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Oscillation and nonoscillation of solutions for second order linear differential equations

By HUEI-LIN HONG (Tauyuan) CHEH-CHIH YEH (Tauyuan),
HORNG-JAAN LI (Chang-Hua) and HSIANG-BIN HSU (Tauyuan City)

Abstract. Oscillation and nonoscillation criteria are established for the second order linear differential equation

$$[p(t)x'(t)]' + q(t)x(t) = 0, \quad t \geq t_0,$$

under the hypothesis that $p(t) > 0$ and

$$\int^{\infty} \frac{dt}{a(t)p(t)} = \infty,$$

where $a(t) \in C^2([t_0,\infty);(0,\infty))$ is given. These results improve some oscillation criteria of Hille, Wintner and Opial.

1. Introduction

In this paper, we consider the second order linear differential equation

$$\left[p(t)x'(t)\right]' + q(t)x(t) = 0 \tag{E}$$

and

$$\left[p_1(t)x'(t)\right]' + q_1(t)x(t) = 0, \tag{E_1}$$

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where $p(t), p_1(t) \in C^1\big([t_0,\infty),(0,\infty)\big)$ and $q(t), q_1(t) \in C\big([t_0,\infty),\mathbb{R}\big)$ for some $t_0 \geq 0$. Suppose that there exist two functions $a(t), a_1(t) \in C^2([t_0,\infty);(0,\infty))$ such that

$$\int^\infty \frac{dt}{a(t)p(t)} = \infty \quad \text{and} \quad \int^\infty \frac{dt}{a_1(t)p_1(t)} = \infty.$$

A solution of (E) is oscillatory if it has arbitrarily large zeros, and otherwise it is nonoscillatory. Equation (E) is oscillatory if all its solutions are oscillatory, and nonoscillatory if all its solutions are nonoscillatory.

In 1984, Harris [3] improved the Leighton oscillation criterion [6] and the Sturm comparison theorem by using a generalized Riccati transformation

$$v(t) = A(t)p(t)\left\{\frac{x'(t)}{x(t)} + F(t)\right\},$$

where $F \in C^1$ is a given function and $A(t) = \exp\big\{-2\int^t F(s)ds\big\}$. The following two theorems are due to Harris [3] and Li and Yeh [8], respectively.

Theorem A. *If*

$$\int^\infty \frac{1}{A(t)p(t)}dt = \int^\infty A(t)\big\{q(t) + p(t)F^2(t) - \big[p(t)F(t)\big]'\big\}dt = \infty,$$

then (E) is oscillatory.

Theorem B. *Let $a \in C^2([t_0,\infty),(0,\infty))$ be a given function and $f(t) = -\frac{a'(t)}{2a(t)}$. Then equation (E) is oscillatory if and only if the equation*

$$[a(t)p(t)w'(t)]' + a(t)\left\{q(t) + p(t)f^2(t) - [p(t)f(t)]'\right\}w(t) = 0$$

is oscillatory.

Moreover, Li and Yeh [8] obtained the following result:

Theorem C. *Let $a \in C^2([t_0,\infty),(0,\infty))$ be a given function and $f(t) = -\frac{a'(t)}{2a(t)}$. If*

$$\int^\infty \frac{dt}{a(t)p(t)} = \int^\infty a(t)\big[q(t) + p(t)f^2(t) - \big(p(t)f(t)\big)'\big]dt = \infty,$$

then (E) is oscillatory.

It is clear that Theorem C cannot be applied under the condition

$$\phi(t) := \int_t^\infty \psi(s)\,ds < \infty, \tag{C_0}$$

where $\psi(s) = a(s)[q(s) + p(s)f(s)^2 - (p(s)f(s))']$.

In 1987, Yan [16] gave some excellent oscillation criteria for equation

$$x''(t) + q(t)x(t) = 0 \tag{E_2}$$

which extended some oscillation criteria of Fite [1], Hille [2], Kamenev [4], Leighton [7], Opial [11], and Wintner [13]–[15]. The purpose of this paper is to establish a necessary and sufficient condition for the nonoscillatory criterion of (E) which is a natural extension of Theorem 2.1 in Yan [16]. Using this necessary and sufficient condition, we can extend the Hille–Wintner comparison theorem for equation of the form (E_2) to equation of the type (E).

2. Nonoscillation and oscillation criteria for equation (E)

Throughout this paper, we let $f(t) = -\frac{a'(t)}{2a(t)}$, $f_1(t) = -\frac{a_1'(t)}{2a_1(t)}$,
$\psi(t) = a(t)[q(t) + p(t)f(t)^2 - (p(t)f(t))']$,
$\psi_1(t) = a_1(t)[q_1(t) + p_1(t)f_1(t)^2 - (p_1(t)f(t))']$,

$$\phi(t) := \int_t^\infty \psi(s)\,ds$$

and

$$\phi_1(t) := \int_t^\infty \psi_1(s)\,ds,$$

where $a(t), a_1(t) \in C^2([t_0,\infty),(0,\infty))$ are given. In other to prove our main results, we need the following lemma which is due to Li and Yeh [9].

