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Greguš type common fixed point theorems in metric spaces of hyperbolic type

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Abstract. In this paper we investigate a class of pairs of Greguš type mapppings I and T on a metric space (X,d) which satisfy the following condition: $d(Tx,Ty)\le \alpha d(Ix,Iy)+\beta\max\{d(Ix,Tx),d(Iy,Ty)\}+\gamma\max\{d(Ix,Iy),$ $d(Ix,Tx),d(Iy,Ty),b(d(Ix,Ty)+d(Iy,Tx))\}$ for all x, y in X, where α, β, γ are constants such that $\alpha>0$, $\beta>0$, $\gamma\ge 0$ and $\alpha+\beta+\gamma=1$, and $3b\le 1+2\alpha\beta$. The main result is that if X is a complete metric space of hyperbolic type, I is continuous, $\mathrm{Co}\,(T(X))\subset I(X)$ and I and T are compatible mappings of type (T), then I and T have a unique common fixed point and at this point T is continuous.

1. Introduction

Let X be a Banach space and C a non-empty closed and convex subset of X. In [15] Greguš proved the following result.

Theorem 1 (Greguš [15]). *Let $T:C\to C$ be a mapping satisfying the following condition:*

$$d(Tx,Ty)\le ad(x,y)+pd(x,Tx)+pd(y,Ty)$$

for all x, $y\in C$, where $0<a<1$, $p\ge 0$ and $a+2p=1$. Then T has a unique fixed point.

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This result has been found very useful and has many interesting generalizations and applications (c.f. [1]–[16], [18]–[21]). Generalizing Greguš result and the result of FISHER and SESSA [14], DAVIS [9] proved the following common fixed point theorem.

Theorem 2 (DAVIS [9]). *Let I and T be two mappings of C into itself, satisfying the inequality*

$$d(Tx, Ty) \leq \alpha d(Ix, Iy) + \beta \max\{(Ix, Tx), d(Iy, Ty)\} \\ + \gamma \max\{d(Ix, Iy), d(Ix, Tx), d(Iy, Ty)\}$$

for all x, $y \in C$, where α, β, γ are constants such that $\alpha > 0$, $\beta > 0$, $\gamma \geq 0$ and $\alpha+\beta+\gamma = 1$. If I is linear, non-expansive and weakly commuting with T in C, and if $I(C)$ contains $T(C)$, then T and I have a unique common fixed point in C and at this point T is continuous.

Further, DAVIS and SESSA in [10] relaxed hypotheses for I, and weakly commutativity replaced by compatibility.

The purpose of this paper is to introduce and to study a class of pairs of Greguš-type mappings I and T from an arbitrary non-empty set Y into a metric space (X, d) of hyperbolic type satisfying the following condition:

$$d(Tx, Ty) \leq \alpha d(Ix, Iy) + \beta \max\{d(Ix, Tx), d(Iy, Ty)\} \\ + \gamma \max\left\{d(Ix, Iy), d(Ix, Tx), d(Iy, Ty), \frac{1 + 2\alpha\beta}{3}(d(Ix, Ty) + d(Iy, Tx))\right\}$$

for all x, y in Y, where $\alpha > 0$, $\beta > 0$, $\gamma \geq 0$ and $\alpha + \beta + \gamma = 1$. We introduce an improved technique and prove that if Y is a subset of X such that $T(Y)$ is bounded, $\mathrm{Co}(T(Y)) \subset I(Y)$, I is continuous on Y and I and T are compatible mappings of type (T), then T and I have a unique common fixed point in Y and at this point T is continuous.

2. Main results

Throughout our consideration we suppose that (X, d) is a metric space which contains a family L of metric segments (isometric images of real line segments) such that

(a) each two points x, y in X are endpoints of exactly one member $\mathrm{seg}[x,y]$ of L,

(b) if u, x, $y \in X$ and if $z \in \mathrm{seg}[x,y]$ satisfies $d(x,z) = \lambda d(x,y)$ for $\lambda \in [0,1]$, then

$$d(u,z) \leq (1-\lambda)d(u,x) + \lambda d(u,y). \tag{1}$$

Space of this type is said to be a *metric space of hyperbolic type* (Takahashi [15] uses the term "convex metric space"). Further, a non-empty subset C of X is of hyperbolic type if a subspace (C,d) is of hyperbolic type. For $A \subset X$, $\mathrm{Co}(A)$ will denote the intersection of all subsets of X which are of hyperbolic type and contain A. A class of metric spaces of hyperbolic type includes all normed linear spaces, as well as all spaces with hyperbolic metric (see [16], for a discussion).

Now we prove the following lemma in metric spaces of hyperbolic type.

