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On 2-groups of almost maximal class

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Abstract. Let G be a 2-group of order 2^n, $n \geqslant 6$, and nilpotency class $n-2$. The invariants of such groups determined by their group algebras over the field of two elements are given in the paper.

1. Introduction

Let p be a prime. We say that a finite p-group G of order p^n has the *coclass* k if the nilpotency class of G is equal to $n-k$. The p-groups of coclass $k=1$ and $k=2$ are called the p-groups of *maximal class* and of *almost maximal class* respectively. The natural partition of the class of all p-groups into a series of well-behaved families of p-groups of fixed coclass has proved to be one of the most fruitful ideas of classifying p-groups (see [7]). But the full description of p-groups by generators and relations within each family was obtained only for a small number of families. It is well known for $(p,k) \in \{(2,1),(3,1)\}$ and can be derived for $(p,k) \in \{(2,2),(2,3)\}$ from [8] (see also [5], [6]).

Here we study the internal structure of 2-groups of nilpotency coclass 2 counting the important invariants of these groups. The work, though motivated by the Modular Isomorphism Problem (MIP) for finite p-groups is

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of independent interest. Recall that the MIP asks whether, for given finite p-groups G and H, the isomorphism of the group algebras FG and FH over a field F of characteristic p implies the isomorphism of G and H. The problem was posed more than 50 years ago and for a long time there was rather small progress in studying it. The state of the problem in the end of the nineties in a more general context is briefly described in [3] (see also [11] and [1], [2], [10], [12], [14], [15]). One of the last results known to the authors are that of Wursthorn saying that the groups of order 2^6 and 2^7 are determined by their modular group algebras over $GF(2)$ (see [3], [15]). He used computer programs in studying the problem for groups of small order, which are classified and have descriptions usable by computers. In December 2002 the authors were informed that the problem was solved positively for group algebras of all p-groups over the field $F = GF(p)$ ([4])[1].

Throughout, p will denote a fixed prime. If G is a p-group then $G' = (G,G)$ is the commutator subgroup of G, $\Phi(G)$ is the Frattini subgroup, G^p is the subgroup generated by p-th powers of all elements of G and $\Omega_i(G)$ is the subgroup generated by all elements of order not bigger than p^i. By $\gamma_i(G)$, $i \geqslant 2$, we mean the i-th term of the lower central series of G ($\gamma_2(G) = G'$); by $\gamma_1(G)$ we mean the subgroup $C_G(\gamma_2(G) \mod \gamma_4(G))$; if $g \in G$ and $X \subset G$, then (X,g) denotes the set of commutators $\{(x,g) : x \in X\}$. By $d(G)$ we denote the minimal number of generators of G, that is $d(G) = \log_p(|G/\Phi(G)|)$. The center of G we denote by $\zeta(G)$. If $g,h \in G$ then $g^h = h^{-1}gh$ and C_g is the conjugacy class of g, i.e. $C_g = \{g^h : h \in G\}$. The set of all conjugacy classes of G will be denoted by $Cl(G)$. If S is a normal subset of G then $Cl_G(S)$ is the set of all conjugacy classes of G contained in S.

We use $(x,y) = x^{-1}y^{-1}xy$ for the group commutator of elements x and y of a group. The group commutator of weight n, $n > 2$, we define inductively by $(x_1, x_2, \ldots, x_n) = ((x_1, x_2, \ldots, x_{n-1}), x_n)$.

For a p-group G we will denote by $R(G)$ the *Roggenkamp parameter*

$$\sum_{i=1,\ldots,t} d(C_G(g_i)),$$

where $g_1, \ldots, g_t$ is a set of representatives of all conjugacy classes of G.

[1] The proof given in the paper appeared to be incorrect, so the MIP remains open.

This parameter was introduced by Roggenkamp and described in [14], [15], where it is proved that $R(G)$ is determined by the modular group algebra. We will also use notation $R_G(S)$ for

$$\sum_{i=1,\dots,s} d(C_G(h_i)),$$

where $h_1, \dots, h_s$ is a set of representatives of the conjugacy classes of G contained in the normal subset S of G.