Lemma 1. *Suppose that there exists a function $a(t) \in C^2([t_0,\infty);(0,\infty))$ such that*

$$\int^\infty \frac{dt}{a(t)p(t)} = \infty$$

and

$$\phi(t) := \int_t^\infty \psi(s)\,ds < \infty \quad \textit{for all } t \geq t_0,$$

then the following four statements are equivalent:

(i) *Equation* (E) *is nonoscillatory.*

(ii) *There is a function* $w \in C([T,\infty);\mathbb{R})$ *for some* $T \geq t_0$ *such that*

$$w(t) = \int_t^\infty \frac{w(s)^2}{a(s)p(s)}\,ds + \int_t^\infty \psi(s)\,ds \quad \textit{for } t \geq T. \tag{C_1}$$

In particular, if $x(t)$ *is a nonoscillatory solution of* (1)*, then* $w(t)$ *can be taken as*

$$w(t) = \frac{a(t)p(t)x'(t)}{x(t)}, \quad \textit{for } t \geq T.$$

(iii) *There is a function* $v \in C([T,\infty);\mathbb{R})$ *for some* $T \geq t_0$ *such that*

$$|v(t)| \geq \left| \int_t^\infty \frac{v(s)^2}{a(s)p(s)}\,ds + \int_t^\infty \psi(s)\,ds \right| \quad \textit{for } t \geq T. \tag{C_2}$$

(iv) *There is a function* $u \in C^1([T,\infty);\mathbb{R})$ *for some* $T \geq t_0$ *satisfying*

$$u'(t) + \psi(t) + \frac{u(t)^2}{a(t)p(t)} \leq 0 \quad \textit{for } t \geq T. \tag{C_3}$$

Throughout this section we suppose that $\alpha_0(t) \in C([t_0,\infty);\mathbb{R})$ is a given function and (C_0) holds. We define the function sequence

$$\{\alpha_n(t)\}_{n=0}^\infty, \quad \text{for } t \geq t_0,$$

as follows (if it exists):

$$\alpha_n(t) = \int_t^\infty \frac{\alpha_{n-1}^+(s)^2}{a(s)p(s)}\,ds + \alpha_0(t), \quad n = 1, 2, \ldots, \tag{1}$$

where $\alpha^+(t) = \frac{1}{2}\big[\alpha(t) + |\alpha(t)|\big]$.

Clearly, $\alpha_1(t) \geq \alpha_0(t)$ and this implies that $\alpha_1^+(t) \geq \alpha_0^+(t)$. By induction,

$$\alpha_{n+1}(t) \geq \alpha_n(t), \quad n = 1, 2, \ldots . \tag{2}$$

That is, the function sequence $\{\alpha_n(t)\}$ is nondecreasing on $[t_0,\infty)$.

Theorem 2. *Suppose that $\alpha_0(t) \le \phi(t)$. If equation (E) is nonoscillatory, then there exists $t_1 \ge t_0$ such that*

$$\lim_{n\to\infty} \alpha_n(t) := \alpha(t) < \infty \quad \text{for } t \ge t_1. \tag{3}$$

Proof. Suppose that (E) is nonoscillatory. Thus, it follows from Lemma 1 that there exists $w \in C[t_1, \infty)$ such that

$$w(t) = \int_t^\infty \frac{w(s)^2}{a(s)p(s)}\,ds + \int_t^\infty \psi(s)\,ds$$

on $[t_1, \infty)$ for some $t_1 \ge t_0$. Thus, $w(t) \ge \alpha_0(t)$, and hence $w^+(t) \ge \alpha_0^+(t)$ for $t \ge t_1$. This implies

$$\begin{aligned} w(t) &= \int_t^\infty \frac{w(s)^2}{a(s)p(s)}\,ds + \int_t^\infty \psi(s)\,ds \\ &\ge \int_t^\infty \frac{w^+(s)^2}{a(s)p(s)}\,ds + \alpha_0(t) \ge \int_t^\infty \frac{\alpha_0^+(s)^2}{a(s)p(s)}\,ds + \alpha_0(t) \\ &= \alpha_1(t) \quad \text{for } t \ge t_1. \end{aligned}$$

By induction,

$$w(t) \ge \alpha_n(t), \quad n = 0, 1, 2, \dots,\ t \in [t_1, \infty). \tag{4}$$

It follows from (2) and (4) that the function sequence $\{\alpha_n(t)\}$ is bounded above on $[t_1, \infty)$. Hence (3) holds. □

Corollary 3. *Suppose that $\alpha_0(t) \le \phi(t)$. If either*

(i) *there exists a positive integer m such that $\alpha_n(t)$ is defined for $n = 1, 2, \dots, m-1$, but $\alpha_m(t)$ does not exist; or*

(ii) *$\alpha_n(t)$ is defined for $n = 1, 2, \dots$, but for arbitrarily large $T^* \ge t_0$, there is $t^* \ge T^*$ such that*

$$\lim_{n\to\infty} \alpha_n(t^*) = \infty,$$

then equation (E) is oscillatory.

Theorem 4. *Suppose that $\alpha_0(t) \ge |\phi(t)|$. If there exists $t_1 \ge t_0$ such that*

$$\lim_{n\to\infty} \alpha_n(t) = \alpha(t) < \infty \quad \text{for } t \ge t_1, \tag{5}$$

then equation (E) is nonoscillatory.