Lemma 1. *Let Y be an arbitrary non-empty set and (X,d) a metric space of hyperbolic type. Let I and T be maps from Y into X satisfying the following condition:*

$$\begin{aligned} d(Tx,Ty) \leq \alpha d(Ix,Iy) + \beta \max\{d(Ix,Tx), d(Iy,Ty)\} \\ + \gamma \max\{d(Ix,Iy), d(Ix,Tx), d(Iy,Ty), b(d(Ix,Ty)+d(Iy,Tx))\} \end{aligned} \tag{2}$$

for all x, $y \in Y$, where α, β, γ, b are constanst such that $\alpha > 0$, $\beta > 0$, $\gamma \geq 0$, $3b \leq 1 + 2\alpha\beta$ and

$$\alpha + \beta + \gamma = 1. \tag{3}$$

If $\mathrm{Co}(T(Y)) \subset I(Y)$, then

$$\inf\{d(Ix,Tx) : x \in Y\} = 0. \tag{4}$$

Proof. To show that I and T satisfy (4), it is sufficies to show that for any point x_0 in Y there exists a point $z = z(x_0)$ in Y such that

$$d(Iz,Tz) \leq \lambda d(Ix_0,Tx_0),$$

where $\lambda < 1$ and λ does not depend of x_0.

Pick $x_0 \in Y$. Then Ix_0 and Tx_0 are defined. Choose an element $x_1 \in Y$ such that $Ix_1 = Tx_0$. Such a choice is permissible since $T(Y) \subset I(Y)$. Similarly, choose $x_2 \in Y$ such that $Ix_2 = Tx_1$. Then choose $x_3 \in Y$ such that $Ix_3 = Tx_2$, and so on. Thus, inductively we choose the sequence $\{x_n\}$ in Y defined by

$$Ix_{n+1} = Tx_n \qquad (\text{for } n = 0, 1, 2, \ldots).$$

For notation simplicity, for each $n \geq 0$ set

$$r_n = d(Ix_n, Tx_n), \quad s_n = d(Ix_n, Tx_{n+1}).$$

If $r_n = 0$ for some n, then $Tx_n = Ix_n$ and we have finished the proof of Lemma. So we assume that $r_n > 0$ for all $n \geq 0$. From (2), with $x = x_n$ and $y = x_{n+1}$, we have for each $n \geq 0$, as $Ix_{n+1} = Tx_n$, as $Ix_{n+1} = Tx_n$,

$$\begin{aligned} r_{n+1} = d(Tx_n, Tx_{n+1}) &\leq \alpha d(Ix_n, Tx_n) \\ &+ \beta \max\{d(Ix_n, Tx_n), d(Ix_{n+1}, Tx_{n+1})\} \\ &+ \gamma \max\{d(Ix_n, Tx_n), d(Ix_{n+1}, Tx_{n+1}), bd(Ix_n, Tx_{n+1})\}. \end{aligned}$$

Thus we have

$$r_{n+1} \leq \alpha r_n + \beta \max\{r_n, r_{n+1}\} + \gamma \max\{r_n, r_{n+1}, bs_n\}. \tag{5}$$

Since (3) implies that

$$\alpha\beta \leq \frac{(1-\gamma)^2}{4} \leq \frac{1-\gamma}{4} \leq \frac{1}{4},$$

we have $b \leq \frac{1}{2}$. By the triangle inequality we get $s_n \leq (r_n + r_{n+1}) \leq 2 \cdot \max\{r_n, r_{n+1}\}$. Thus, from (5), it follows

$$r_{n+1} \leq \alpha r_n + (\beta + \gamma) \max\{r_n, r_{n+1}\}. \tag{6}$$

If we assume that $r_n < r_{n+1}$ for some n, then from (6) and (3) we have, as $\alpha > 0$,

$$r_{n+1} < \alpha r_{n+1} + (\beta + \gamma) r_{n+1} = r_{n+1},$$

a contradiction. Therefore, $r_{n+1} \leq r_n$ for all $n \geq 0$. This implies

$$r_n \leq r_0 = d(Ix_0, Tx_0) \quad (\text{for } n = 1, 2, \ldots). \tag{7}$$

From (2), with $x = x_n$ and $y = x_{n+2}$, we get, by (7),

$$\begin{aligned} s_{n+1} = d(Tx_n, Tx_{n+2}) &\leq \alpha s_n + \beta r_0 \\ &+ \gamma \max\{s_n, r_0, b(d(Ix_n, Tx_{n+2}) + r_0)\}. \end{aligned} \tag{8}$$

