Publ. Math. Debrecen
65/1-2 (2004), 65–79

The approximate solution of nonlinear operational equations

By ĐURĐICA TAKAČI (Novi Sad) and ARPAD TAKAČI (Novi Sad)

Abstract. The approximate solution of a class of differential equations, in the field of Mikusiński operators is constructed by using the method of successive approximations. It is shown that a sequence of approximate solutions converges uniformly in the field of Mikusiński operators to the exact solution of the considered equation. The obtained results are applied to a class of partial integro-differential equations.

1. Notations and notions

The set of continuous functions $\mathcal{C}_+$ with supports in $[0,\infty)$, with the usual addition and the multiplication given by the convolution

$$f * g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau, \quad t \geq 0,$$

is a commutative ring without unit element. By the Titchmarsh theorem, $\mathcal{C}_+$ has no divisors of zero, hence its quotient field, called the Mikusiński operator field, and denoted by $\mathcal{F}$, can be defined. Its elements are called *operators*; they are quotients of the form

$$\frac{f}{g}, \quad f \in \mathcal{C}_+,\ 0 \not\equiv g \in \mathcal{C}_+,$$

Mathematics Subject Classification: 44A40, 65J10.
Key words and phrases: Mikusiński operators, differential equations.

where the last division is observed in the sense of convolution (see [1]).

Clearly, every continuous function $a = a(t)$ with support in $[0, \infty)$ can be observed as a (unique) operator of the form $(a * g)/g$ (where g is an arbitrary nonzero element from $\mathcal{C}_+$); we shall simply denote this operator by a. Then we say that the operator a represents the continuous function $a(t)$ and write $a = \{a(t)\}$. In view of this remarks, the multiplication in $\mathcal{F}$ of two continuous functions $a = a(t)$ and $b = b(t)$ from $\mathcal{C}_+$ will be simply denoted by ab; this product is thus the operator c representing the continuous function $c(t) = a*b(t)$, $t \geq 0$. We shall denote by $\mathcal{F}_c$ the proper subset of $\mathcal{F}$ consisting of the operators representing continuous functions.

As examples of operators, we have the integral operator $\ell \in \mathcal{F}_c$ representing the constant function 1 (on $[0, \infty)$), and its powers ℓ^α, $\alpha \geq 1$:

$$\ell = \{1\}, \qquad \ell^\alpha = \left\{ \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right\}, \quad \alpha \geq 1.$$

Also, among the most important operators are the inverse operator to ℓ, the differential operator s, and I is the identity operator, i.e.,

$$\ell s = I.$$

Neither s nor I are operators from $\mathcal{F}_c$.

For the theory of differential equations, the following relation, connecting the operator representing the nth derivative of an $n$times derivable function $x = x(t)$ with the operator x is essential:

$$\left\{ x^{(n)}(t) \right\} = s^n x - x(0)s^{n-1} - \cdots - x^{(n-1)}(0)I.$$

Two operators a and b can be compared only if they represent continuous real valued functions, say $a = \{a(t)\}$ and $b = \{b(t)\}$, and then, by definition, we have

$$a \leq b \quad \text{iff} \quad a(t) \leq b(t), \qquad \text{for each } t \geq 0,$$

$$a \leq_T b \quad \text{iff} \quad a(t) \leq b(t), \quad \text{for each } 0 \leq t \leq T$$

(see [1], p. 237).

Two operational functions $a(x) = \{a(t, x)\}$, $b(x) = \{b(t, x)\}$ representing continuous real valued function of two variables $t \in [0, T]$ and $x \in [c, d]$, and are compared as

$$a(x) \leq_T b(x), \quad x \in [c, d],$$

iff

$$a(t,x) \le b(t,x) \quad \text{for every pair } (t,x) \in [0,T] \times [c,d].$$

The absolute value of an operator $a \in \mathcal{F}_c$, representing a continuous real valued function $a(t)$, $t \ge 0$, i.e., $a = \{a(t)\}$, is the operator $|a| = \{|a(t)|\}$. Also, we put $|a(x)| = \{|a(x,t)|\}$ if a represents a continuous function of two variables.