PROOF. If (5) holds, then it follows from (2) and (5) that

$$\alpha_n(t) \le \alpha(t),\ n = 0, 1, 2, \ldots, \quad \text{for } t \ge t_1.$$

Applying the monotone convergence theorem,

$$\alpha(t) = \int_t^\infty \frac{\alpha^+(s)^2}{a(s)p(s)}\, ds + \alpha_0(t)\ (\ge 0), \quad \text{for } t \ge t_1.$$

Thus,

$$\begin{aligned} \alpha^+(t) = \alpha(t) &= \int_t^\infty \frac{\alpha^+(s)^2}{a(s)p(s)}\, ds + \alpha_0(t) \\ &\ge \int_t^\infty \frac{\alpha^+(s)^2}{a(s)p(s)}\, ds + |\phi(t)| \\ &\ge \left| \int_t^\infty \frac{\alpha^+(s)^2}{a(s)p(s)}\, ds + \phi(t) \right| \quad \text{for } t \ge t_1. \end{aligned}$$

It follows from Lemma 1 that (E) is nonoscillatory. Thus, our proof is complete. □

Corollary 5. *Suppose that $\alpha_0(t) \ge |\phi(t)|$. If (E) is oscillatory, then either*

(i) *there exists a positive integer m such that $\alpha_n(t)$ is defined for $n = 1, 2, \ldots, m-1$, but $\alpha_m(t)$ does not exist; or*

(ii) *$\alpha_n(t)$ is defined for $n = 1, 2, \ldots$, but, for arbitrarily large $T^* \ge t_0$, there is $t^* \ge T^*$ such that*

$$\lim_{n\to\infty} \alpha_n(t^*) = \infty.$$

If $\phi(t) \ge 0$, then it follows from Theorems 2 and 4 that we have the following two corollaries.

Corollary 6. *Suppose that $\alpha_0(t) = \phi(t) \ge 0$. Then (E) is nonoscillatory if and only if there exists $t_1 \ge t_0$ such that*

$$\lim_{n\to\infty} \alpha_n(t) = \alpha(t) < \infty \quad \textit{for } t \ge t_1.$$

Corollary 7. *Suppose that* $\alpha_0(t) = \phi(t) \geq 0$. *Then* (E_1) *is oscillatory if and only if either*

(i) *there exists a positive integer* m *such that* $\alpha_n(t)$ *is defined for* $n = 1, 2, \ldots, m-1$, *but* $\alpha_m(t)$ *does not exist; or*

(ii) $\alpha_n(t)$ *is defined for* $n = 1, 2, \ldots,$ *but, for arbitrarily large* $T^* \geq t_0$, *there is* $t^* \geq T^*$ *such that*

$$\lim_{n\to\infty} \alpha_n(t^*) = \infty.$$

Remark 1. For $a(t) = 1$, Corollaries 6 and 7 reduce to Theorems 2.1 and 2.2 in Yan [16], respectively.

Theorem 8. *If there exists a function* $B(t) \in C([t_1, \infty); \mathbb{R})$ *for some* $t_1 \geq t_0$ *such that*

$$|\phi(t)| + \int_t^\infty \frac{B^+(s)^2}{a(s)p(s)}\, ds \leq B^+(t) \quad \text{for } t \geq t_1, \tag{6}$$

then equation (E) *is nonoscillatory.*

Proof. Let $\alpha_0(t) = |\phi(t)|$. Then, by (6), $0 \leq \alpha_0(t) \leq B^+(t)$ for $t \geq t_1$. Thus

$$\begin{aligned} \alpha_1(t) &= \int_t^\infty \frac{\alpha_0^+(s)^2}{a(s)p(s)}\, ds + \alpha_0(t) \\ &\leq \int_t^\infty \frac{B^+(s)^2}{a(s)p(s)}\, ds + |\phi(t)| \\ &\leq B^+(t) \quad \text{for } t \geq t_1. \end{aligned}$$

By induction,

$$\alpha_n(t) \leq B^+(t), \quad n = 0, 1, 2, \ldots, \ t \in [t_1, \infty).$$

This and (2) imply that (5) holds. Thus, it follows from Theorem 4 that (E) is nonoscillatory. □

Taking $B(t) = 2|\phi(t)|$ in the above theorem, we obtain the following corollary which improves a result of Wintner [14].

Corollary 9. *If*

$$\int_t^\infty \frac{\phi(s)^2}{a(s)p(s)}\,ds \le \frac{|\phi(t)|}{4}$$

then equation (E) *is nonoscillatory.*

Theorem 10. *Suppose that there exists a function* $B(t) \in C([t_0,\infty);\mathbb{R})$ *with the property that for arbitrarily large* $T^* \ge t_0$*, there is* $t^* \ge T^*$ *such that* $B(t^*) > 0$*. If*

$$B(t) \le \phi(t) \text{ and } kB(t) \le \int_t^\infty \frac{B^+(s)^2}{a(s)p(s)}\,ds \quad \text{for all sufficient large } t, \tag{7}$$

where $k > \frac{1}{4}$ *is a constant, then equation* (E) *is oscillatory.*

Proof. Let $\alpha_0(t) = \phi(t)$. This and (7) imply that $\alpha_0^+(t) = \phi^+(t) \ge B^+(t)$ on $[t_1,\infty)$ for some $t_1 \ge t_0$. Thus

$$\begin{aligned}\alpha_1(t) = \int_t^\infty \frac{\alpha_0^+(s)^2}{a(s)p(s)}\,ds + \alpha_0(t) &\ge \int_t^\infty \frac{B^+(s)^2}{a(s)p(s)}\,ds + B(t)\\ &\ge (k+1)B(t) := C_1B(t), \quad \text{where } C_1 = k+1.\end{aligned}$$