Write $t_n = d(Ix_n, Tx_{n+2})$ for $n = 0, 1, 2, \ldots$ Then from (2) and (7) we obtain

$$\begin{aligned} t_{n+1} = d(Tx_n, Tx_{n+3}) &\le \alpha t_n + \beta r_0 \\ &+ \gamma \max\{t_n, r_0, b(d(Ix_n, Tx_{n+3}) + s_{n+1})\}. \end{aligned} \tag{9}$$

Since by the triangle inequality and (7) we have

$$d(Ix_n, Tx_{n+3}) \le d(Ix_n, Tx_{n+2}) + r_{n+3} \le t_n + r_0$$

and as $s_{n+1} \le 2r_0$, from (9) we get

$$t_{n+1} \le \alpha t_n + \beta r_0 + \gamma \max\{t_n, r_0, b(t_n + 3r_0)\}. \tag{10}$$

By the triangle inequality and (7), we have that $t_n \le 3r_0$ for each $n \ge 0$. Thus, $\limsup t_n = t \le 3r_0$. Taking the limit superior in (10) we get

$$t \le \alpha t + \beta r_0 + \gamma \max\{t, r_0, b(t + 3r_0)\}. \tag{11}$$

Assume that $t \ge r_0$. Then, if from (11) $t \le \alpha t + \beta r_0 + \gamma t$, then we get $t \le r_0$.

If from (11), $t \le \alpha t + \beta r_0 + \gamma b(t + 3r_0)$, then, as $\gamma bt \le \frac{\gamma}{2} t$ and by (3),

$$3\gamma b r_0 \le (\gamma + 2(\alpha\gamma)\beta) r_0 \le \left(\gamma + \frac{\beta}{2}\right) r_0,$$

we get that

$$t \le \alpha t + \beta r_0 + \frac{\gamma}{2} t + \left(\gamma + \frac{\beta}{2}\right) r_0.$$

Hence we get

$$t \le \frac{3\beta + 2\gamma}{2\beta + \gamma} r_0 = \left(2 - \frac{\beta}{2\beta + \gamma}\right) r_0 < \left(2 - \frac{\beta}{3}\right) r_0. \tag{12}$$

Thus, there exists n_0 such that $d(Ix_n, Tx_{n+2}) = t_n < 2r_0 - (\beta/4) r_0$ for all $n \ge n_0$. Substituting this in (8) we get that

$$s_{n+1} \le \alpha s_n + \beta r_0 + \gamma \max\left\{s_n, r_0, 3br_0 - \frac{b\beta}{4} r_0\right\} \tag{13}$$

for all $n \ge n_0$. Since $s_n \le r_n + r_{n+1} \le 2r_0$ for all $n \ge 0$, we have that $\limsup s_n = s \le 2r_0$. Taking the upper limit in (13) we get

$$s \le \alpha s + \beta r_0 + \gamma \max\left\{s, r_0, (1 + 2\alpha\beta) r_0 - \frac{b\beta}{4} r_0\right\}. \tag{14}$$

Assume that $s \geq r_0$. Then, if from (14), $s \leq \alpha s + \beta r_0 + \gamma s$, then we get $s \leq r_0$.

If from (14), $s \leq \alpha s + \beta r_0 + (\gamma + 2\alpha\beta\gamma) r_0 - \frac{b\beta\gamma}{4} r_0$, then it follows

$$s \leq \left(1 + \frac{2\alpha\beta\gamma}{\beta+\gamma} - \frac{b\beta\gamma}{4(\beta+\gamma)}\right) r_0.$$

Thus, there exists $k > n_0$ such that

$$s_k \leq \left(1 + \frac{2\alpha\beta\gamma}{\beta+\gamma}\right) r_0. \tag{15}$$

Let $z \in Y$ be such that $Iz \in \operatorname{seg}[Tx_k, Tx_{k+1}]$ and such that

$$d(Iz, Tx_k) = \frac{1}{2} d(Tx_k, Tx_{k+1}) = \frac{1}{2} r_{k+1} \leq \frac{1}{2} r_0. \tag{16}$$

Since X is of hyperbolic type, from (1), with $\lambda = \frac{1}{2}$ and $u = Tx_{k+1}$, we get

$$d(Iz, Tx_{k+1}) \leq \frac{1}{2} d(Tx_k, Tx_{k+1}) \leq \frac{1}{2} r_0. \tag{17}$$

Again from (1), but now with $u = Tx_{k-1}$, we obtain

$$\begin{aligned} d(Iz, Tx_{k-1}) &\leq \frac{1}{2}(d(Tx_{k-1}, Tx_k) + d(Tx_{k-1}, Tx_{k+1})) \\ &= \frac{1}{2}(r_k + s_k) \leq \frac{1}{2}(r_0 + s_k). \end{aligned} \tag{18}$$