If the operators a and b are from $\mathcal{F}_c$, then it holds

$$|a+b| \le |a| + |b|,$$

$$|ab| = \left| \int_0^t a(\tau) b(t-\tau)\, d\tau \right| \le |a|\,|b|,$$

and

$$|a| \le_T \alpha(T)\ell, \quad \text{where} \quad \alpha(T) = \max_{t \in [0,T]} |a(t)|.$$

In this paper we shall use only the type I convergence (shortly: convergence) in the field $\mathcal{F}$ (see [2], p. 155). By definition, a sequence of operators $(a_n)_{n\in\mathbb{N}}$ converges to an operator a iff there exists an operator $q \ne 0$, such that $(qa_n)_{n\in\mathbb{N}}$ is a sequence of continuous functions on $[0,\infty)$ which converges uniformly on every finite interval to the continuous function qa.

2. Introduction

We consider the partial integro-differential equation of the form

$$\begin{aligned} &\sum_{i=0}^{n_1} \sum_{j=0}^{2} A_{i,j}(x) \frac{\partial^{i+j} u}{\partial t^i \partial x^j}(t,x) \\ &\qquad = \int_0^t \left(\sum_{i=0}^{n_2} \sum_{j=0}^{1} \mathcal{B}_{i,j}(t-\tau,x) \frac{\partial^{i+j} u}{\partial t^i \partial x^j}(\tau,x) \right) d\tau \\ &\qquad\quad + \int_0^t \left(\sum_{i=0}^{n_3} C_i(x) u(t-\tau,x) \frac{\partial^i u}{\partial t^i}(\tau,x) \right) d\tau, \end{aligned} \tag{1}$$

with the appropriate initial conditions

$$\frac{\partial^i u}{\partial t^i}(0,x) = \psi_i(x), \quad i = 0,1,\dots,\max(n_1,n_2,n_3)-1, \tag{2}$$

$$\frac{\partial^j u}{\partial x^j}(t,x_0) = \phi_j(t), \quad j = 0,1, \tag{3}$$

for $x \in [x_0, x_0+h]$ and $t \geq 0$, such that $A_{n_1,2} \neq 0$, $n_1 > \max(n_2,n_3)$, $n_1,n_2,n_3 \in \mathbb{N}_0$.

In (1), the coefficients $A_{i,j}$, $i = 0,1,2,\dots,n_1$, $j = 0,1,2$, and C_i, $i = 1,2,\dots,n_2$, depend only on the variable x, while the coefficients $\mathcal{B}_{i,j}$, $i = 0,1,2,\dots,n_3$, $j = 0,1$, depend on variables t and x. In (2) and (3), ψ_i, $i = 1,\dots,\max(n_1,n_2,n_3)-1$, ϕ_j, $j = 0,1$, are continuous functions of the variable t and x, respectively. Further on (without loss of generality), we shall take $\psi_i = 0$, for each $i = 1,\dots,\max(n_1,n_2,n_3)-1$.

In $\mathcal{F}$, (1) with the conditions (2) corresponds to the equation

$$\begin{aligned}\sum_{i=0}^{n_1}\sum_{j=0}^{2} A_{i,j}(x)s^i u^{(j)}(x) = \sum_{i=0}^{n_2}\sum_{j=0}^{1} s^i B_{i,j}(x)u^{(j)}(x)\\ + \sum_{i=0}^{n_3} C_i(x)s^i u^2(x),\end{aligned} \tag{4}$$

where s is the differential operator, and $B_{i,j}$ are operational functions

$$B_{i,j}(x) = \{\mathcal{B}_{i,j}(t,x)\}, \quad i = 1,2,\dots,n_2,\ j = 0,1.$$

In $\mathcal{F}$, the conditions (3) correspond to the conditions

$$\frac{\partial^j u}{\partial x^j}(x_0) = \phi_j, \quad j = 0,1, \tag{5}$$

where the operators ϕ_j, correspond to the functions $\phi_j(t)$, $j = 0,1$.

