 (D. Raz [26], A. Szilárd, S. Yu, K. Zhang, J. Shallit [33])**.** *Every $k+1$-bounded language is k-polyslender.* □

The next statement was first proved by M. Latteux and G. Thierrin [22], and later, independently, by D. Raz [26].

Theorem 3.2 (M. Latteux and G. Thierrin [22], D. Raz [26])**.** *Every polyslender context-free language is bounded.* □

We shall use the following simple observation.

Proposition 3.3. *Let L be the union of the languages $L_1, \ldots, L_k$ $(k \geq 1)$. Then L is polyslender if and only if all of $L_1, \ldots, L_k$ are polyslender. In particular, L is slender if and only if all of $L_1, \ldots, L_k$ are slender.* □

Now we consider the following recursive definition. A language $L \subseteq X^*$ is called a *non-crossing* 1*-multiple paired loop language* iff it is of the form

$L = \{uv^n wx^n y : n \geq 0\}$ for some words $u, v, w, x, y \in X^*$ with $vx \neq \lambda$. Inductively, for every pair k, ℓ of positive integers, L is a *non-crossing* $k+\ell$*-multiple paired loop language* iff one of the following conditions holds:

(i) $L = \{uv^n L' x^n y : n \geq 0\}$ for some non-crossing $k + \ell - 1$-multiple paired loop language L' and words u, v, x, y with $vx \neq \lambda$;

(ii) $L = L_1 L_2$, where L_1 is a non-crossing k-multiple paired loop language and L_2 is a non-crossing ℓ-multiple paired loop language.

In addition, non-crossing 1-multiple paired loop languages are simply called *paired loop languages.* $L \subseteq X^*$ is called a k*-multiple loop language* iff there exist $u_1, v_1, \ldots, u_k, v_k, u_{k+1} \in X^*$ such that, $L = u_1 v_1^* \ldots u_k v_k^* u_{k+1}$. 1-multiple loop languages are simply called *loop languages.*

For slender regular languages, we have the following characterization, first proved by M. Kunze, H. J. Shyr and G. Thierrin [21], and later, independently, by J. Shallit [29], [30], [31], and more later, also independently, by G. Păun and A. Salomaa [24] ([30] and [31] are an extended abstract form and a revised form, respectively, of [29]).

Theorem 3.4 (M. Kunze, H. J. Shyr and G. Thierrin [21], J. Shallit [29], [30], [31] and G. Păun, A. Salomaa [24])**.** *A regular language is slender if and only if it is a finite disjoint union of loop languages.* □

The next extension of the above result also holds.

Theorem 3.5 (A. Szilárd, S. Yu, K. Zhang, J. Shallit [33])**.** *Given a nonnegative integer k, a regular language is k-polyslender if and only if it is a finite disjoint union of $(k+1)$-multiple loop languages.* □

The next theorem was proved by M. Lattux and G. Thierrin [22] and later, independently, by L. Ilie [17] and D. Raz [26]. It was also conjectured by G. Păun and A. Salomaa [24].

Theorem 3.6 (M. Latteux and G. Thierrin [22], L. Ilie [17], D. Raz [26])**.** *Every slender context-free language is a finite disjoint union of paired loop languages.* □

Following S. Ginsburg [11], for any pair of words $x, y \in X^*$ and $Z \subseteq X^*$ we put

$$(x, y) \star Z = \{x^n Z y^n : n \geq 0\}.$$

Theorem 3.7 (S. Ginsburg [11]). *The family of bounded context-free languages is the smallest family of languages containing all finite languages and closed with respect to the following operations: finite union, finite product, $(x, y) \star Z$, where x and y are words.* □

Using this result, the following characterization can be derived for polyslender languages.

Theorem 3.8 (P. Dömösi and A. Matescu [9]). *A context-free language is k-polyslender if and only if it is a finite union of non-crossing $k+1$-multiple paired loop languages.* □

We note that L. Ilie, G. Rozenberg and A. Salomaa [18] also give a characterization of polyslender languages which is essentially equivalent to the above statement. □

4. Main results

Throughout this section we assume that our alphabet (X) has at least two letters (say $a, b \in X$).