If G is a finite p-group then by [9] the number of conjugacy classes of maximal elementary abelian subgroups of given rank is determined by FG. By the *Quillen parameter* of G we mean a series $Q(G) = (q_1, q_2, \dots)$, where q_i is the number of conjugacy classes of maximal elementary abelian subgroups of rank i. Since 2-groups of nilpotency coclass $\leqslant 2$ do not contain elementary abelian subgroups of rank bigger than 4, we will write the Quillen parameter in the form (q_1, q_2, q_3, q_4).

In the paper we count the isomorphism class of the centers, the numbers of conjugacy classes, the Quillen parameters and the Roggenkamp parameters of all 2-groups of nilpotency coclass 2. From this follows that the last two parameters determine almost all the groups.

Theorem 1.1. *Let G and H be groups of order 2^n, $n \geqslant 8$, and nilpotency class $n-2$. If $Q(G) = Q(H)$ and $R(G) = R(H)$ then either $G \simeq H$ or $\{G, H\} \subset \{G_9, G_{13}, G_{14}\}$ or $\{G, H\} = \{G_{24}, G_{25}\}$.*

It appears also that the numbers of conjugacy classes does not give more information about differences between these groups and the isomorphism class of the center allows only to exclude G_9 from the first set in the theorem.

Using the fact that every metacyclic p-group G is determined by the modular group algebra FG over the field $F = GF(p)$ ([2], [12]) and some additional arguments one can prove the following theorem.

Theorem 1.2. *Let $F = GF(2)$ be the field of 2 elements and let G be a group of order 2^n, $n \geqslant 8$, and nilpotency class $n-2$. Then $FG \simeq FH \Rightarrow G \simeq H$, provided $\{G, H\} \neq \{G_{24}, G_{25}\}$*

As we were informed by W. Kimmerle for $n = 8$ it was checked using a computer that FG_{24} is not isomorphic to FG_{25}.

2. Presentations of groups of almost maximal class

Following [5], [6] and [8], 2-groups of almost maximal class of order 2^n, $n > 4$, are classified. However, the full list of presentations of these groups by generators and relations was not published. In [5] and [6] some groups were omitted. In [8] there is given the list of pro-2 presentations of pro-2 groups of coclass ≤ 3. Using both approaches first we give the list of all groups of nilpotency coclass 2 and order 2^n, $n > 4$. According to [8] they are divided into five families with numbers 7, 8, 9, 50, 59. We will use the same numbers of the families and denote them by Fam k, where $k \in \{7, 8, 9, 50, 59\}$. For our purposes the presentations derived from the pro-2 presentations of pro-2 groups are more convenient than ones that were given in James' paper.

2.1. 2-groups of almost maximal class with cyclic commutator subgroup

Theorem 2.1 ([5], Theorem 5.1)**.** *There are precisely* 6 *groups of order* 2^n, $n \geqslant 4$, *and class* $n-2$ *with* $G/\gamma_2(G)$ *elementary abelian and* $\gamma_2(G)$ *cyclic. They form* Fam 59 *and are given by the following presentation:*

$$\langle x, y, t : x^{2^{n-2}} = t^2 = 1,\ y^2 = z_1,\ x^y = x^{-1}z_2,\ x^t = xz_3,\ t^y = tz_4 \rangle$$

where the values of z_i, $1 \leqslant i \leqslant 4$, *for particular groups* G_m, $1 \leqslant m \leqslant 6$, *are such as in the following table.*

	G_1	G_2	G_3	G_4	G_5	G_6
z_1	1	1	$x^{2^{n-3}}$	$x^{2^{n-3}}$	1	$x^{2^{n-3}}$
z_2	1	$x^{2^{n-3}}$	1	1	1	1
z_3	1	1	1	1	$x^{2^{n-3}}$	$x^{2^{n-3}}$
z_4	1	1	1	$x^{2^{n-3}}$	1	1