By induction,

$$\begin{gathered}\alpha_n(t) \ge C_nB(t) \quad \text{for } t \in [t_1,\infty),\\ \text{where } C_n = 1 + kC_{n-1}^2,\ n = 1,2,\dots.\end{gathered} \tag{8}$$

Clearly, $C_n > C_{n-1}$, $n = 1,2,3,\dots$. Now we show that $\lim_{n\to\infty} C_n = \infty$. Suppose the increasing sequence $\{C_n\}$ is bounded above. Hence, $\lim\limits_{n\to\infty} C_n$ exists, say, $\lim_{n\to\infty} C_n = \beta \in \mathbb{R}$. From $C_n = 1 + kC_{n-1}^2$,

$$\beta = 1 + k\beta^2. \tag{9}$$

Since $k > \frac{1}{4}$, the equation (9) has no real root. This contradiction proves that $\lim_{n\to\infty} C_n = \infty$. This and (8) imply that

$$\lim_{n\to\infty} \alpha_n(t^*) = \infty,$$

where $t^* \ge t_1$ satisfies $B(t^*) > 0$. Thus, it follows from Corollary 3 that (E) is oscillatory. $\square$

Taking $B(t) = \phi(t)$ in the above theorem, we obtain the following corollary which improves an OPIAL's result [11].

Corollary 11. *Suppose that for arbitrarily large $T^* \geq t_0$, there is $t^* \geq T^*$ such that $\phi(t^*) > 0$. If there exists $\epsilon > 0$ such that*

$$\int_t^\infty \frac{\phi^+(s)^2}{a(s)p(s)}\,ds \geq \frac{(1+\epsilon)}{4}\phi(t) \quad \textit{for all sufficient large } t,$$

then equation (E) is oscillatory.

Remark 2. YAN [16] proved Corollaies 9 and 11 under the stronger condition: $P(t) := \int_t^\infty p(s)\,ds \geq 0$.

Corollary 12. *Suppose that $\alpha_0(t) \leq \phi(t)$. Let $\pi(t) \in C^1\big([t_0,\infty),(0,\infty)\big)$ satisfy $\pi'(t) = \frac{1}{a(t)p(t)}$. If one of the following conditions is satisfied:*

(i) $\alpha_0(t) \geq \frac{C_0}{\pi(t)}$ *forsufficiently large t,*

(ii) $\int_t^\infty \frac{\alpha_0^+(s)^2}{a(s)p(s)}\,ds \geq C_0\alpha_0(t)$ *for $t \geq t_0$,*

(iii) $\lim_{t\to\infty} \alpha_0(t) \geq 0$, *and $\alpha_0'(t) \leq -\frac{C_0}{a(t)p(t)\pi(t)^2}$ for sufficiently large t,*

where $C_0 > \frac{1}{4}$ is a constant, then (E) is oscillatory.

PROOF. If (i) is satisfied, then

$$\begin{aligned}\alpha_1(t) &= \int_t^\infty \frac{\alpha_0^+(s)^2}{a(s)p(s)}\,ds + \alpha_0(t) \geq \int_t^\infty \frac{C_0^2}{a(s)p(s)\pi(t)^2}\,ds + \alpha_0(t)\\ &= \frac{C_0^2}{\pi(t)} + \alpha_0(t) \geq \frac{C_0^2}{\pi(t)} + \frac{C_0}{\pi(t)} = \frac{C_1}{\pi(t)},\end{aligned}$$

where $C_1 = C_0^2 + C_0$. Thus, by induction,

$$\alpha_n(t) \geq \frac{C_n}{\pi(t)}, \tag{10}$$

where $n = 1, 2, \ldots$, and $C_n = C_{n-1}^2 + C_0$. It is easy to see that $C_n > C_{n-1}$, $n = 1, 2, \ldots$. Now we show that $\lim_{n\to\infty} C_n = \infty$. Suppose the increasing sequence $\{C_n\}$ is bounded above. Hence, $\lim_{n\to\infty} C_n$ exists, say, $\lim_{n\to\infty} C_n = \beta \in \mathbb{R}$. From $C_n = C_{n-1}^2 + C_0$,

$$\beta = \beta^2 + C_0. \tag{11}$$

Since $C_0 > \frac{1}{4}$, the equation (11) has no real root. This contradiction proves that $\lim_{n\to\infty} C_n = \infty$. Thus, it follows from (10) that

$$\lim_{n\to\infty} \alpha_n(t) = \infty, \quad t \in [t_0, \infty).$$

Thus, by (ii) of Corollary 3, (E) is oscillatory.

If (ii) is satisfied, then (E) is oscillatory by taking $B(t) = \alpha_0(t)$ in Theorem 10.