Lemma 4.1. *Let $L \subseteq X^*$ be a palindromic language (i.e., $L \subseteq \mathrm{Pal}(X)$). Then L is context-free if and only if $\mathrm{Pal}(X) \setminus L$ is context-free. Furthermore, if we write L in the form*

$$L = \bigcup_{a \in X \cup \{\lambda\}} \{pap^R : p \in L(a)\}$$

(where by Theorem 2.6, the languages $L(a)$ are regular and uniquely determined by L), then

$$\mathrm{Pal}(X) \setminus L = \bigcup_{a \in X \cup \{\lambda\}} \{pap^R : p \in X^* \setminus L(a)\}.$$

Proof. Easy by Theorem 2.6 and Proposition 2.1. □

Consider again the set Q $(= Q(X))$ of all primitive words over X. We put $Q^{(1)} = Q \cup \{\lambda\}$ and $Q^{(i)} = \{q^i : q \in Q\}$ for $i > 1$. The next result shows a distant analogy with Rice's theorem in recursion theory if we let "context-free" correspond to "recursive" (see, e.g., [12]).

Theorem 4.2. *Let $H \subseteq \{1,2,3,\dots\}$ and let $L(H) = \mathrm{Pal} \cap \{\bigcup_{i\in H} Q^{(i)}\}$. Then $L(H)$ is context-free if and only if either $H = \emptyset$ or $H = \{1,2,3,\dots\}$.*

First proof. If $H = \emptyset$ or $H = \{1,2,3,\dots\}$ then obviously L is context-free (because then either $L = \emptyset$ or $L = \mathrm{Pal}$, respectively). Otherwise, by Lemma 4.1, we can suppose $1 \notin H$ (since in the opposite case we can take $\{1,2,3,\dots\} \setminus H$ instead of H), and let $k \in H$ be arbitrary, so $k \geq 2$. Now suppose indirectly that $L(H)$ is context-free, so it has to satisfy Bar–Hillel's lemma (Theorem 2.2), with some constant $p \geq 1$. Clearly

$$a^{p+1}b^{p+1}a^{p+1} \in \mathrm{Pal} \cap Q,$$

so putting

$$z = (a^{p+1}b^{p+1}a^{p+1})^k,$$

we have

$$z \in \mathrm{Pal} \cap Q^{(k)} \subseteq L(H).$$

Obviously $|z| > p$, so by Bar–Hillel's lemma, there is a factorization $z = uvwxy$ such that, $|vx| > 0$, $|vwx| \leq p$ and $z' = uwy \in L(H)$, i.e., we have taken $i = 0$ as "iteration exponent" in (3) (in Bar–Hillel's lemma). Since $1 \notin H$, z' should be nonprimitive. Consider now the following cyclic permutation $\bar{z}$ of z:

$$\bar{z} = (a^{2p+2}b^{p+1})^k.$$

The original cancellation of v and x from z appears in a circularly shifted way in $\bar{z}$, yielding some word $\bar{z}'$ which is a cyclic permutation of z'. Furthermore, since $|vwx| \leq p$, the shifted cancellation reduces either the length of *exactly one* "a-portion" a^{2p+2} or that of *exactly one* "b-portion" b^{p+1}, *or both* at a time, and in each case, *the reduced length/lengths* is/are still *positive*. Therefor e in each case, since $k > 1$, clearly $\bar{z}'$ has to be primitive. Finally, by the obvious invariance of primitivity under cyclic permutation, z' has to be primitive, too, a contradiction. □

Remarks. 1. Here we have used a strongly restricted version of Bar–Hillel's lemma, in which the iteration exponent can only be 0. More generally, we can even observe that *this restricted version of Bar–Hillel's lemma is still a powerful device,* it could be used instead of the full version, in most "classical", typical non-context-freeness proofs, known from the literature.

2. In the above first proof, we did not use the requirement that $z' = uwy$ should be a palindrome, too.

Second proof. The difference relative to the first proof begins after choosing an arbitrary nonempty subset H of the set $\{2, 3, 4, \dots\}$ and indirectly supposing that $L(H)$ is context-free. We start with a few. easy observations on the words z of the form

$$z = d(k_1, \ell_1) \dots d(k_t, \ell_t),$$

where $d(k, \ell)$ is a shorthand with the meaning

$$d(k, \ell) = ab^k ab^\ell a, \quad \text{for} \quad k, \ell \geq 1$$

(so clearly $d(k, \ell)$ is always a *primitive word*), further, $t \geq 2$, and $k_1, \dots, k_t$, $\ell_1, \dots, \ell_t \geq 1$. With the further shorthand $d_j = d(k_j, \ell_j)$, $j = 1, \dots, t$, we can write z in the following, more concise form