From (18) and (15) we have

$$d(Iz, Tx_{k-1}) \leq \left(1 + \frac{\alpha\beta\gamma}{\beta+\gamma}\right) r_0. \tag{19}$$

Again from (1), but now with $u = Tz$, we get

$$d(Iz, Tz) \leq \frac{1}{2}(d(Tz, Tx_k) + d(Tz, Tx_{k+1})). \tag{20}$$

Put

$$M = \max\{d(Iz, Tz), d(Ix_0, Tx_0)\}.$$

Then from (2), (16) and (17) we have, as $Ix_{k+1} = Tx_k$,

$$d(Tz, Tx_{k+1}) \leq \frac{\alpha}{2} M + \beta M + \gamma \max\left\{\frac{M}{2}, M, b\left(\frac{M}{2} + d(Tx_k, Tz)\right)\right\}. \tag{21}$$

Since $b \le \frac{1}{2}$, by the triangle inequality and (16) we get

$$b\left(\frac{M}{2}+d(Tx_k,Tz)\right) \le b\left(\frac{M}{2}+d(Tx_k,Iz)+d(Iz,Tz)\right) \le \frac{1}{2}\left(\frac{M}{2}+\frac{M}{2}+M\right)=M.$$

Thus from (21) and (3) we obtain

$$d(Tz,Tx_{k+1}) \le \left(1-\frac{\alpha}{2}\right)M. \tag{22}$$

Again from (2) and (16) we have

$$\begin{aligned} d(Tz,Tx_k) &\le \alpha d(Iz,Tx_{k-1})+\beta M \\ &+\gamma\max\left\{d(Iz,Tx_{k-1}),M,b\left(\frac{M}{2}+d(Ix_k,Tz)\right)\right\}. \end{aligned} \tag{23}$$

Since by the triangle inequality

$$\begin{aligned} d(Ix_k,Tz) = d(Tx_{k-1},Tz) &\le d(Tx_{k-1},Iz)+d(Iz,Tz) \\ &\le d(Iz,Tx_{k-1})+M, \end{aligned}$$

from (23) we get

$$\begin{aligned} d(Tz,Tx_k) &\le \alpha d(Iz,Tx_{k-1})+\beta M \\ &+\gamma\max\left\{d(Iz,Tx_{k-1}),M,\frac{1+2\alpha\beta}{3}\left(\frac{3}{2}M+d(Iz,Tx_{k-1})\right)\right\}. \end{aligned} \tag{24}$$

Consider now two possible cases.

Case I. Assume that from (24) we have

$$d(Tz,Tx_k) \le \alpha d(Iz,Tx_{k-1})+\beta M+\gamma d(Iz,Tx_{k-1}).$$

Then by (19) we get

$$\begin{aligned} d(Tz,Tx_k) &\le (\alpha+\beta+\gamma)M+\frac{\alpha\beta\gamma(\alpha+\gamma)}{\beta+\gamma}M \\ &= M+\frac{(\alpha\gamma)\alpha\beta+(\beta\gamma)\alpha\gamma}{\beta+\gamma}M. \end{aligned} \tag{25}$$

Since (3) implies that $4\alpha\gamma \le (1-\beta)^2 < 1$, $4\beta\gamma \le (1-\alpha)^2 < 1$, from (25) we get

$$d(Tz, Tx_k) < M + \frac{\alpha\beta + \alpha\gamma}{4(\beta+\gamma)} M = \left(1 + \frac{\alpha}{4}\right) M.$$

Hence we have

$$d(Tz, Tx_k) < \left(1 + \frac{\alpha}{2}\right) M - \frac{\alpha}{4} M. \tag{26}$$

Thus, from (20) and by (26) and (22), we get

$$d(Iz, Tz) < \left(1 - \frac{\alpha}{8}\right) \max\{d(Iz, Tz), d(Ix_0, Tx_0)\}.$$

Hence we have, as $\alpha > 0$,

$$d(Iz, Tz) < \left(1 - \frac{\alpha}{8}\right) d(Ix_0, Tx_0). \tag{27}$$

Case II. Assume now that (24) implies

$$d(Tz, Tx_k) \le \alpha d(Iz, Tx_{k-1}) + \beta M + \frac{\gamma + 2\alpha\beta\gamma}{3}\left(\frac{3}{2}M + d(Iz, Tx_{k-1})\right).$$