Finally, if (iii) is satisfied, then

$$\begin{aligned} -\alpha_0(t) &\le \int_t^\infty \alpha_0'(s)\, ds \\ &\le -C_0 \int_t^\infty \frac{1}{a(s)p(s)\pi(s)^2}\, ds = -\frac{C_0}{\pi(t)}. \end{aligned}$$

Hence, (i) is satisfied, and hence (E) is oscillatory. □

Next we consider equation (E_1). Suppose that $\beta_0(t) \in C([t_0,\infty);\mathbb{R})$ is a given function. Similarly, we define the function sequence

$$\{\beta_n(t)\}_{n=0}^\infty, \quad \text{for } t \ge t_0,$$

as follows (if it exists):

$$\beta_n(t) = \int_t^\infty \frac{\beta_{n-1}^+(s)^2}{a(s)p(s)}\, ds + \beta_0(t), \quad n = 1, 2, \ldots. \tag{12}$$

Clearly, $\beta_1(t) \ge \beta_0(t)$ and this implies that $\beta_1^+(t) \ge \beta_0^+(t)$. By induction,

$$\beta_{n+1}(t) \ge \beta_n(t), \quad n = 1, 2, \ldots. \tag{13}$$

That is, the function sequence $\{\beta_n(t)\}$ defined in (12) is nondecreasing on $[t_0, \infty)$.

Using Theorems 2 and 4, we can give another proof of the following Hille–Wintner comparison theorem which is due to Li and Yeh [9].

Theorem 13. *Assume that*

$$0 < a(t)p(t) \le a_1(t)p_1(t),\ |\phi_1(t)| \le \phi(t) \text{ for all sufficiently large } t. \tag{14}$$

If (E) is nonoscillatory, then (E_1) is nonoscillatory; or equivalently, if (E_1) is oscillatory, then also (E) is oscillatory.

Proof. Let $\alpha_0(t)=\phi(t)$, $\beta_0(t)=|\phi_1(t)|$. Suppose that (E) is nonoscillatory. It follows from Theorem 2 that there exists $t_1 \geq t_0$ such that

$$\lim_{n\to\infty} \alpha_n(t) := \alpha(t) < \infty \quad \text{for } t \geq t_1. \tag{15}$$

Clearly, by (14),

$$\beta_0(t) = |\phi_1(t)| \leq \phi(t) = \alpha_0(t),$$

and hence $\beta_0^+(t) \leq \alpha_0^+(t)$ for $t \geq t_1$. This and (14) imply that, for $t \geq t_1$,

$$\begin{aligned}\beta_1(t) &= \int_t^\infty \frac{\beta_0^+(s)^2}{a_1(s)p_1(s)}\,ds + \beta_0(t)\\ &\leq \int_t^\infty \frac{\alpha_0^+(s)^2}{a(s)p(s)}\,ds + \alpha_0(t) = \alpha_1(t).\end{aligned}$$

By induction,

$$\beta_n(t) \leq \alpha_n(t), \quad n = 0,1,2,\dots,\ t \in [t_1,\infty). \tag{16}$$

Therefore, by (13), (15) and (16),

$$\beta(t) := \lim_{n\to\infty} \beta_n(t) \leq \lim_{n\to\infty} \alpha_n(t) = \alpha(t) < \infty,\ t \in [t_1,\infty).$$

Thus, by Theorem 4, (E_1) is nonoscillatory. Hence, the proof is complete. □

Theorem 14. *Suppose that $\alpha_0(t) \leq \phi(t)$. If equation (E) is nonoscillatory, then*

$$\limsup_{t\to\infty} [\alpha(t) - \phi(t)] \exp\left(4\int^t \frac{\phi^+(s)}{a(s)p(s)}\,ds\right) < \infty, \tag{17}$$

where $\alpha(t)$ satisfies (3).

Proof. Assume that (E) is nonoscillatory, then (E_a) is nonoscillatory. Let $w(t)$ be a solution of (E_a) and

$$v(t) = \frac{a(t)p(t)w'(t)}{w(t)}.$$

It follows from Lemma 1 that

$$v(t) = u(t) + \phi(t), \quad \text{for } t \geq t_1 \geq t_0,$$

where $u(t) = \int_t^\infty \frac{v(s)^2}{a(s)p(s)}\, ds$. Then

$$u'(t) = -\frac{v(t)^2}{a(t)p(t)} = -\frac{(u(t)+\phi(t))^2}{a(t)p(t)} \leq 0.$$

We claim that

$$u'(t) + 4\frac{\phi^+(t)}{a(t)p(t)}u(t) \leq 0. \tag{18}$$

Since $u(t) > 0$ and $u'(t) \leq 0$, then (18) holds if $\phi(t) \leq 0$. If $\phi(t) \geq 0$, then $(u(t)+\phi(t))^2 \geq 4\phi^+(t)u(t)$. This implies that (18) holds. Clearly, (18) implies that

$$u(t) \leq u(t_1)\exp\left(-4\int_{t_1}^t \frac{\phi^+(s)}{a(s)p(s)}\, ds\right) \quad \text{for } t \geq t_1. \tag{19}$$

On the other hand, we have $v(t) = u(t) + \phi(t) \geq \phi(t) \geq \alpha_0(t)$, thus

$$u(t) = \int_t^\infty \frac{v(s)^2}{a(s)p(s)}\, ds \geq \int_t^\infty \frac{\alpha_0^+(s)^2}{a(s)p(s)}\, ds.$$

This implies that

$$v(t) = u(t) + \phi(t) \geq \int_t^\infty \frac{\alpha_0^+(s)^2}{a(s)p(s)}\, ds + \alpha_0(t) = \alpha_1(t) \quad \text{for } t \geq t_1.$$