$$z = d_1 \dots d_t.$$

Now it is easy to see that for an arbitrary $s \geq 2$, z is an *s-th power* if and only if s divides t, and

$$z = (d_1 \dots d_{t/s})^s.$$

So for z to be an *s-th power*, it is necessary that in the sequence

$$d_1 \dots d_t,$$

every member should have *at least s occurrences*. If here t is odd, then necessarily even $s \geq 3$. Now we apply Theorem 2.3 to $L(H)$. To this end, in z let $t \in H$, and let

$$d_1, \dots, d_t = d(m, m), \quad \text{i.e.,} \ z = d(m, m)^t,$$

where

$$m \geq 3(c - 1)/2, \quad \text{i.e.,} \ 2m + 3 \geq 3c,$$

and c (≥ 2) is the constant from Theorem 2.3. Clearly

$$z \in \mathrm{Pal} \cap Q^{(t)} \subseteq L(H).$$

So we can apply Theorem 2.3 to the word z of $L(H)$, excluding in z exactly those positions containing an 'a', and choosing iteration exponent $i = 2$

in (3) (in the theorem). The iteration (with $i = 2$) modifies one or two of the t d-factors of z, and we obtain a modified word $z' \in L(H)$. If now t is odd, then (see above) for z' to be nonprimitive, there should be at least three modified (longer) d-factors in z', which is impossible, So let t be even. Then, since z' should be a palindrome, too, in z necessarily exactly two, symmetrically positioned d-factors, say d_j and d_{t+1-j}, for some $1 \leq j \leq t/2$, will be modified into some $(d_j)'$ and $(d_{t+1-j})'$ such that,

$$(d_j)' = ((d_{t+1-j})')^R,$$

so these two, modified d-factors differ *from one another, and from all the other, unchanged* d-factors, too. (Namely, for some $1 \leq m' \leq m$, either $(d_j)' = d(m+m', m)$, and $(d_{t+1-j})' = d(m, m+m')$, or vice versa.) So each of the two different, modified d-factors has only one occurrence in z', therefore (see the above), z' cannot be nonprimitive, a contradiction again. $\square$

Conjecture. *We conjecture that the statement of Theorem 4.2 remains valid if we omit "*$\mathrm{Pal}\cap$*" from the definition of $L(H)$.*

From this "new" conjecture our "old" conjecture (see [5]) that Q is not context-free, would easily follow as a special case. However at present we cannot even prove that Q is not context-free. We can (easily) prove only (similarly to the above proof of Theorem 4.2) that $i \geq 2$ and $Q^{(i)} \subseteq L \subseteq X^* \setminus Q$ imply that L is not context-free.

Problem. *Characterize those palindromic regular, context-free, context-sensitive, and phrase-structure (type* 0 *or recursively enumerable) languages, consisting of primitive words. Furthermore, characterize those linear, indexed, and linear indexed languages, that have the same property.*

Now we prove two Lemmas.

Lemma 4.3. *Let $L = L_1 \cup \ldots \cup L_k (k \geq 1)$. If L is dense then one of the L_i is dense.*

Proof. Supposing the contrary, let $w_j \in X^*$ such that, $L_j \cap X^* w_j X^* = \emptyset$ $(j = 1, \ldots, k)$. Then, by L being dense, $L \cap X^* w_1 \cdots w_k X^* \neq \emptyset$. Therefore for some $i, 1 \leq i \leq k$, $L_i \cap X^* w_1 \ldots w_k X^* = L_i \cap X^* w_1 \ldots w_i \ldots w_k X^* \neq \emptyset$, which implies $L_i \cap X^* w_i X^* \neq \emptyset$, a contradiction. $\square$

Lemma 4.4. *Let* $L = \bigcup_{a \in X \cup \{\lambda\}} \{uau^R : u \in L(a)\}$ *be an arbitrary palindromic language. (The languages* $L(a)$ *are uniquely determined by* L*.) If* L *is dense, then* $L(a)$ *is dense for some* $a \in X \cup \{\lambda\}$*.*

PROOF. By Lemma 4.3, $\{uau^R : u \in L(a)\}$ is dense for some $a \in X \cup \{\lambda\}$. Let $w \in X^*$. Therefore, there exist $x, y \in X^*$ and $u \in L(a)$ such that $xww^Rww^Ry = uau^R$. Consequently, ww^R is a subword of $L(a)$ or $L(a)^R$. In fact, in either case, ww^R is a subword of $L(a)$. Hence $X^*wX^* \cap L(a)$ is not empty. This completes the proof of the lemma. $\square$

Given a language $L \subseteq X^*$, let $\deg(L) = \{i \geq 0 : p^i \in L$ for some $p \in Q\}$. (We put $z0 = \lambda$ for any word z.) We prove the following partial result concerning our above mentioned open problem.