In [5], [6] and [7], the authors considered the partial differential equation of the form (1), where $\mathcal{B}_{i,j} = 0$, $i = 1,\dots,n_2$, $j = 0,1$, and $C_i = 0$, $i = 1,\dots,n_3$, with the conditions (2) and the boundary conditions instead of initial conditions (3). The corresponding problem in the field $\mathcal{F}$ is a linear ordinary differential equation with boundary conditions. By using the factorization method in $\mathcal{F}$, the approximate solutions were constructed and the error of approximation was estimated.

In [3], the Volterra linear equation and the following equation

$$x'(t) + \lambda x(t) + \int_0^t k(t-\tau)x(\tau)\,d\tau = f(t), \quad \text{with} \quad x(0) = x_0,$$

where k and f are continuous functions, were analyzed. Using some algebraic considerations, in particular proving that the space $\mathcal{C}_+$ is a Jacobson algebra, Heatherly and Huffman proved the existence and uniqueness of the solution of this problems in the field $\mathcal{F}$.

Clearly, (4) is a nonlinear equation in the field $\mathcal{F}$. It can be reduced to a system of simultaneous first order differential equations in $\mathcal{F}$, as in the classical theory of differential equations.

In Section 3, we apply the method of successive approximations to the first order nonlinear differential equation in the field $\mathcal{F}$, similarly as Picard's method in classical sense. For that purpose, we introduce a Lipschitz type condition in the field $\mathcal{F}$. We prove the existence of the solution of the considered problem and construct the approximate solutions in the field $\mathcal{F}$. We also estimate the error of approximation and prove the convergence of the sequence of approximate solutions.

The upper method can be applied to more general cases, e.g., to a partial differential equation containing the nth derivative in x; its corresponding equation in $\mathcal{F}$ would then be an nth order ordinary differential equation, and not just of order two as is (4). The other possible generalization is to replace the conditions in (2) or (3) with certain generalized functions (for example, δ distributions).

3. The first order nonlinear differential equation

Let us consider the nonlinear first order differential equation in the field $\mathcal{F}$

$$u'(x) = D(x)u(x) + E(x)u^2(x), \tag{6}$$

with the condition

$$u(x_0) = u_0, \tag{7}$$

where u is an unknown operator function, D and E are continuous operational functions on the interval $(x_0 - h, x_0 + h)$, $h > 0$, while the given operator u_0 on the right-hand side of (7) is not necessarily representing a continuous function, but we assume that there exists an $r \in \mathbb{N}$, such that $\ell^r u_0$ represents a continuous function.

We shall assume that the operational functions D and E can be respectively written in the form

$$D(x) = \ell^{r_1} A(x) + \ell^{r_2} B(x), \quad E(x) = \ell^{r_3}, \tag{8}$$

where ℓ is the integral operator, r_1, r_2 and r_3 are natural numbers, A is a continuous parametric function, and B is an operational function representing a continuous one, both on the interval $(x_0 - h, x_0 + h)$, for some $h > 0$.

If we denote by f the right hand side of (6) (with D and E from (8)) then the problem (6), (7) in the field $\mathcal{F}$ can be written as

$$u'(x) = f(x, u), \quad x \in (x_0 - h, x_0 + h), \tag{9}$$

with the condition

$$u(x_0) = u_0. \tag{10}$$

Let us denote the domain of the operational function $f(x, u)$ by

$$\Omega = (x_0 - h, x_0 + h) \times (a, b), \tag{11}$$

where a and b are operators from $\mathcal{F}_c$, and $a < b$. This means that we are looking for the solutions u from $\mathcal{F}_c$, such that $a < u < b$.