By induction,

$$v(t) = u(t) + \phi(t) \geq \alpha_n(t), \quad n = 0, 1, 2, \ldots,\ t \in [t_1, \infty). \tag{20}$$

Therefore, (19) and (20) imply that

$$\alpha_n(t) - \phi(t) \leq u(t) \leq u(t_1)\exp\left(-4\int_{t_1}^t \frac{\phi^+(s)}{a(s)p(s)}\, ds\right),$$

and hence

$$[\alpha_n(t) - \phi(t)]\exp\left(4\int_{t_1}^t \frac{\phi^+(s)}{a(s)p(s)}\, ds\right) \leq u(t_1), \quad n = 0, 1, 2, \ldots,\ t \in [t_1, \infty).$$

This and (3) imply that

$$\exp\left(4\int_{t_1}^t \frac{\phi^+(s)}{a(s)p(s)}\, ds\right)$$

$$= \lim_{n\to\infty}[\alpha_n(t) - \phi(t)]\exp\left(4\int_{t_1}^t \frac{\phi^+(s)}{a(s)p(s)}\, ds\right) \leq u(t_1), \quad t \in [t_1, \infty).$$

This implies that (17) holds. Thus we complete this proof. □

Corollary 15. *Suppose that $\alpha_0(t) \le \phi(t)$. If either*

(i) *$\alpha_n(t)$ exists for $n = 1, 2, \dots, m$, and*

$$\limsup_{t\to\infty}[\alpha_m(t) - \phi(t)] \exp\left(4\int^t \frac{\phi^+(s)}{a(s)p(s)}\, ds\right) = \infty; \text{ or}$$

(ii) *(3) holds and*

$$\limsup_{t\to\infty}[\alpha(t) - \phi(t)] \exp\left(4\int^t \frac{\phi^+(s)}{a(s)p(s)}\, ds\right) = \infty,$$

then equation (E) is oscillatory.

Theorem 16. *Suppose that $\alpha_0(t) \le \phi(t)$. If*

$$\int^\infty \exp\left(-4\int^s \frac{\phi^+(u)}{a(u)p(u)}\, du\right) ds < \infty, \quad \int^\infty \phi(s)\, ds < \infty, \tag{21}$$

and there exists a nonnegative integer m such that

$$\int^\infty \alpha_m(s)\, ds = \infty, \tag{22}$$

then (E) is oscillatory.

Proof. Assume that (E) is nonoscillatory, then (1) is nonoscillatory. Let $w(t)$ be a solution of (1) and

$$v(t) = \frac{a(t)p(t)w'(t)}{w(t)}.$$

As the proof of Theorem 14, we obtain that

$$\begin{gathered} \alpha_n(t) - \phi(t) \le u(t_1) \exp\left(-4\int_{t_1}^t \frac{\phi^+(s)}{a(s)p(s)}\, ds\right), \\ n = 0, 1, 2, \dots, \ t \in [t_1, \infty). \end{gathered} \tag{23}$$

Integrating (23) from t_1 to t and let $t \to \infty$, we have

$$\int_{t_1}^\infty \alpha_n(s)\, ds \le u(t_1) \int_{t_1}^\infty \exp\left(-4\int_{t_1}^s \frac{\phi^+(u)}{a(u)p(u)}\, du\right) ds + \int_{t_1}^\infty \phi(s)\, ds.$$

Noting (21) and (22), we get a condradiction.

Hence (E) is oscillatory. □

Remark 3. For the oscillatory criteria of (E) adopting coefficients p and q only, we refer to Lee, Yeh and Gau [10].

3. Examples

Example 1. Consider the following differential equation

$$x''(t) + \left(\frac{1}{t^3} + \frac{1}{4t^2}\right) x(t) = 0. \tag{E_3}$$

We take that $p(t) = 1,\ q(t) = \frac{1}{t^3} + \frac{1}{4t^2}$ and $a(t) = t$. Thus, $f(t) = -\frac{a'(t)}{2a(t)} = -\frac{1}{2t}$ and $\psi(t) = a(t)[q(t) + p(t)f^2(t) - (p(t)f(t))'] = \frac{1}{t^2}$ and

$$\phi(t) = \int_t^\infty \psi(s)ds = \int_t^\infty \frac{1}{s^2}ds = \frac{1}{t} < \infty.$$

Next, we let $\alpha_0(t) = \phi(t) = \frac{1}{t}$ and define

$$\alpha_n(t) = \int_t^\infty \frac{\alpha_{n-1}^+(s)^2}{a(s)p(s)}\, ds + \alpha_0(t), \quad n = 1, 2, \ldots,$$

where $\alpha^+(t) = \frac{1}{2}[\alpha(t) + |\alpha(t)|]$. Therefore,

$$\alpha_1(t) = \int_t^\infty \frac{\frac{1}{s^2}}{s} ds + \frac{1}{t} = \frac{1}{2t^2} + \frac{1}{t},$$

$$\alpha_2(t) = \int_t^\infty \frac{\left(\frac{1}{2s^2} + \frac{1}{s}\right)^2}{s} ds + \frac{1}{t} = \frac{1}{16t^4} + \frac{1}{3t^3} + \frac{1}{2t^2} + \frac{1}{t}$$

and

$$\alpha_3(t) = \frac{1}{2048t^8} + \frac{1}{168t^7} + \frac{25}{864t^6} + \frac{11}{120t^5} + \frac{11}{48t^4} + \frac{1}{3t^3} + \frac{1}{2t^2} + \frac{1}{t}.$$

It is clearly that, for t sufficiently large,

$$1 > \alpha_{n+1}(t) \geq \alpha_n(t), \quad n = 1, 2, \ldots$$

and there exist $t_1 \geq t_0$ and function $\alpha(t)$ such that

$$\lim_{n\to\infty} \alpha_n(t) = \alpha(t), \quad \text{for } t \geq t_1.$$

Hence, it follows from Theorem 4 that (E_3) is nonoscillatory.