Then by (19) we get

$$\begin{aligned} d(Tz, Tx_k) &\le \alpha\left(1 + \frac{\alpha\beta\gamma}{\beta+\gamma}\right) M + \beta M + \frac{\gamma + 2\alpha\beta\gamma}{3}\left(\frac{5}{2}M + \frac{\alpha\beta\gamma}{\beta+\gamma}M\right) \\ &= \left(\alpha + \beta + \frac{5}{6}\gamma\right) M + \frac{5\alpha\beta\gamma(\beta+\gamma) + \alpha\beta\gamma(3\alpha + \gamma + 2\alpha\beta\gamma)}{3(\beta+\gamma)} M. \end{aligned}$$

Thus we have

$$d(Tz, Tx_k) \le \left(\alpha + \beta + \frac{5}{6}\gamma\right) M + \frac{L}{6(\beta+\gamma)} M, \tag{28}$$

where

$$\begin{aligned} L = {} & 10(\beta\gamma)\alpha\beta + \big(2(\beta\gamma)\alpha\gamma + 4(\alpha\gamma)\beta\gamma + 4(\alpha\beta)\gamma^2\big) + 6(\alpha\beta)\alpha\gamma + 2(\beta\gamma)\alpha\gamma \\ & + 2(\alpha\gamma)(\beta\gamma)\alpha\beta + 2(\alpha\beta)(\beta\gamma)\alpha\gamma. \end{aligned}$$

Since $\alpha\beta$, $\alpha\gamma$, $\beta\gamma \le \frac{1}{4}$, we get

$$L \le \frac{5}{2}\alpha\beta + \frac{5}{2}\alpha\gamma + \beta\gamma + \gamma^2 + \frac{1}{8}\alpha\beta + \frac{1}{8}\alpha\gamma$$

$$= 3\alpha(\beta+\gamma) + \gamma(\beta+\gamma) - \frac{3}{8}\alpha(\beta+\gamma).$$

Now from (28) we get

$$d(Tz, Tx_k) \le \left(\alpha + \beta + \frac{5}{6}\gamma\right) M + \left(\frac{\alpha}{2} + \frac{\gamma}{6} - \frac{1}{16}\alpha\right) M.$$

Hence, by (3), we have

$$d(Tz, Tx_k) \le \left(1 + \frac{\alpha}{2}\right) M - \frac{1}{16}\alpha M. \tag{29}$$

Thus, from (20) and by (29) and (22), we get

$$d(Iz, Tz) < \left(1 - \frac{\alpha}{32}\right) \max\left\{d(Iz, Tz), d(Ix_0, Tx_0)\right\},$$

and hence

$$d(Iz, Tz) < \left(1 - \frac{\alpha}{32}\right) d(Ix_0, Tx_0). \tag{30}$$

Taking in consideration (27) and (30), we conclude that in both cases

$$d(Iz, Tz) \le \left(1 - \frac{\alpha}{32}\right) d(Ix_0, Tx_0)$$

holds. □

Generalizing the Jungck's notion of compatibility of two mappings, Pathak *et al.* in [20] introduced the concept of compatible mappings of type (T) (type (I)) in normed spaces.

Definition 1. Let I and T be mappings from a normed space X into itself. The mappings I and T are said to be *compatible of type* (T) if

$$\limsup_{n\to\infty} \|ITx_n - Ix_n\| + \limsup_{n\to\infty} \|ITx_n - TIx_n\| \le \limsup_{n\to\infty} \|TIx_n - Tx_n\|,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. (Note that originally in [20] write lim instead of lim sup, but we think that lim sup is more convenient.)

Every compatible pair of mappings is a compatible pair of type (T) (see Proposition 2.1 in [20]), but not conversily (see Example 2.1 in [20]).

Now we prove our main result.

Theorem 1. *Let Y be an arbitrary non-empty set and (X, d) a metric space of hyperbolic type. Let I and T be two mappings from Y into X, satisfying (2), where α, β, γ, b are as in Lemma 1, and such that $\mathrm{Co}(T(Y)) \subset I(Y)$. Further, suppose that*

(I) *$T(Y)$ or $I(Y)$ is complete, then I and T have a coincidence point, say $u \in Y$, i.e. such that $Tu = Iu$, and if $Tv = Iv$ for some other point $v \in Y$, then $Iv = Iu$;*

(II) *Y is a subset of X such that $T(Y)$ is bounded, I is continuous on Y and I and T are compatible mappings of type (T), then T and I have a unique common fixed point in Y and at this point T is continuous.*