The function f is an operational function of two variables x and u in the field $\mathcal{F}$, defined on the domain Ω. Since by supposition A and B are continuous functions on Ω, then the operational function f represents a function of three variables on the domain Ω_T (for some $T > 0$) given by

$$\Omega_T = (0, T) \times (x_0 - h, x_0 + h) \times (a, b).$$

The problem (9), (10) can be expressed in the field $\mathcal{F}$ as

$$u(x) = u_0(x) + \int_{x_0}^{x} f(\xi, u(\xi))\, d\xi. \tag{12}$$

In (12), the integral of an operational function in $\mathcal{F}$ appears; it is analogous to the Riemann integral of a continuous function (see [2]). In particular, the integral of the continuous operational function f in (12) is a continuous operational function in $\mathcal{F}$, on the domain Ω given by (11).

3.1. The Lipschitz condition in the field $\mathcal{F}$

In the theory of ordinary differential equations the main role play the functions satisfying the Lipschitz condition. Thus we need to express the Lipschitz condition in the field $\mathcal{F}$, where absolute values of operators or operational functions representing continuous functions are considered.

Definition 1. An operational function f on a domain Ω, given by (11), satisfies the Lipschitz condition "with the factor ℓ^k, $k \in \mathbb{N}_0$", in the field $\mathcal{F}$, if there exists a positive numerical constant L such that

$$\left|\ell^k(f(x,u_2) - f(x,u_1))\right| \leq_T L\,\ell\left|\ell^{k_1}(u_2 - u_1)\right|, \tag{13}$$

for $k_1 \leq k \in \mathbb{N}$, and every $(x,u_1),(x,u_2) \in \Omega$, $T > 0$.

In (13), we consider the absolute value of operational functions, meaning that $\ell^k(f(x,u_2) - f(x,u_1))$ and $\ell^{k_1}(u_2 - u_1)$ represent continuous functions of three variables (namely t, x and u), and of two variables (namely t and x), respectively. If the operational functions f and u represent continuous functions, then $k = k_1 = 0$, i.e., $\ell^k = \ell^{k_1} = I$, where I is the identity operator. In this case we simply use

Definition 2. An operational function f, representing continuous function, satisfies the Lipschitz condition in the field $\mathcal{F}$ on a domain Ω, given by (11), if there exists a positive numerical constant L such that

$$|f(x,u_2) - f(x,u_1)| \leq_T L\,\ell\,|u_2 - u_1|\,, \tag{14}$$

for every $(x,u_1),(x,u_2) \in \Omega$, $T > 0$.

In the rest of the paper we shall assume that the operational function f from the right-hand side of (6) is representing a continuous function, and, moreover, satisfies the Lipschitz condition in the field $\mathcal{F}$ on a domain Ω, i.e.,

$$\begin{aligned}
|f(x_2,u_2) - f(x_1,u_1)| &= \Big|(\ell^{r_1}A(x) + \ell^{r_2}B(x))u_2 + \ell^{r_3}u_2^2 \\
&\quad - \big((\ell^{r_1}A(x) + \ell^{r_2}B(x))u_1 + \ell^{r_3}u_1^2\big)\Big| \\
&= \Big|\ell^{r_1}A(x) + \ell^{r_2}B(x) + \ell^{r_3}(u_2 + u_1)\Big|\,|u_2 - u_1|
\end{aligned}$$

$$\leq_T L\,|u_2 - u_1|\,\ell,$$

where $L = \max_{(t,x,u)\in\Omega_T} |\mathcal{L}(t,x,u)|$, with

$$\{\mathcal{L}(t,x,u)\} = \ell^{r_1} A(x) + \ell^{r_2} B(x) + \ell^{r_3}(u_2 + u_1).$$