Example 2. Consider the equation (E_3) and let $B(t) = \frac{2}{t}$. We also have $p(t) = 1,\ a(t) = t$ and $\phi(t) = \frac{1}{t}$. Then,

$$|\phi(t)| + \int_t^\infty \frac{B^+(s)^2}{a(s)p(s)} ds = \frac{1}{t} + \int_t^\infty \frac{4}{s^3} ds = \frac{1}{t} + \frac{2}{t^2} \le \frac{2}{t}$$

for t sufficiently large. Hence, it follows from Theorem 8 that (E_3) is nonoscillatory.

Example 3. Consider (E_3) and the following differential equation

$$\big(tx'(t)\big)' + \frac{1}{t^2} x(t) = 0. \tag{E_4}$$

We take that $p(t)$, $q(t)$, $a(t)$, $\phi(t)$ and $\phi(t)$ are the same as in Example 1 and $p_1(t) = t$, $q_1(t) = \frac{1}{t^2}$, $a_1(t) = 1$. Thus, $f_1(t) = -\frac{a_1'(t)}{2a_1(t)} = 0$, $\psi_1(t) = a_1(t)[q_1(t) + p_1(t)f_1^2(t) - (p_1(t)f_1(t))'] = \frac{1}{t^2}$ and

$$\phi_1(t) = \int_t^\infty \psi_1(s) ds = \frac{1}{t} < \infty.$$

Thus,

$$0 < a(t)p(t) \le a_1(t)p_1(t), \quad |\phi(t)| \le \phi_1(t)$$

for t sufficiently large. Since (E_3) is nonoscillatory, it follows from Theorem 14 that (E_4) is nonoscillatory.

Example 4. Consider the following differential equation

$$x''(t) + \Big(\frac{\alpha \sin \beta t}{t^\gamma} + \frac{\mu}{t^2}\Big) x(t) = 0, \tag{E_5}$$

where $\alpha, \beta \ne 0$, $\gamma > 0$ and $\mu \in \mathbb{R}$ are constants. Let

$$a(t) = t^{-2\lambda} \exp\Big(- \frac{2\alpha}{\beta} \int_t^\infty \frac{\cos \beta s}{s^\gamma} ds \Big),$$

where $\lambda > \max\big\{ -\frac{1}{2}, \frac{1-2\gamma}{2} \big\}$. Then

$$a(t) = t^{-2\lambda} + O(t^{-2\lambda-\gamma}), \quad f(t) = \frac{\lambda}{t} - \frac{\alpha \cos \beta t}{\beta t^\gamma}$$

and

$$\psi(t) = [t^{-2\lambda} + O(t^{-2\lambda-\gamma})]$$
$$\times \left[\frac{\lambda^2+\lambda+\mu}{t^2} + \frac{\alpha^2}{2\beta^2 t^{2\gamma}} + \frac{\alpha^2 \cos 2\beta t}{2\beta^2 t^{2\gamma}} - \frac{\alpha(2\lambda+\gamma)\cos\beta t}{\beta t^{\gamma+1}}\right].$$

We separtate into five cases.

Case (a). If $0<\gamma<1$, then

$$\phi(t) = \frac{\alpha^2}{2\beta^2(2\lambda+2\gamma-1)} t^{-2\lambda-2\gamma+1} + O(t^{-2\lambda+m}),$$

where $m=\max\{-1, 1-3\gamma\}$. Let $\lambda > \frac{3-4\gamma}{2}$, then

$$\int_t^\infty \frac{\phi_+^2(s)}{a(s)p(s)} ds = \frac{1}{4\gamma+2\lambda-3}\left(\frac{\alpha^2}{2\beta^2(2\lambda+2\gamma-1)}\right) t^{-4\gamma-2\lambda+3}$$
$$+ O(t^{-4\gamma-2\lambda+3}) \geq \phi(t)$$

for all sufficiently large t. By Corollary 11, equation (5) is oscillatory.

Case (b). If $\gamma=1$ and $\mu > \frac{1}{4} - \frac{\alpha^2}{2\beta^2}$, then

$$\phi(t) = \frac{2\beta^2(\lambda^2+\lambda+\mu)+\alpha^2}{2\beta^2(1+2\lambda)} t^{-2\lambda-1} + O(t^{-2\lambda-2}),$$

and hence

$$\int_t^\infty \frac{\phi_+^2(s)}{a(s)p(s)} ds = \frac{1}{1+2\lambda}\left[\frac{2\beta^2(\lambda^2+\lambda+\mu)+\alpha^2}{2\beta^2(1+2\lambda)}\right]^2 t^{-2\lambda-1} + O(t^{-2\lambda-2})$$
$$= \left[\frac{1}{4} + \frac{1}{(1+2\lambda)^2}\left(\mu - \frac{1}{4} + \frac{\alpha^2}{2\beta^2}\right)\right]\phi(t) + O(t^{-2\lambda-2}).$$

By Corollary 11, equation (5) is oscillatory.