Proof. By Lemma 1 it follows that I and T satisfy (4). Thus we can choose a sequence $\{x_n\}$ in Y such that

$$\lim_{n\to\infty} d(Ix_n, Tx_n) = 0. \tag{31}$$

We show that $\{Ix_n\}$ in X is a Cauchy sequence. Denote

$$r_n = d(Ix_n, Tx_n).$$

Using the triangle inequality, from (2) we have

$$\begin{aligned} d(Ix_m, Ix_n) \le r_m + d(Tx_m, Tx_n) + r_n \le r_m \\ + r_n + \alpha d(Ix_m, Ix_n) + \beta \max\{r_m, r_n\} \\ + \gamma \max\{d(Ix_m, Ix_n), r_m, r_n, b(d(Ix_m, Tx_n) + d(Ix_n, Tx_m))\}. \end{aligned}$$

Using again the triangle inequality and take in consideration that $\max\{r_m, r_n\} \le r_m + r_n$, we get

$$\begin{aligned} d(Ix_m, Ix_n) \le (1+\beta)(r_m + r_n) + \alpha d(Ix_m, Ix_n) \\ + \gamma \max\{d(Ix_m, Ix_n), r_m, r_n, b(2d(Ix_m, Ix_n) + r_n + r_m)\}. \end{aligned} \tag{32}$$

Since $b \le \frac{1}{2}$, from (32) we get

$$\begin{aligned} d(Ix_m, Ix_n) \le (1+\beta)(r_m + r_n) + \alpha d(Ix_m, Ix_n) \\ + \gamma \max\{d(Ix_m, Ix_n), r_m, r_n, (d(Ix_m, Ix_n) + \tfrac{1}{2}(r_m + r_n))\}. \end{aligned}$$

Hence we have, by (3),

$$d(Ix_m, Ix_n) \le \frac{1+\beta+\gamma/2}{\beta}(r_m + r_n). \tag{33}$$

Taking the limit in (33) when $n > m \to +\infty$ we get, by (31),

$$\lim_{n>m\to\infty} d(Ix_m, Ix_n) = 0.$$

Thus, $\{Ix_n\}$ is a Cauchy sequence in X.

We show (I). Suppose that $I(Y)$ is complete. Then there exists a point $p \in I(Y)$ such that

$$\lim_{n\to\infty} Ix_n = p. \tag{34}$$

From (31) it also follows that $\lim_{n\to\infty} Tx_n = p$.

Let $u \in Y$ be such that $Iu = p$. This can done since $T(Y) \subset I(Y)$. We prove that $Tu = p$. From (2) we have

$$d(Tu, Tx_n) \leq \alpha d(Iu, Ix_n) + \beta \max\{d(Iu, Tu), d(Ix_n, Tx_n)\}$$
$$+ \gamma \max\{d(Iu, Ix_n), d(Iu, Tu), d(Ix_n, Tx_n), b(d(Iu, Tx_n) + d(Ix_n, Tu))\}.$$

Passage to the limit as n tends to infinity yelds, by (34),

$$d(Tu, p) \leq (\beta + \gamma) d(Tu, p). \tag{35}$$

Since $\beta + \gamma = 1 - \alpha < 1$, from (35) it follows that $d(Tu, p) = 0$. Hence $Tu = p$. Thus we have

$$Tu = Iu = p. \tag{36}$$

Further, if we assume that $T(Y)$ is complete, we get the same conclusion.

Assume that $v \in Y$ is such that $Iv = Tv$. Then we get, as $2b \leq 1$,

$$\begin{aligned} d(Iv, Iu) = d(Tv, Tu) &\leq \alpha d(Iv, Iu) + \gamma \max\{d(Iu, Iv), 2bd(Iv, Iu)\} \\ &= (\alpha + \gamma) d(Iv, Iu). \end{aligned}$$

This implies, as $\alpha + \gamma = 1 - \beta < 1$, that $d(Iv, Iu) = 0$. Hence $Iv = Iu$.

Now we prove (II). Since I is continuous, from (34) it follows

$$\lim_{n\to\infty} IIx_n = Ip \quad \text{and} \quad \lim_{n\to\infty} ITx_n = Ip. \tag{37}$$

Since I and T are compatible mapping of type (T), by Definition 1, and by (34), (36) and (37) we have

$$d(Ip, p) + \limsup_{n\to\infty} d(TIx_n, Ip) \leq \limsup_{n\to\infty} d(TIx_n, Tu). \tag{38}$$

From (2) and by (36) we get, as $b \leq \frac{1}{2}$,

$$d(TIx_n, Tu) \leq \alpha d(IIx_n, Iu) + \beta d(IIx_n, TIx_n)$$
$$+ \gamma \max\{d(IIx_n, Iu), d(IIx_n, TIx_n), \tfrac{1}{2}(d(IIx_n, Tu) + d(Iu, TIx_n))\}.$$