4. The method of successive approximations in $\mathcal{F}$

In the following considerations let us denote the domain of the function f by

$$\tilde{\Omega} = \big\{(x,u) \;\big|\; |x - x_0| \leq h,\ |\ell^{k_1}(u - u_0)| \leq_T Mh\ell\big\}, \tag{15}$$

where M is a positive constant. Note that the first absolute value in (15) is considered in the usual sense. The operators u and u_0 are not necessarily from $\mathcal{F}_c$, therefore the power $k_1 \in \mathbb{N}$ of the operator ℓ^{k_1} is chosen sufficiently large so that $\ell^{k_1}(u - u_0)$ represents a continuous function. If u and u_0 represent continuous functions, then we simply put

$$\tilde{\Omega} = \big\{(x,u) \;\big|\; |x - x_0| \leq h,\ |u - u_0| \leq_T Mh\ell\big\}. \tag{16}$$

Next we shall apply the method of successive approximations in the field $\mathcal{F}$, analogously as is done in classical analysis, in order to obtain the existence of the solution of the problem (9), (10). Firstly, let us choose an arbitrary continuous operational function $q_0(x)$, satisfying $q_0(x_0) = u_0$, where $(x_0, q_0) \in \tilde{\Omega}$. Then we define the sequence of operators $(q_n)_{n\in\mathbb{N}}$ in the field $\mathcal{F}$ by

$$q_n(x) = u_0 + \int_{x_0}^{x} f(\xi, q_{n-1}(\xi))\, d\xi, \quad x \in (x_0 - h, x_0 + h). \tag{17}$$

It holds

Proposition 1. *For every $n \in \mathbb{N}$, the operational function $q_n(x)$ is continuous, provided that f is a continuous function.*

Proof. The operational function f is a continuous one, and by supposition there exists an operator ℓ^k such that the operator $\ell^k u_0$ is from $\mathcal{F}_c$. Therefore, the operator function

$$q_1(x) = u_0 + \int_{x_0}^{x} f(\xi, q_0(\xi))\, d\xi,$$

is also continuous. By using the mathematical induction, we obtain that $q_n(x)$ is a continuous operational function on $(x_0 - h, x_0 + h)$, for every $n \in \mathbb{N}$.

If the operational function f in the problem (9), (10) represents a continuous function of three variables t, x and u, and, moreover, u_0 is an operator from $\mathcal{F}_c$, then every operational function $q_n(x)$, $n \in \mathbb{N}$, represents a continuous function of two variables t and x. $\square$

We can prove now

Theorem 1. *Let us suppose that for the problem* (9), (10) *the following conditions are satisfied:*

1. *the operational function f represents a continuous function on the domain $\tilde{\Omega}$, given by* (16)*;*
2. *the operational function f satisfies the Lipschitz condition* (14) *in the field $\mathcal{F}$;*
3. *it holds $|f(x,u)| \leq_T M\ell$, where $M = \max_{0 \leq t \leq T,\, |x-x_0| \leq h} |f_1(t,x,u)|$, and f_1 is a function satisfying $f(x,u) = \{f_1(t,x,u)\}$.*

Then, every operational function q_n, $n \in \mathbb{N}$, satisfies the inequality

$$|q_n(x) - u_0| \leq_T Mh\ell, \quad \text{for } |x - x_0| \leq h,\ 0 \leq t \leq T.$$

Proof. The operational function f represents a continuous function on the domain $\tilde{\Omega}$, given by (16), and each operational functions q_n, $n \in \mathbb{N}$, also represents a continuous function. Therefore we can use the absolute value and the estimations as in classical analysis

$$|q_n(x) - u_0| \leq \int_{x_0}^{x} |f(\xi, q_n(\xi))| \, d\xi \leq_T M|x - x_0|\ell,$$

$$|x - x_0| \leq h,\ T > 0.$$

Now we can prove the uniform convergence of the sequence $(q_n(x))_{n\in\mathbb{N}}$. $\square$

Theorem 2. *If the conditions of Theorem 1 are satisfied, then the sequence of operational functions $(q_n(x))_{n\in\mathbb{N}}$, given by* (17)*, converges uniformly in the field $\mathcal{F}$.*

Proof. As one can expect, the proof is similar to the well known proof in the case of ordinary differential equations. The difference is that, instead of functions of one variable, we deal with operational functions representing continuous functions of two variables.

