Publ. Math. Debrecen
43 / 1–2 (1993), 29–39

Solving a quartic discriminant form equation

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Abstract. All algebraic integers of discriminant and norm a power of 2 and 3 only in the quartic field $x^4 + 12x^2 + 18$ are calculated. This involves solving a discriminant form equation of Mahler type.

1. Introduction

We calculate in this paper all the algebraic integers in the number field K generated by a root, θ, of x^4+12x^2+18 which have discriminant a product of powers of two and three only. The method used follows that of [6] and [7]. Effective finiteness results for discriminant form equations of Mahler type where first given by GYŐRY in a series of papers culminating in [7]. An earlier non effective finiteness results was given by BIRCH and MERRIMAN in [1]. GYŐRY's result, from which the following method is derived, is that if $f \in \mathbb{Z}[X]$ is monic of degree $n \geq 3$ and $D(f) \in S$, where S is the set of numbers divisible by a finite set of primes only. Then f is $\mathbb{Z}$ equivalent to a polynomial f^* such that

$$\overline{|f^*|} \leq C(S, n).$$

The resulting large bound will be reduced by the methods of DE WEGER, however a complex linear form arises and there is the need to extend the usual L^3 algorithm to the complex case.

In particular we shall prove the following theorem and its corollary.

Theorem 1. *All the algebraic integers in the field above with discriminant a power of two and three only are given by*

$$\alpha = r + s\theta$$

where $r \in \mathbb{Z}$ and s is a power of two and three only.

The author was supported by a SERC postgraduate studentship whilst carrying out this research.

Corollary 1. *All algebraic integers in the field K defined above with discriminant and norm a power of two and three only are roots of polynomials of the form*

$$x^4 + 2^{2a}3^{2b}12x^2 + 2^{4a}3^{4b}18.$$

where $a, b \in \mathbb{N}$.

The discriminant form equation considered here is actually an index form equation of Mahler type. The papers of Gaál, Pethő and Pohst, [3] and [4], consider the solution of index form equations of Thue type. As far as we know these are the only other index form equations considered so far, except for ones coming from Thue-Mahler equations, see [5] and [8]. This work is part of the authors Phd Thesis at Kent and he would like to thank his suporvisor J. Merriman.

2. The structure of the field

We consider the field of degree 4 generated by a root of the polynomial

$$g = x^4 + 12x^2 + 18.$$

Let $g(\theta) = 0$, then all the roots of g are given by

$$\theta_1 = \theta,\ \theta_2 = -\theta,\ \theta_3 = 3\theta + \theta^3/3,\ \theta_4 = -3\theta - \theta^3/3,$$

all of which are complex. Let $K = \mathbb{Q}(\theta)$, then the group $Gal(K/\mathbb{Q})$ is cyclic, of order four, and is generated by the permutation of roots given by the cycle (1324). We have $D_K = 2^{11}3^2 = 18432$. The only units in K of finite order are ± 1 and by Dirichlets units theorem there is only one unit, η, of infinite order. This can easily be calculated (see [10] or [2]) to be $1 + \theta^2/3$.

If a rational prime, p, decomposes in K into r_p prime ideals, $\wp_i$, then as K is a Galois field we have that $e_i = e_p$ for all i, where e_i is the ramification index of the prime ideal $\wp_i$. We have $r_p e_p f_p = n = 4$.

Now g does not factorize in $\mathbb{Q}_2$ or $\mathbb{Q}_3$ so by [2, p 271] we have $r_p = 1$ for p equal to 2 and 3. Now by [10, p 396] we have that $4 - f_3 \leq v_3(D_K) = 2$. But we can find an element in K of norm 18 so we must have $e_2 = 4$, $f_2 = 1$ and $e_3 = 2$, $f_3 = 2$.

It can be easily shown, see [10] that the ideal class group is cyclic of order 2 and is generated by the image of $\wp_2$. As a byproduct we find the following generators for these ideals, which will be of use later on

$$\begin{aligned} \wp_2^2 &= (-2 - \theta^2/3) = (\beta_1), \\ \wp_3^2 &= (3) = (\beta_2), \\ \wp_2\wp_3 &= (-3\theta - \theta^3/3). \end{aligned}$$

Now let $\alpha \in K$ be an algebraic integer. Denote the conjugates of α by $\alpha = \alpha_1, \alpha_2, \alpha_3, \alpha_4$. Then $D(\alpha)$ is divisible by D_K. Our purpose is to find all such α with $D(\alpha) = 2^{u_1}3^{u_2}$. Then $\mathbb{Q}(\alpha)$ is contained in K. We give bounds which allow us to calculate the solution of this problem in the next section.