Theorem 4.5. *Let* $L \subseteq X^*$ *be a dense palindromic context-free language. Then* $L \cap (X^+ \setminus Q)$ *is not empty. More exactly, we have that* $\deg(L)$ *is infinite.*

PROOF. Let $L = \bigcup_{a \in X \cup \{\lambda\}} \{uau^R : u \in L(a)\}$. By Lemma 4.4 and Theorem 2.6, there exists an $a \in X \cup \{\lambda\}$ such that $L(a)$ is a dense regular language. Let $A = (S, X, \delta, s_0, F)$ be a deterministic automaton accepting $L(a)$. Put $Sp = \{\delta(s, p) : s \in S\}$ $(p \in X^*)$. Let $v \in X^*$ be a word such that $|Sv| = \min\{|Sx| : x \in X^*\}$. Since $L(a)$ is dense, there exist $u, w \in X^*$ such that $r = uvw \in L(a)$. It is obvious that $|Sr| = |Sv| = \min\{|Sx| : x \in X^*\}$. Since $S(ar^Rr)2 \subseteq Sar^Rr \subseteq Sr$ and $|Sr| = \min\{|Sx| : x \in X^*\}$, $S(ar^Rr)2 = Sar^Rr = Sr$. Hence there exists $k, k \geq 1$ such that $S(ar^Rr)^k$ and Sr coincide. Now consider $r(ar^Rr)^{ik} \in X^*$ for any i, $i \geq 1$. Then $\delta(s_0, r(ar^Rr)^{ik}) = \delta(\delta(s_0, r), (ar^Rr)^{ik}) = \delta(s_0, r) \in F$ and $r(ar^Rr)^{ik} \in L(a)$. Notice that $r(ar^Rr)^{ik}a[(ar^Rr)^{ik}]^Rr^R \in \{uau^R : u \in L(a)\} \subseteq L$. Therefore, $r(ar^Rr)^{ik}a[(ar^Rr)^{ik}]^Rr^R = r(ar^Rr)^{ik}a(r^Rra)^{ik}r^R =$
$(rar^R)^{ik}rar^R(rar^R)^{ik} = (rar^R)^{2ik+1} \in L$, for every $i \geq 1$. Hence $\deg(L)$ is infinite. $\square$

Using Theorem 4.5, we get two further partial results concerning our open problem above. (A *primitive palindrome* is defined as a primitive word which is a palindrome at the same time.)

Corollary 4.6. *There exists no dense palindromic context-free language consisting of primitive words.* □

The second corollary actually represents a completely different proof of a special case ($\{H\} = 1$) of Theorem 4.2 (since of course, for any language L, $L \cap Q$ is context-free if and only if $L \cap Q^{(1)}$ is context-free).

Corollary 4.7. *The language of all primitive palindromes is not context-free.* □

Now we prove another property of the language of all primitive palindromes.

Proposition 4.8. $Q \cap \mathrm{Pal}$ *cannot be generated by any self-embedding grammar.*

Proof. Consider a grammar $G = (V, X, S, P)$ with $S \to ASB \in P$ for some $A, B \in (V \cup X)^*$ such that $L(G)$ is nonempty, moreover, for an appropriate pair w, z with $wz \in X^+$, $A \overset{*}{\Rightarrow} w$ and $B \overset{*}{\Rightarrow} z$. Suppose that, contrary to our statement, $L(G) = Q \cap \mathrm{Pal}$. First we prove $w = z^R$.