Firstly, let us recall that from Proposition 1 it follows that every operational function $q_n(x)$, $n \in \mathbb{N}$, represents a continuous function on the domain $\tilde{\Omega}$. Let us suppose that for the operator function q_0 there exists a numerical constant K such that in the field $\mathcal{F}$ it holds

$$|q_0'(x) - f(x, q_0(x))| \leq_T K\ell, \quad |x - x_0| \leq h,$$

where $K = \max_{0\leq t\leq T,\, |x-x_0|\leq h} |Q_0(t,x)|$, and

$$\{Q_0(t,x)\} = q_0'(x) - f(x, q_0(x)).$$

The operator function q_1 also represents a continuous function (of variables t and x), and therefore we have the following estimation:

$$\begin{aligned} |q_0(x) - q_1(x)| &= \left| q_0(x) - u_0 - \int_{x_0}^{x} f(\xi, q_0(\xi))\, d\xi \right| \\ &\leq \int_{x_0}^{x} |q_0'(\xi) - f(\xi, q_0(\xi))|\, d\xi \\ &\leq_T K|x - x_0|\ell. \end{aligned} \tag{18}$$

Using the inequalities in (18) and the assumed Lipschitz condition for f, we have

$$\begin{aligned} |q_2(x) - q_1(x)| &\leq \int_{x_0}^{x} |f(\xi, q_1(\xi)) - f(\xi, q_0(\xi))|\, d\xi \\ &\leq_T L \int_{x_0}^{x} |q_1 - q_0|\ell\, d\xi \\ &\leq_T \frac{LK|x - x_0|^2}{2!}\ell^2, \quad |x - x_0| \leq h. \end{aligned}$$

Next, using the mathematical induction, we get

$$\begin{aligned} |q_n(x) - q_{n-1}(x)| &\leq_T \frac{L^{n-1}K|x - x_0|^n}{n!}\ell^n \\ &\leq_T \frac{L^{n-1}K|x - x_0|^n T^{n-1}}{n!(n-1)!}\ell, \quad n \in \mathbb{N}, \end{aligned} \tag{19}$$

for $|x - x_0| \le h$.

Let us consider the following infinite series of functions in the field $\mathcal{F}$

$$q_0(x) + \sum_{n=1}^{\infty} (q_n(x) - q_{n-1}(x)). \tag{20}$$

It represents a functional series of functions of two variables, and from (19) it follows that it converges uniformly to a continuous function $a(t, x)$. However, the partial sum of (20) is equal to $q_n(x)$, and therefore the sequence of operational functions $(q_n(x))_{n\in\mathbb{N}}$ converges uniformly to a continuous function $a(x)$. $\square$

In fact, we have

Theorem 3. *If the conditions of Theorem 1 are satisfied, then the sequence of operational functions* $(q_n(x))_{n\in\mathbb{N}}$*, given by* (17)*, converges uniformly to the exact solution* $u(x)$ *of the problem* (9), (10).

Proof. By Theorem 2, there exists the limit

$$a(x) := \lim_{n\to\infty} q_n(x) = a(x),$$

Then we have

$$\begin{aligned} a(x) &= \lim_{n\to\infty} q_n(x) = u_0 + \int_{x_0}^{x} \lim_{n\to\infty} f(\xi, q_{n-1}(\xi))\, d\xi \\ &= u_0 + \int_{x_0}^{x} f(\xi, \lim_{n\to\infty} q_{n-1}(\xi))\, d\xi = u_0 + \int_{x_0}^{x} f(\xi, u(\xi))\, d\xi \\ &= u(x). \end{aligned}$$ $\square$