3. Generating the bounds

We now show that we are in the following situation.

$$\begin{gathered} \alpha_j - \alpha_i = \sigma\gamma_{j,i}, \quad 1 \le i, j \le 4, \quad i \ne j \\ \sigma = \epsilon\beta_1^{a_1}\beta_2^{a_2}, \\ \gamma_{j,i} = \delta_{j,i}\eta^{v_{j,i}}\beta_1^{w_{1,j,i}}\beta_2^{w_{2,j,i}}, \end{gathered} \tag{1}$$

where ϵ is a unit, $a_i, w_{l,j,i} \in \mathbb{N}$, $v_{j,i} \in \mathbb{Z}$ and the $\delta_{j,i}$'s are integers of K such that

$$(\delta_{j,i}) = \wp_2^{r_{1,j,i}}\wp_3^{r_{2,j,i}}, \quad 0 \le r_{l,j,i} < h_K.$$

Let $U = \max(w_{l,j,i})$ and $V = |v_{j,i}|$. Set $H = \max(U, V)$. Then for explicitly given contants V_0 and K_0 we show that if $H \ge V_0$, then $H < K_0$.

To prove this we follow the paper of Győry [7]. By the unique factorization of ideals there exists integers $U_{l,j,i} \in \mathbb{N}$ such that for all $1 \le i, j \le 4$ we have

$$(\alpha_j - \alpha_i) = \wp_2^{U_{1,j,i}}\wp_3^{U_{2,j,i}}.$$

By Euclids algorithm there exists $u_{l,j,i}$ and $r_{l,j,i}$ such that $U_{l,j,i} = h_K u_{l,j,i} + r_{l,j,i}$. Therefore there exists units $\epsilon_{j,i}$ such that:

$$\alpha_j - \alpha_i = \delta_{j,i}\epsilon_{j,i}\beta_1^{u_{1,j,i}}\beta_2^{u_{2,j,i}}.$$

There are a finite number of possible $\delta_{j,i}$ and we set,

$$c_1 = \max_{\delta_{j,i}}\left(\max_{1\le i,j\le 4}(\overline{|\delta_{j,i}|})\right) = 3.2004.$$

For $l = 1, 2$ we set $a_l = \min u_{l,j,i}$ and $w_{l,j,i} = a_l$. Also, let $\epsilon_{j,i}/\epsilon_{3,1} = \eta^{v_{j,i}}$. So we have equation (1) where we have set $v_{3,1} = 0$. Choose $q \ne 1$ such that $V = |v_{q,1}|$, if necessary we can obviously relabel the $v_{j,i}$'s to achieve this. Trivially, $q \ne 3$. We now have that $\gamma_{q,1} + \gamma_{3,q} = \gamma_{3,1}$. Define Λ_q by

$$\Lambda_q = (\gamma_{3,q}/\gamma_{3,1}) - 1 = -\gamma_{q,1}/\gamma_{3,1}.$$

Fix, once and for all, a determination of the complex logarithm. We consider the linear form

$$\Delta_q^{(i)} = \ln\left(\frac{\delta_{3,q}^{(i)}}{\delta_{3,1}^{(i)}}\right) + v_{3,q}\ln\eta^{(i)} + (w_{1,3,q} - w_{1,3,1})\ln\beta_1^{(i)} + (w_{2,3,q} - w_{2,3,1})\ln\beta_2^{(i)} + a_0 2\pi\sqrt{-1}.$$

We define

$$B = \max(|v_{3,q}|, |w_{1,3,q} - w_{1,3,1}|, |w_{2,3,q} - w_{2,3,1}|, |a_0|).$$

Note that we have $|a_0| \leq 3H+1$ and so $B \leq 3H+1$. We now obtain from Waldschmidt's Theorem, [13], that

$$|\Delta_q^{(i)}| > \exp(-c_2(\ln H + c_3)).$$

where $c_2 = 0.9669 \cdot 10^{35}$ and $c_3 = 3.8666$.