Let k be a positive integer with $k > |w|, |z|$. Moreover, let a, b be a pair of distinct letters in X. Then $a^k b a^k \in Q \cap \mathrm{Pal}$. In addition, by using $S \to ASB \in P$, $A \overset{*}{\Rightarrow} w$ and $B \overset{*}{\Rightarrow} z$, $wa^k ba^k z \in Q \cap \mathrm{Pal}$ also holds. By $k > |w|, |z|$, this directly implies $w = z^R$. On the other hand, then $azwazwa \in \mathrm{Pal}$ for every $a \in X$. In addition, $(azw)2 \notin Q$ trivially holds. But then, using Borwein's Lemma (Theorem 2.4), $zw \notin a^+$ implies $azwazwa \in Q$. Let us suppose $wz \notin a^+$. (Otherwise we can consider another letter of X instead of a.) Then $azwazwa \in Q \cap \mathrm{Pal}$. Simultaneously, $S \overset{*}{\Rightarrow} wazwazwaz$, a contradiction. This ends the proof. □

Now by using the above argument we can prove the following

Proposition 4.9. Q *cannot be generated by any self-embedding grammar.*

Proof. Consider again a grammar $G = (V, X, S, P)$ with $S \to ASB \in P$ for some $A, B \in (V \cup X)^*$ such that $L(G)$ is nonempty, moreover, for an appropriate pair w, z with $wz \in X^+$, $A \overset{*}{\Rightarrow} w$ and $B \overset{*}{\Rightarrow} z$. Suppose that, contrary to our statement, $L(G) = Q$. Let us consider a letter $a \in X$ with $wz \notin a^+$. Then, using Borwein's Lemma (Theorem 2.4), $azwazwa \in Q$

because $azwazw \notin Q$ obviously holds. Simultaneously, $S \overset{*}{\Rightarrow} wazwazwaz$, a contradiction. The proof is complete. $\square$

By Theorems 2.4 and 3.4 we immediately get the following statement.

Proposition 4.10. *Every palindromic regular language is slender.* $\square$

It is easy to see that the non-regular context-free palindromic language $\{pp^R : p \in X^*\}$ is non-slender. Therefore, the above statement cannot be extended to context-free languages. However, we prove the following two characterizations.

Theorem 4.11. *Every slender palindromic context-free language is a finite union of languages of the form* $\{uv^n w(v^R)^n u^R : n \geq 0,\ w \text{ is a palindrome}\}$.

Proof. Consider a slender palindromic context-free language L. Then by Theorem 2.6 we have $L = \bigcup_{a \in X \cup \{\lambda\}} \{pap^R : p \in L(a)\}$, where the $L(a)$ $(a \in X \cup \{\lambda\})$ are regular languages. By Proposition 3.3, these languages have to be slender. Therefore, by Theorem 3.4, they have to be finite unions of loop languages. Thus we have that for every $a \in X \cup \{\lambda\}$ there are words $u_i, v_i, w_i \in X^*$, $i = 1, \dots, m$ such that,

$$\{pap^R : p \in L(a)\} = \bigcup_{i=1}^{m} \{u_i v_i^n w_i a w_i^R (v_i^R)^n u_i^R : n \geq 0\}.$$

This completes the proof. $\square$

Theorem 4.12. *Every polyslender palindromic context-free language is a finite union of languages of the form* $\{w_1 z_1^{n_1} w_2 z_2^{n_2} w_3 \dots w_t z_t^{n_t} p (z_t^R)^{n_t} w_t^R \dots w_3^R (z_2^R)^{n_2} w_2^R (z_1^R)^{n_1} w_1^R : t \geq 1,\ n_1, \dots, n_t \geq 0,\ p \text{ is a palindrome}\}$.

Proof. Let L be a polyslender palindromic context-free language. In view of Theorem 2.6 we have $L = \bigcup_{a \in X \cup \{\lambda\}} \{pap^R : p \in L(a)\}$, where the $L(a)$ $(a \in X \cup \{\lambda\})$ are regular languages. By Proposition 3.3, these languages have to be polyslender. In consequence of Theorem 3.5, they are finite unions of multiple loop languages. Hence, for every $a \in X \cup \{\lambda\}$ there are words $u_{i,j}, v_{i,k} \in X^*$, $i = 1, \dots, m$, $j = 1, \dots, m_i + 1$, $k = 1, \dots, m_i$

such that,

$$\{pap^R : p \in L(a)\} = \bigcup_{i=1}^{m} \{u_{i,1}v_{i,1}^{n_{i,1}}u_{i,2}v_{i,2}^{n_{i,2}} \dots u_{i,m_i}v_{i,m_i}^{n_{i,m_i}}u_{i,m_i+1}au_{i,m_i+1}^R(v_{i,m_i}^R)^{n_{i,m_i}} \dots \dots u_{i,3}^R(v_{i,2}^R)^{n_{i,2}}u_{i,2}^R(v_{i,1}^R)^{n_{i,1}}u_{i,1}^R : n_{i,k} \geq 0,\ k = 1, \dots, m_i\}.$$

The proof is complete. □

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