Thus we obtained that the sequence of operational functions $(q_n(x))_{n\in\mathbb{N}}$ given by (17) converges uniformly to the exact solution $u(x)$ of the problem (9). Therefore the sum of the infinite series given by (20) is $u(x)$, i.e., we have

$$u(x) = q_0(x) + \sum_{n=1}^{\infty} (q_n(x) - q_{n-1}(x)). \tag{21}$$

5. The approximate solution of the problem (9), (10)

Let us construct the approximate solution of the problem (9), (10) by using the Picard's method of successive approximations in the field of Mikusiński operators. Namely, we shall construct the sequence of approximate solutions $(u_n(x))_{n\in\mathbb{N}}$ of the problem

$$u'(x) = f(x,u), \quad x \in [x_0, x_0+h), \quad u(x_0) = u_0,$$

as

$$u_n(x) = u_0 + \int_{x_0}^{x} f(\xi, u_{n-1})\, d\xi, \quad n \in \mathbb{N}, \tag{22}$$

where the integral in the previous expression is defined in the field of $\mathcal{F}$ (see [2], p. 67).

The approximate solutions given by (22) are of the same form as the operational function (17). Thus, we obtain the sequence of approximate solutions of the problem (9), (10), i.e., the sequence $(u_n)_{n\in\mathbb{N}}$ of operational functions in the field $\mathcal{F}$.

In order to estimate the error of approximation between the exact solution u of problem (9), (10) and its approximate one u_n, $n \in \mathbb{N}$, given by (22), let us suppose that the operational function $u(x) - u_n(x)$, $n \in \mathbb{N}$, represents a continuous function. Then the approximate solution can be estimated, by using (21), and $u_n(x) = q_n(x)$ as

$$\begin{aligned}
|u(x) - u_n(x)| &= \left| \sum_{i=1}^{\infty} (q_i(x) - q_{i-1}(x)) - \sum_{i=1}^{n} (q_i(x) - q_{i-1}(x)) \right| \\
&\le \sum_{i=n+1}^{\infty} |q_i(x) - q_{i-1}(x)| \\
&\le_T \ell \sum_{i=n+1}^{\infty} \frac{L^{i-1} K |x - x_0|^i T^{i-1}}{i!(i-1)!} \\
&\le_T \ell \frac{L^n |x - x_0|^{n+1} T^n}{(n+1)(n!)^2} R,
\end{aligned}$$

where R is a positive number such that

$$R \ge \sum_{i=1}^{\infty} \frac{L^{i-1} K |x - x_0|^{i-1} T^{i-1}}{i!(i-1)!}. \tag{23}$$

If the operational function $u(x) - q_n(x)$, $n \in \mathbb{N}$, does not represent a continuous function, but there exists a $k \in \mathbb{N}$ such that $\ell^k(u(x) - q_n(x))$ represents a continuous function, then the error of approximation in the field $\mathcal{F}$ can be estimated with the factor ℓ^k as

$$|\ell^k(u(x) - q_n(x))| \leq_T \ell \frac{L^n |x - x_0|^{n+1} T^n}{(n+1)(n!)^2} R,$$

where the constant R is defined in (23).

6. An example

In order to illustrate the exposed method, we consider the following integro-differential equation

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^4 u(t,x)}{\partial x^2 \partial t^2} - \int_0^t \frac{\partial^2 u(t-\tau, x)}{\partial t^2} u(\tau, x)\, d\tau, \qquad x \in [0,1],\ t \geq 0, \tag{24}$$

with the conditions

$$u(0,x) = 0, \quad \frac{\partial u(0,x)}{\partial t} = 0, \tag{25}$$

and

$$u(t,0) = 1, \quad \frac{\partial u(t,0)}{\partial x} = 1. \tag{26}$$

Note that the problem (24), (25), (26) is of the type (1), (2), (3).