We now apply Yu's Theorem, [14]: to do this we choose $b, c, \wp$ such that $w_{t,b,c} = U$, where $t = 1$ if $\wp = \wp_2$ and $t = 2$ if $\wp = \wp_3$. Now choose an a such that $w_{t,a,c} = 0$: one can do this as due to the action of the cyclic Galois group. $\gamma_{i,c}$, $\gamma_{j,c}$ and $\gamma_{b,c}$ are pairwise non-conjugate if and only if $i \neq j \neq b \neq i$. If the Galois group were not cyclic we could still apply Yu's Theorem with a bit more work.

Let $A = \gamma_{a,b}/\gamma_{a,c}$ and $B = \gamma_{b,c}/\gamma_{a,c}$ so that we have $A - 1 = B$ and $\mathrm{ord}_\wp(B) \geq h_K U - 1 = 2U - 1$. Now as $U \geq 1$ we have that $\wp \mid B$ and hence $\mathrm{ord}_\wp(A) = 0$, so $\delta_{a,b}/\delta_{a,c} = 1$. Set $T = 1$ if $t = 2$ or $T = 2$ if $t = 1$, then we have

$$A = \eta^{v_{a,b} - v_{a,c}} \beta_T^{w_{T,a,b} - w_{T,a,c}}.$$

We now apply Yu's theorem to obtain $\mathrm{ord}_\wp(A-1) \leq c_5(\ln D + c_4)$ where $D = \max(|v_{a,b} - v_{a,c}|, |w_{T,a,b} - w_{T,a,c}|)$. So $D < 2H$ and hence

$$2U - 1 \leq c_5(\ln H + \ln 2 + c_4),$$

giving

$$U \leq c_5(\ln H + c_6) = T_1(H),$$

where calculation shows that $c_5 = 0.9875 \cdot 10^{19}$ and $c_6 = 4.8520$.

Choose k such that $1 \leq k \leq 4$ and $|\gamma_{q,1}^{(k)}| = \min_l |\gamma_{q,1}^{(l)}|$. If we set $\epsilon_q = \eta^{v_{q,1}}$ then we can find a constant $c_7 = 0.88137$ and an index g such that

$$|\ln|\epsilon_q^{(g)}|\ | \geq c_7 V.$$

Choose $c_8 \leq c_7/(4-1)$. We take $c_8 = 0.2203$. Then we have five cases to consider:

Case 1: $V \leq U$. Obviously we have $H < T_1(H)$ and so by the Lemma of Pethő and De Weger, [9], we have

$$H < 2c_5c_6 + 2c_5 \ln(c_5) = T_2.$$

Case 2: $U \leq V$, $\ln|\gamma_{q,1}^{(k)}| > -c_8V$ and $\ln|\epsilon_q^{(g)}| \geq c_7V$. Define the constants c_9 and c_{10} as follows:

$$c_9 = \max_{i=1,4}\left(\sum_{j=1}^{2} \ln \frac{|N(\beta_j)|}{|\beta_j^{(i)}|}\right) = 4.3356.$$

$$c_{10} = \max_{q}\left(\max_{i=1,4}\left(\ln \frac{|N(\delta_{q,1})|}{|\delta_{q,1}^{(i)}|}\right)\right) = 2.6085.$$

Then we have $c_7V \leq Uc_9 + c_{10} + (4-1)c_8V$. So if we set $c_{11} = c_7 - 3c_8$ we have $V \leq (c_9T_1(H) + c_{10})/c_{11}$. Hence, again by Pethő and De Weger's Lemma, we obtain

$$H \leq 2(c_{10} + c_9c_5c_6 + c_9c_5 \ln(c_9c_5/c_{11}))/c_{11} = T_3.$$

Case 3: $U \leq V$, $\ln|\gamma_{q,1}^{(k)}| > -c_8V$ and $\ln|\epsilon_q^{(g)}| \leq -c_7V$. We define c_{12} by

$$c_{12} = \sum_{i=1}^{2} \ln \overline{|\beta_i|} = 1.4452.$$

Then we have that $c_7V \leq c_8V + c_{12}U + \ln c_1$, which means that

$$V \leq (c_{12}U + \ln c_1)/(c_7 - c_8) = c_{13} + c_{14}T_1(H).$$

So we obtain,

$$H \leq 2(c_{13} + c_{14}c_5c_6 + c_{14}c_5 \ln(c_{14}c_5)) = T_4.$$

Cases 4 and 5: $U \leq V$, $\ln|\gamma_{q,1}^{(k)}| \leq -c_8V$. Now as $v_{3,1} = 0$ we have that

$$\ln|\Lambda_q^{(k)}| \leq -c_8V - \ln|\gamma_{3,1}^{(k)}| \leq -c_8V + 3(c_{12}U + \ln c_1).$$