Using that $d(Iu, TIx_n) \leq d(Iu, Ip) + d(TIx_n, Ip)$ and taking the limit superior as $n \to \infty$, we obtain, by (36) and (37),

$$\limsup_{n\to\infty} d(TIx_n, Tu) \leq \alpha d(Ip, p) + \beta \limsup d(TIx_n, Ip)$$
$$+ \gamma(d(Ip, p) + \limsup_{n\to\infty} d(TIx_n, Ip)).$$

Hence we get

$$\limsup_{n\to\infty} d(TIx_n, Tu) \leq (\alpha + \gamma) d(Ip, p) + (\beta + \gamma) \limsup_{n\to\infty} d(TIx_n, Ip). \tag{39}$$

From (38) and (39), and using (3),we obtain

$$\beta d(Ip, p) + \alpha \limsup_{n\to\infty} d(TIx_n, Ip) \leq 0.$$

This implies, as $\alpha > 0$, $\beta > 0$, that $d(Ip, p) = 0$ and $\lim_{n\to\infty} d(TIx_n, Ip) = 0$. Hence $Ip = p$. Again from (2) we have

$$d(Tp, Tx_n) \leq \alpha d(Ip, Ix_n) + \beta \max\{d(Ip, Tp), d(Ix_n, Tx_n)\}$$
$$+ \gamma \max\{d(Ip, Ix_n), d(Ip, Tp), d(Ix_n, Tx_n), \tfrac{1}{2}(d(Ip.Tx_n) + d(Ix_n, Tp))\}.$$

Taking the limit as $n \to \infty$ we get

$$d(Tp, p) \leq (\beta + \gamma)\, d(Tp, p).$$

Hence, as $\beta + \gamma = 1 - \alpha < 1$, we have $d(Tp, p) = 0$; hence $Tp = p$. Thus we proved that

$$Tp = p = Ip,$$

i.e. p is a common fixed point of T and I. Let q be also a common fixed point of T and I. Then, from $d(p, q) = d(Tp, Tq)$ and (2), we obtain

$$d(p, q) \leq (\alpha + \gamma) d(p, q).$$

Since $\beta > 0$ implies that $\alpha + \gamma = 1 - \beta < 1$, we have $d(p, q) = 0$ and so T and I have a unique common fixed point.

Now we show that T is continuous at p. Let $\{y_n\}$ be a sequence in $Y \subset X$ with the limit p. From (2) we have, as $b \leq \frac{1}{2}$,

$$\begin{aligned} d(p, Ty_n) = d(Tp, Ty_n) &\leq \alpha d(Ip, Iy_n) + \beta d(Iy_n, Ty_n) \\ &+ \gamma \max\{d(Ip, Iy_n), d(Iy_n, Ty_n), \tfrac{1}{2}(d(Ip, Ty_n) + d(Iy_n, Tp))\} \\ &\leq \alpha d(p, Iy_n) + \beta(Iy_n, p) + d(p, Ty_n)) + \gamma(d(Iy_n, p) + d(p, Ty_n)) \end{aligned}$$

and hence, letting n go to infinity, we obtain

$$\limsup_{n\to\infty} d(p, Ty_n) \leq (\beta + \gamma) \limsup_{n\to\infty} d(p, Ty_n).$$

As $\beta + \gamma < 1$, the last inequality implies

$$\limsup_{n\to\infty} d(p, Ty_n) = 0,$$

and this means that T is continuous at u. □

Remark 1. If $b = 0$, then the condition (2) becomes the contractive condition of Davis [9], but Theorem 1, with $b = 0$, is a generalization of Theorem in [9].

Remark 2. Theorem 1 with $\gamma = 0$ is a generalization of Theorem of Fisher and Sessa [14] and Jungck [16].

Remark 3. The condition that $\mathrm{Co}(T(Y))$ is contained in $I(Y)$ is necessary in our Theorem 1. This shows the following example.

Example 1. Let X be the set of reals with the usaul distance and $Y = [0, 1]$. Define $T, I : Y \to Y$ as follows:

$$\begin{aligned} Tx = 1 \quad &\text{for } 0 \leq x \leq 1/2 \quad \text{and} \quad Tx = 0 \quad \text{for } 1/2 < x \leq 1; \\ Ix = 0 \quad &\text{for } 0 \leq x \leq 1/2 \quad \text{and} \quad Ix = 1 \quad \text{for } 1/2 < x \leq 1. \end{aligned}$$

Then all the assumptions of our Theorem are trivially satisfied except that $\mathrm{Co}(T(Y)) \subseteq I(Y)$, but T and I do not have common fixed points.

Remark 4. The following example shows that our Theorem 1 as a genuine generalizaton of the theorems [6], [12]–[14], [16] and [19].