Case (c). If $\gamma=1$ and $\mu \leq \frac{1}{4} - \frac{\alpha^2}{2\beta^2}$, we let

$$\lambda > -\frac{1}{2} + \sqrt{\frac{1}{4} - \left(\mu + \frac{\alpha^2}{2\beta^2}\right)},$$

then there exists a constant $\theta > 0$ such that

$$\phi(t) \le \frac{2\beta^2(\lambda^2+\lambda+\mu)+\alpha^2}{2\beta^2(1+2\lambda)} t^{-2\lambda-1} + \theta t^{-2\lambda-2}.$$

Let

$$B(t) = Kt^{-2\lambda-1} + Mt^{-2\lambda-2},$$

where

$$K = \frac{1+2\lambda}{2} - \sqrt{\frac{1}{4} - \left(\mu + \frac{\alpha^2}{2\beta^2}\right)} > 0 \quad \text{and} \quad M > \frac{(1+\lambda)\theta}{1+\lambda-K} > 0,$$

then

$$\begin{aligned} |\phi(t)| + \int_t^\infty \frac{B_+^2(s)}{a(s)p(s)} ds &\le Kt^{-2\lambda-1} + \frac{(1+\lambda)\theta + KM}{1+\lambda} t^{-2\lambda-2} + O(t^{-2\lambda-3}) \\ &\le Kt^{-2\lambda-1} + Mt^{-2\lambda-2} = B_+(t). \end{aligned}$$

By Theorem 8, equation (5) is oscillatory.

Case (d). If $\gamma > 1$ and $\mu > \frac{1}{4}$, then

$$\phi(t) = \frac{\lambda^2+\lambda+\mu}{1+2\lambda} t^{-2\lambda-1} + O(t^{-2\lambda-h}),$$

where

$$h = \min\{\gamma+1, 2\gamma-1\}.$$

Thus,

$$\begin{aligned} \int_t^\infty \frac{\phi_+^2(s)}{a(s)p(s)} ds &= \frac{1}{1+2\lambda} \left(\frac{\lambda^2+\lambda+\mu}{1+2\lambda}\right)^2 t^{-2\lambda-1} + O(t^{-2\lambda-h}) \\ &= \left[\frac{1}{4} + \frac{1}{(1+2\lambda)^2}\left(\mu - \frac{1}{4}\right)\right] \phi(t) + O(t^{-2\lambda-h}). \end{aligned}$$

By Corollary 11, equation (5) is oscillary.

Case (e). If $\gamma > 1$ and $\mu \le \frac{1}{4}$, we let

$$\lambda > -\frac{1}{2} + \sqrt{\frac{1-\mu}{4}} \quad \text{and} \quad q = \min\{\gamma+2, 2\gamma\} > 2,$$

then

$$\phi(t) = \frac{\lambda^2+\lambda+\mu}{1+2\lambda} t^{-2\lambda-1} + O(t^{-2\lambda-q+1}) \le \frac{\lambda^2+\lambda+\mu}{1+2\lambda} t^{-2\lambda-1} + \theta t^{-2\lambda-q+1}$$

for some constant $\theta > 0$. Let

$$B(t) = K^* t^{-2\lambda-1} + M^* t^{-2\lambda-q+1},$$

where

$$K^* = \frac{1+2\lambda-\sqrt{1-4\mu}}{2} > 0 \quad \text{and} \quad M^* > \frac{(2\lambda+q-1)\theta}{2\lambda+q-1-2K^*} > 0,$$

then

$$\begin{aligned}|\phi(t)| + \int_t^\infty \frac{B_+^2(s)}{a(s)p(s)} ds &\le K^* t^{-2\lambda-1} + \left(\theta + \frac{2K^*M^*}{2\lambda+q-1}\right) t^{-2\lambda-q+1} \\ &+ O(t^{-2\lambda-2q+3}) \le K^* t^{-2\lambda-1} + M^* t^{-2\lambda-q+1} = B_+(t).\end{aligned}$$

By Theorem 8, equation (5) is oscillatory.

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HUEI-LIN HONG
DEPARTMENT OF INFORMATION MANAGEMENT
LUNGHWA UNIVERSITY OF SCIENCE AND TECHNOLOGY
KUEISHAN TAUYUAN
333 TAIWAN
REPUBLIC OF CHINA

E-mail: hong@mail.lhu.edu.tw

CHEH-CHIH YEH
DEPARTMENT OF INFORMATION MANAGEMENT
LUNGHWA UNIVERSITY OF SCIENCE AND TECHNOLOGY
KUEISHAN TAUYUAN
333 TAIWAN
REPUBLIC OF CHINA

E-mail: ccyeh@mail.lhu.edu.tw

HORNG-JAAN LI
GENERAL EDUCATION CENTER
CHIEN KUO INSTITUTE OF TECHNOLOGY
CHANG-HUA
500 TAIWAN
REPUBLIC OF CHINA

HSIANG-BIN HSU
NATIONAL YANG MING SENIOR HIGH SCHOOL
TAUYUAN CITY
330 TAIWAN
REPUBLIC OF CHINA

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