Case 4: $3(c_{12}U + \ln c_1) \geq 9c_8V/10$. Then we have that

$$V \leq 30(c_{12}U + \ln c_1)/(9c_8) = c_{16} + c_{15}T_1(H)$$
$$H \leq 2(c_{16} + c_5c_6c_{15} + c_5c_{15} \ln(c_5c_{15})) = T_5.$$

Case 5: $3(c_{12}U + \ln c_1) < 9c_8V/10$. We have that $\ln|\Lambda_q^{(k)}| < -c_8V/10$ that is $|\epsilon^{\Delta_q^{(k)}} - 1| < \epsilon^{-c_8V/10}$. Now if we assume that $V \geq V_0 = 10\ln(3)/c_8$ then we have

$$\exp(-c_2(\ln H + c_3)) < |\Delta_q^{(k)}| \leq 2|\epsilon^{\Delta_q^{(k)}} - 1| < 2e^{-c_8V/10}.$$

So we have that

$$H < 10(\ln 2 + c_2c_3 + c_2 \ln H)/c_8 = c_{17} + c_{18}\ln H,$$
$$H < 2(c_{17} + c_{18}\ln c_{18}) = T_6.$$

Summarissing our conclusions in all five cases, we find that, if $H \geq V_0 = 49$ then we have, $H < \max(T_2, T_3, T_4, T_5, T_6) = 0.7744 \cdot 10^{39} = K_0$.

4. Reduction of the bounds

The P-adic Reduction Step

We follow [11] in our treatment of p-adic logarithms, the definition of which is given there. Choose $a, b, c, t, T, \wp$ as in the previous application of YU's Theorem. Let $\psi = v_{a,b} - v_{a,c}$ and $\xi = w_{T,a,b} - w_{T,a,c}$. Then since $-1 \leq \mathrm{ord}_\wp(\delta_{b,c}/\delta_{a,c}) \leq 1$,

$$2U - 1 \leq \epsilon_\wp \mathrm{ord}_p(z-1) \leq 2U + 1.$$

There are two cases to consider.

Case A: $t = 1$.

$$\ln_2 \eta = \frac{1}{4}\ln_2 \eta^4 = 2 + \theta^2(1+2) + 4 + \dots,$$
$$\ln_2 \beta_2 = \frac{1}{2}\ln_2 \beta_2^2 = 4 + 8 + \dots \in \mathbb{Q}_2.$$

Case B: $t = 2$.

$$\ln_3 \eta = \frac{1}{8}\ln_3 \eta^8 = 2.3 + \theta^2(1 + 2.3) + 9 + \dots\ .$$
$$\ln_3 \beta_1 = \frac{1}{4}\ln_3 \beta_1^4 = 3 + 9 + \dots \in \mathbb{Q}_3.$$

In both cases we have, for $a_p, b_p, c_p \in \mathbb{Q}_p$ and $b_p \equiv 1 \bmod p$,

$$2U - 1 \leq \epsilon_p \mathrm{ord}_p(\psi(a_p + b_p\theta^2) + \xi c_p) \leq 2U + 1.$$

Now by an argument of [11, p22] we obtain

$$\mathrm{ord}_p(\Lambda_{p,i}) \geq (2U-1)/\epsilon_p - \frac{1}{2}\mathrm{ord}_p(18432).$$

where $\Lambda_{p,1} = a_p\psi + c_p\xi$ and $\Lambda_{p,2} = b_p\psi$. Hence we obtain the following reduction step.