Table 1

Theorem 2.2 ([5], Theorem 5.2)**.** *The number of groups of order* 2^n, $n \geqslant 5$, *and class* $n-2$ *with* $G/\gamma_2(G)$ *of exponent* 4, $\gamma_2(G)$ *cyclic and* $\gamma_1(G)/\gamma_2(G)$ *cyclic is:*

$$\begin{cases} 3, & \text{if } n = 5, \\ 6, & \text{if } n > 5. \end{cases}$$

They form Fam 9 *and are given by the following presentation:*

$$\langle x, y, t : x^{2^{n-2}} = t^2 = 1,\ y^2 = z_1,\ x^y = x^{-1}z_2t,\ x^t = xz_3,\ t^y = tz_4\rangle$$

where the values of z_i, $1 \leqslant i \leqslant 4$, *for particular groups* G_m, $7 \leqslant m \leqslant 12$, *are such as in the following table (for* G_9, G_{11} *and* G_{12} *we have* $n > 5$*).*

	G_7	G_8	G_9	G_{10}	G_{11}	G_{12}
z_1	1	$x^{2^{n-3}}$	1	1	1	$x^{2^{n-3}}$
z_2	1	1	$x^{2^{n-4}}$	1	$x^{2^{n-4}}$	$x^{2^{n-4}}$
z_3	1	1	1	$x^{2^{n-3}}$	$x^{2^{n-3}}$	$x^{2^{n-3}}$
z_4	1	1	$x^{2^{n-3}}$	$x^{2^{n-3}}$	1	1

Table 2

Theorem 2.3 ([5], Theorem 5.3(a))**.** *The number of groups of order* 2^n, $n \geqslant 5$, *and class* $n-2$ *with* $G/\gamma_2(G)$ *of exponent* 4, $\gamma_2(G)$ *cyclic and* $\gamma_1(G)/\gamma_2(G)$ *elementary abelian is:*

$$\begin{cases} 3, & \text{if } n = 5,\\ 4, & \text{if } n > 5. \end{cases}$$

They form Fam 50 *and are given by the following presentation:*

$$\langle x, y : x^{2^{n-2}} = 1,\ y^4 = z_1,\ x^y = x^{-1}z_2\rangle,$$

where the values of z_1 *and* z_2 *are such as in the following table (for* G_{16} *we have* $n > 5$*).*

	G_{13}	G_{14}	G_{15}	G_{16}
z_1	1	1	$x^{2^{n-3}}$	1
z_2	1	$x^{2^{n-3}}$	1	$x^{2^{n-4}}$

Table 3

2.2. 2-groups of almost maximal class with 2-generated commutator subgroup

Theorem 2.4 ([5], [8])**.** *The number of groups of order* 2^n, $n \geqslant 5$, *and class* $n-2$ *with* $G/\gamma_2(G)$ *of exponent* 4, $\gamma_2(G)$ 2-*generated and* $\gamma_1(G)/\gamma_2(G)$

elementary abelian is

$$\begin{cases} 3, & \text{if } n = 5, \\ 4, & \text{if } n = 6, \\ 9, & \text{if } n > 6,\ n \text{ odd}, \\ 10, & \text{if } n > 6,\ n \text{ even}. \end{cases}$$

They form Fam 8.

The description of the groups of Fam 8 by generators and defining relations is more complicated than in the previous families. Let $|G| = 2^n = 2^{2k+2+\epsilon}$, $n \geqslant 6$, $\epsilon \in \{0,1\}$. Each of the groups of Fam 8 can be described as the group

$$\begin{aligned}&\langle x_1, x_2, y : \\ &x_1^{2^{k+\epsilon}} = x_2^{2^k} = 1,\ y^4 = t_1,\ x_1^y = x_1x_2,\ x_2^y = x_1^{-2}x_2^{-1}t_2,\ x_2^{x_1} = x_2t_3\rangle,\end{aligned} \tag{1}$$

where $t_1, t_3 \in \{1, z_1\}$, $t_2 \in \{1, z_1, z_2, z_1z_2\}$, $\epsilon \in \{0,1\}$ and:
if $\epsilon = 0$, then $z_1 = x_2^{2^{k-1}}$, $z_2 = x_1^{2^{k-1}}$;
if $\epsilon = 1$, then $z_1 = x_1^{2^k}$, $z_2 = x_2^{2^{k-1}}$.
The values of t_i $(i = 1,2,3)$, for the groups G_m, $18 \leqslant m \leqslant 27$, are given in the following table. Note that for $\epsilon = 1$ the groups G_{24} and G_{25} are isomorphic.