The equation (24) with the conditions (25) corresponds, in the field $\mathcal{F}$, to the equation

$$su(x) = s^2 u''(x) - s^2 u^2(x).$$

Thus in the field $\mathcal{F}$ we obtain the following problem:

$$u''(x) = \ell u(x) + u^2(x), \quad u'(0) = \ell, \quad u(0) = \ell. \tag{27}$$

The equation in (27) reduces to the following system of simultaneous first order equation

$$u'(x) = z(x), \quad z' = \ell u(x) + u^2(x). \tag{28}$$

Applying the method of successive approximations to the system (28), using the conditions (26), we get

$$u_0 = u(0) = \ell, \quad z_0 = u'(0) = \ell,$$

and thus for the first approximation

$$u_1(x) = \int_0^x z_0 \, d\xi = \int_0^x \ell \, d\xi = \ell x,$$

$$z_1(x) = \int_0^x (\ell u_0(x) + u_0^2(x)) \, d\xi = \int_0^x (\ell^2 + \ell^2) \, d\xi = 2\ell^2 x.$$

The second and third approximation will be equal to

$$u_2(x) = \int_0^x z_1 \, d\xi = \int_0^x 2\ell^2 \xi \, d\xi = \ell^2 x^2,$$

$$z_2(x) = \int_0^x (\ell^2 \xi + \ell^2 \xi^2) \, d\xi = \ell^2 \left(\frac{x^2}{2} + \frac{x^3}{3} \right),$$

and

$$u_3(x) = \int_0^x z_2 \, d\xi = \ell^2 \left(\frac{x^3}{3!} + \frac{2x^4}{4!} \right).$$

As shown before, the sequence of approximate solutions $(u_n)_{n \in \mathbb{N}}$ converges to the exact solution of the problem (27). Clearly one can continue the upper procedure in order to obtain an approximate solution u_n with given precision. Note that each u_n is an operator function representing a continuous function.

Acknowledgements. The authors want to thank for the financial support of grants MNTRS-1866 and CEEPUS-SK-42.

References

[1] J. Mikusiński, Operational Calculus, Vol. I, *PWN – Polish Scientific Publishers, Warszawa and Pergamon Press, Oxford*, 1983.

[2] J. Mikusiński and T. K. Boehme, Operational Calculus, Vol. II, *PWN – Polish Scientific Publishers, Warszawa and Pergamon Press, Oxford*, 1987.

[3] H. E. Heatherly and J. P. Huffman, Algebraic properties of the Mikusiński convolution algebra, *Acta Mathematica Hungarica* **89**(3) (2000), 179–187.

[4] G. Dajun, V. Lakshmikantham and L. Xinzhi, Nonlinear Integral Equations in Abstract Spaces, *Kluwer Academic Publishers*, *Dordrecht/Boston/London*, 1996.

[5] Đ. Takači and A. Takači, An algorithm for the approximate solution of operator difference equation, *Numerical Functional Analysis and Optimization* **17**(9–10) (1996), 991–1003.

[6] Đ. Takači and A. Takači, The factorization method in the field of Mikusiński operators, *Publicationes Mathematicae Debrecen* **50** (1997), 287–303.

[7] Đ. Takači and A. Takači, Construction of the solutions of difference equation in the field of Mikusiński operators, *Rocky Mountain Journal of Mathematics* **27**, No. 4 (1997), 1245–1264.

ĐURĐICA TAKAČI
DEPARTMENT OF MATHEMATICS AND INFORMATICS
FACULTY OF SCIENCE
UNIVERSITY OF NOVI SAD
TRG DOSITEJA OBRADOVIĆA 4, 21000 NOVI SAD
SERBIA AND MONTENEGRO

E-mail: djtak@im.ns.ac.yu, takacsi@eunet.yu

ARPAD TAKAČI
DEPARTMENT OF MATHEMATICS AND INFORMATICS
FACULTY OF SCIENCE
UNIVERSITY OF NOVI SAD
TRG DOSITEJA OBRADOVIĆA 4, 21000 NOVI SAD
SERBIA AND MONTENEGRO

E-mail: takaci@im.ns.ac.yu, takacsi@eunet.yu

(Received July 15, 2002; file received September 3, 2003)