Lemma 2. *Choose an $m \in \mathbb{N}$ such that $p^m \geq 2K_0$ then $U \leq (m+g)/h$ where for $p = 2$ we have $h = \frac{1}{2}$ and $g = -\frac{23}{4}$ and for $p = 3$ we have $h = 1$ and $g = -\frac{3}{2}$.*

Proof. Assume otherwise that $hU - g \geq m$ and so $\mathrm{ord}_p(\Lambda_{p,2}) \geq hU - g \geq m$. Now as $b_p \equiv 1 \bmod p$ we find that $\mathrm{ord}_p(w) \geq m$ and hence $p^m \mid \psi$. But this would mean that $p^m < |\psi| \leq 2H \leq 2K_0$ which is a contradiction. □

Complex Linear Forms In Logarithms And The Complex Reduction Step

We now give a method to reduce the bounds in Case 5. For each i we let

$$\Delta_q^{(i)} = v_{3,q} \ln(\eta^{(i)}) + a \ln(\beta_1^{(i)}) + b \ln(\beta_2^{(i)}) + u \ln(-3\theta_i - \theta_i^3/3) + a_0 2\pi\sqrt{-1}$$

where the preceding work imposes the following constraints

$$|u| \leq 1,\ |v_{3,q}| \leq H,\ |a| \text{ and } |b| \leq U \leq H,\ |a_0| \leq H + 2U + 1 \text{ and } H < K_0.$$

Also, we have from the above discussion that in Case 5,

$$|\Delta_q^{(i)}| \leq 2e^{-c_8 H/10}.$$

In order to reduce the bound for H by using this inequality one can either take the real and imaginary parts of the linear form separately and use the original L^3 algorithm, or one can note that the L^3 algorithm can be extended to the complex case in a straightforward way. We take the latter approach as experience shows that the former often does not work because one has to "throw" half the information away. So we apply a variant of the real reduction step given in [12] or [11], with a different L^3 algorithm, to reduce the bound for H. We apply the method 12 times for each combination of i and u.

The Reduction Process

Currently we have $U, V \leq K_0 = 0.775 \cdot 10^{39}$. Our reduction process proceeds as follows:

1) Perform the p-adic reduction step to reduce the bound for U.

2) Examine the various cases given in the above discussion and with this reduced bound, we can reduce the bound for V. We will not be able to reduce the bound for V in Case 5.

3) In Case 5 reduce the bound for V using the above process.

4) Repeat steps 1–3 until no further improvement is made.

Firstly we apply Lemma 2 with $p = 2$ to find $2^{135} \geq 2K_0$ and with $p = 3$ we find $3^{85} \geq 2K_0$. And hence the bound for U can be reduced to 258. Note how much this first reduction step reduces the bound for U. Examining the various cases of the preceding discussion gives:

Case		$V \leq$
1	$V \leq U$	258
2	$V \leq 19.676U + 11.85$	5088
3	$V \leq 2.1862U + 1.760$	565
4	$V \leq 21.863U + 17.60$	5658

We now apply the reduction step for Case 5 to get $V \leq 188$. So in all cases $V \leq 5658$ which we take as our new K_0. The whole process is repeated again. Another application of Lemma 2 gives $U \leq 14$ and performing the above analysis we find $V \leq 323$ in all cases. Lemma 2 is performed again to get $U \leq 6$, which leads us to deduce that $V \leq 188$. One more application of Lemma 2 gives $U \leq 5$. But we cannot reduce the bound for V any further so our reduction process halts here with the final bounds: $U \leq 5$, $V \leq 188$.

We now summarise on what we have deduced so far about the α_i's and their related exponents;

$$\begin{aligned} \alpha_j - \alpha_i &= \sigma\gamma_{j,i}, \\ \sigma &= \epsilon\beta_1^{a_1}\beta_2^{a_2}, \\ \gamma_{j,i} &= (-3\theta - \theta^3/3)^{\lambda_{j,i}}\eta^{v_{j,i}}\beta_1^{w_{1,j,i}}\beta_2^{w_{2,j,i}}, \\ D(\alpha_i) &= \pm 2^{u_i}3^{u_2}. \end{aligned}$$

where $u_1, u_2 \in \mathbb{N}$ and $|\lambda_{j,i}| \leq 1$, $|v_{j,i}| \leq 188$ and $0 \leq w_{k,j,i} \leq 5$. Since two α's which differ by a linear equivalence or multiplication by an S-unit are both solutions if one of them is, we shall only look for solutions, distinct under these transformations. Therefore if $\alpha = a + 2^b3^c\phi$ with ϕ an integer of K, $a \in \mathbb{Z}$ and $b, c \in \mathbb{N}$ then we will insist that $a = b = c = 0$; such an α will be called reduced.