Example 2. Let $Y = [0,1]$ be the closed unit interval and $T, I : Y \to Y$ be defined by $Tx = x/4$ and $Ix = x^{1/2}$. Clearly $\mathrm{Co}[T(Y)] \subseteq I(Y)$, I is continuous and T and I are weakly commutative, hence compatile of type (T). As

$$d(Tx,Ty) = 1/4 \cdot |x-y| \leq 1/4 \cdot |x-y| \frac{2}{x^{1/2}+y^{1/2}} = 1/2 \cdot d(Ix,Iy)$$

for all $x, y \in Y$, we conclude that all the hypotheses of Theorem 1 are satisfed and 0 is a unique common fixed point. But I is neither linear nor nonexpansive.

References

[1] Lj. B. Ćirić, On some discontinuous fixed point mappings in convex metric spaces, *Czechoslovak Math. J.* **43**(118) (1993), 319–326.

[2] Lj. B. Ćirić, On a common fixed point theorem of a Greguš type, *Publ. Inst. Math. (Beograd)* (49) **63** (1991), 174–178.

[3] Lj. B. Ćirić, On Divicaro, Fisher and Sessa open questions, *Arch. Math. T.* **29** (1993), 145–152.

[4] Lj. B. Ćirić, On some nonexpansive type mappings and fixed points, *Indian J. Pure Appl. Math.* **24**(3) (1993), 145–149.

[5] Lj. B. Ćirić, Nonexpansive type mappings and a fixed point theorem in convex metric spaces, *Rend. Accad. Naz. Sci.* **XL**, *Mem. Mat.* (5) **XIX**, (1995), 263–271.

[6] Lj. B. Ćirić, On some mappings in metric spaces and fixed point theorems, *Acad. Roy. Belg. Bull. Cl. Sci.* (5) T. **VI** (1995), 81–89.

[7] Lj. B. Ćirić, On nonexpansive type multi-valued mappings, coincidences and fixed points, *Bull. Acad. Serbe Sci. Arts Cl. Sci. Math. Natur. No.* **21** (1996), 9–17.

[8] Lj. B. Ćirić, On a generalization of a Greguš fixed point theorem, *Czechoslovak Math. J.* **50**(125) (2000), 449–458.

[9] R. O. Davis, Another version of a common fixed-point theorem, *Publ. Math. Debrecen* **38** (1991), 237–243.

[10] R. O. Davis and S. Sessa, A common fixed point theorem of Greguš type for compatible mappings, *Facta Univ. (Niš), Ser. Math. Inform.* **7** (1992), 99–106.

[11] D. Delbosco, O. Ferrero and F. Rossati, Teoremi di punto fisso per applicazioni negli spazi di Banach, *Boll. Un. Mat. Ital.* (6) **2-A** (1983), 297–303.

[12] M. L. Divicaro, B. Fisher and S. Sessa, A common fixed point theorem of Greguš type, *Publ. Math. Debrecen* **34** (1987), 83–89.

[13] B. Fisher, Common fixed points on a Banach space, *Chung Yuan J.* **11** (1982), 19–26.

[14] B. Fisher and S. Sessa, On a fixed point theoem of Greguš, *Internat. J. Math. Math. Sci.* **9**, no. 1 (1986), 23–28.

[15] M. Greguš, Jr., A fixed point theorem in Banach space., *Boll. Un. Mat. Ital.* **A 5** (1980), 193–198.

[16] G. Junck, On a fixed point theorem of Fisher and Sessa, *Internat J. Math. Math. Sci.* **13** (1990), 497–779.

[17] W. A. Kirk, Krasnoselskii's iteration process in hyperbolic spaces, *Numer. Func. Anal. Optim.* **4** (1982), 371–381.

[18] B. Y. Li, Fixed point theorems of nonexpansive mappings in convex metric spaces, *App. Math. Mech.* (English edn.) **10** (1989), 183–188.

[19] R. N. Mukherjea and V. Verma, A note on a fixed point theorem of Greguš, *Math. Japon.* **33** (1988), 745–749.

[20] H. K. Pathak, S. M. Kang, Y. J. Cho and J. S. Jung, Greguš type common fixed point theorems for compatible mappings of type (T) and variational inequalities, *Publ. Math. Debrecen* **46** (1995), 285–299.

[21] H. K. Pathak and M. S. Khan, Compatible mappings of type (B) and common fixed point theorems of Greguš type, *Czechoslovak Math. J.* **45** (1995), 685–698.

[22] W. Takahashi, A convexity in metric spaces and nonexpansive mappings I, *Kodai Math. Sem. Rep.* **22** (1970), 142–149.

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