5. Finding the solution

Firstly we shall have to bound the variables a_1, a_2. For $l = 1$ and 2 we have

$$\mathrm{ord}_{\wp_{\ell+1}}\left(\prod_{1\leq i<j\leq 4}\gamma_{j,i}^2\right) \leq 12(\mathrm{ord}_{\wp_{\ell+1}}(\delta_{j,i}) + Ue_l/\mathrm{ord}_{\wp_{\ell+1}}(\beta_l)) = \tau_l.$$

Note that $\tau_1 = 132$, $\tau_2 = 72$, $e_1 = 4$ and $e_2 = 2$. For $l = 1, 2$, choose d_l to be the greatest integer such that

$$u_l e_l - \mathrm{ord}_{\wp_{\ell+1}}\left(\prod_{1\leq i<j\leq 4}\gamma_{j,i}^2\right) \geq 12d_l e_l \tag{2}$$

Now we also have

$$\pm 2^{u_1}3^{u_2} = (\epsilon\beta_1^{a_1}\beta_2^{a_2})^{12}\prod_{1\le i<j\le 4}\gamma_{j,i}^2.$$

So $d_l \ge 0$ and by the definition of d_l we have that $12(d_l+1)e_l$ is greater than the right hand side of equation (2). This gives, using equation (2), that

$$\begin{gathered} 0 \le u_l e_l - 12 d_l e_l \le 12 e_l + \tau_l \\ 0 \le 2a_l - d_l e_l \le e_l. \end{gathered} \tag{3}$$

Choose ξ such that $\beta_1^{a_1}\beta_2^{a_2} = 2^{d_1}3^{d_2}\xi$. Now ξ is an integer of the field K, since $2a_l - d_l e_l \ge 0$ for $l = 1, 2$. Let $\chi_{j,i} = \epsilon\xi\gamma_{j,i}$ so that

$$\alpha_j - \alpha_i = 2^{d_1}3^{d_2}\chi_{j,i}. \tag{4}$$

Note that $\chi_{i,i} = 0$. Let $A_1 = \alpha_1 + \ldots + \alpha_4$ and $\Phi_i = -\sum_{j=1}^4 \chi_{j,i}$. Then $A_1 \in \mathbb{Z}$ and Φ_i is an integer of the field K. From equation (4) we then obtain

$$4\alpha_i = -\sum_{j=1}^4 (2^{d_1}3^{d_2}\chi_{j,i} - \alpha_j) = 2^{d_1}3^{d_2}\Phi_i + A_1.$$

Now if $d_1 > 2$ then we have $A_1 \equiv 0 \bmod 4$ so $\alpha_i = A_2 + 2^{d_1-2}3^{d_2}\Phi_i$ for $A_2 \in \mathbb{Z}$. Which means that $A_2 = 0 = d_2$, $d_1 = 2$, a contradiction since we have assumed α_i reduced. So $d_1 \le 2$. Let $\Phi_i' = 2^{d_1}\Phi_i$. Then $\Phi_i' \equiv A_2 \bmod 4$ for some integer A_2. So $\Phi_i' = A_2 + 4\Phi_i''$, say, and

$$4\alpha_i = A_1 + 3^{d_2}(A_2 + 4\Phi_i'').$$

Hence $A_1 + 3^{d_2}A_2 \equiv 0 \bmod 4$ which leads us to deduce that for some $A_3 \in \mathbb{Z}$,

$$\alpha_i = A_3 + 3^{d_2}(\Phi_i' - A_2) = (A_3 - 3^{d_2}A_2) + 2^{d_1}3^{d_2}\Phi_i$$

Again as the α_i are reduced we see that $d_1 = d_2 = 0$. From this follows, by the inequalities (3), that $a_1 \le 2$ and $a_2 \le 1$.

Note that we also have $u_l \le 12 + \tau_l + 12d_l$ and so $u_1 \le 144$ and $u_2 \le 84$. Now let $\epsilon = \eta^v$ and $\alpha_j - \alpha_i = \epsilon w_{j,i}$. The variable v is then the last variable which needs to be bounded.

Let $T_8 = \max_i(\pm 188|\ln|\eta(i)\|) \approx 165$. We find that

$$\ln\overline{|w_{j,i}^{-1}|} \le T_9 = 166.9, \quad \ln\overline{|w_{j,i}|} \le T_{10} = 175.92.$$

We have then

$$2^{u_1}3^{u_2}\prod_{1\le i<j\le 4} w_{j,i}^{-2} = \eta^{12v}, \quad \epsilon^{-T_{10}} \le \overline{|\eta|}^v \le 2^{12}3^7\epsilon^{T_9}$$

Hence we have $-199 \le v \le 207$.

6. Proof of the theorem

We now have for all i and j such that $i \neq j$ that

$$\alpha_j - \alpha_i = \pm\eta^{a_{j,i}}(-3\theta - \theta^3/3)^{b_{j,i}}\beta_1^{c_{j,i}}\beta_2^{d_{j,i}} \tag{5}$$

where $-387 \leq a_{j,i} \leq 395$, $-1 \leq b_{j,i} \leq 1$, $0 \leq c_{j,i} \leq 7$ and $0 \leq d_{j,i} \leq 6$. This represents over 130000 cases to consider. Each α_i is represented in the following way with $r, s, t, v \in \mathbb{Z}$, $(s, t, v) = 1$ and $\deg(\alpha_i) = 4$:

$$\alpha_i = r + s\theta_i + t\theta_i^2/3 + v\theta_i^3/3.$$

Hence

$$\alpha_1 - \alpha_3 = 4t + \theta(-2s + 10v) + 2t\theta^2/3 + (4v - s)\theta^3/3.$$

By equating coefficients of powers of θ_i, we use this equation, with equation (5), to find a set of possible solution triples (s, t, v). This is easily done (slowly) with a computer algebra system. We then substitute these into the discriminant form to test if each triple is actually a solution. This step takes about 15 Hours CPU time, but after all this computing effort the only solution is $(s, t, v) = \pm(1, 0, 0)$. This proves the Theorem.

Finally, we can also find all quartic integers which lie in the field K with both discriminant and norm a product of powers of two and three only which lie in the field K. We have by the above that such an integer must be of the form $\alpha = 2^c3^d(r + y\theta)$ where $(r, y) = 1$, $y = \pm 2^a 3^b$ and $N(r + y\theta) = 2^n 3^m$ for some $a, b, c, d, n, m \in \mathbb{N}$. Examining the norm equation modulo powers of 2 and 3 gives that $n \leq 1$ and $m \leq 2$. As K is a totally complex field the solution to this is trivial.

Let $R_i = R\epsilon(\theta_i)$ and $I_i = Im(\theta_i)$. Then for $i = 1$ or $i = 2$ or both we have that

$$(r + yR_i)^2 + (yI_i)^2 \leq 18$$

and so $|yI_i| \leq \sqrt{18}$. Hence $|y| \leq 3.2$ and $|r| \leq 4.2$. This gives us an easy way to deduce that the only possible values of r and y are $\pm(0, 1)$. So the only algebraic integers of degree four in K with discriminant and norm a product of powers of two and three only are given by the roots of the polynomial

$$X^4 + 2^{2c}3^{2d}12X^2 + 2^{4c}3^{4d}18.$$

We have hence proved the corollary.

7. Computer implementation

All the calculations where performed using the computer algebra package MAPLE running on a VAX cluster. The main computing power was expended in the final search for the solutions, which we have already noted

took 15 hours CPU time. The reduction from the large bounds to the smaller ones using the L^3 algorithm took only 2 hours of CPU time.

The advantage of using a computer algebra system is that the final step is easy to program, and it is easy to calculate with large integer numbers or high accuracy floating point ones. However, the (only) draw back is that a computer algebra system is very inefficient when compared to an equivalent FORTRAN program, this may lead to a larger than necessary use of CPU time.

The above method will work (with a few minor modifications) for all index form equations of Mahler type. Unfortunately for large degree problems the bounds may become so large that amongst other things the p-adic logarithms cannot be computed to the required accuracy.

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(Received June 17, 1991; revised February 17, 1992)