	G_{18}	G_{19}	G_{20}	G_{21}	G_{22}	G_{23}	G_{24}	G_{25}	G_{26}	G_{27}
t_1	1	z_1	z_1	1	z_1	1	z_1	1	1	z_1
t_2	1	1	z_1	1	1	z_1	z_2	z_1z_2	z_2	z_1z_2
t_3	1	1	1	z_1	z_1	1	1	1	z_1	z_1

Table 4

For $n = 5$ we have only the groups G_{18}, G_{19} and

$$\begin{aligned}G_{17} = \langle x_1, x_2, y \mid\ & \\ & x_1^8 = x_2^4 = 1, y^4 = x_1^4, x_1^y = y^2x_1x_2, x_2^y = x_1^{-2}, x_2^{x_1} = x_2^{-1}x_1^2x_2\rangle;\end{aligned}$$

for $n = 6$ we have only the groups G_{18}, G_{19}, G_{20} and G_{23}.

Accordingly to the definitions from [8], G_{18} is the mainline group of this family with the immediate descendants $G_{19} - G_{22}$ (which are terminal) and G_{23} (which have 4 immediate terminal descendants $G_{24} - G_{27}$ if $\epsilon = 0$, and 3 immediate terminal descendants $G_{24} - G_{27}$ with $G_{24} \cong G_{25}$, if $\epsilon = 1$).

2.3. 2-groups of almost maximal class with 3-generated commutator subgroup

Theorem 2.5 ([5], [8]). *The number of groups of order 2^n, $n \geq 6$, and class $n-2$ with $G/\gamma_2(G)$ of exponent 4, $\gamma_2(G)$ 3-generated and $\gamma_1(G)/\gamma_2(G)$ elementary abelian is*

$$\begin{cases} 2, & \text{if } n = 6, \\ 4, & \text{if } n > 6, \ n \text{ odd}, \\ 12, & \text{if } n > 6, \ n \text{ even}. \end{cases}$$

They form Fam 7.

The groups of Fam 7 have the most complicated description by generators and defining relations among all 2-groups of coclass 2.

Assume first that $|G| = 2^n = 2^{2k+2}$, $n \geqslant 6$, that is $G = G_m$, where $28 \leqslant m \leqslant 39$. Each of these groups can be described as the group

$$\begin{aligned} &\langle x_1, x_2, y : x_1^{2^{k+1}} = 1, \\ &x_2^{2^{k-1}} = x_1^{2^k},\ y^4 = t_1,\ x_1^y = y^2 x_1 x_2 t_2,\ x_2^y = x_1^{-2} t_3,\ x_2^{x_1} = x_2^{-1} t_4 \rangle, \end{aligned} \tag{2}$$

where $t_1, t_2, t_3 \in \{1, z_1\}$, $t_4 \in \{1, z_1, z_2\}$, $z_1 = x_1^{2^k} = x_2^{2^{k-1}}$, $z_2 = x_1^{2^{k-1}} x_2^{2^{k-2}}$. The values of t_i $(i = 1, 2, 3, 4)$ for particular groups are given in the following table:

	G_{28}	G_{29}	G_{30}	G_{31}	G_{32}	G_{33}	G_{34}	G_{35}	G_{36}	G_{37}	G_{38}	G_{39}
t_1	1	1	z_1	z_1	1	1	z_1	z_1	1	1	z_1	z_1
t_2	1	z_1	1	z_1	z_1	1	z_1	1	1	1	z_1	1
t_3	1	1	1	1	1	1	1	1	1	z_1	1	1
t_4	1	1	z_1	z_1	z_1	z_1	1	1	z_2	z_2	z_2	z_2

Table 5

Note that for $n = 6$ we have only groups G_{28} and G_{29}.

For further calculations we derive additional helpful relations. For $28 \leqslant m \leqslant 35$ in a group $G = G_m$ we have

$$x_1^{y^2} = x_1^{-1} x_2 t_1,\ (x_1^2)^{y^2} = x_1^{-2} t_4,\ (x_1^2)^y = x_2 t_4,\ x_2^{y^2} = x_2^{-1} t_4. \tag{3}$$

If $36 \leqslant m \leqslant 39$ then

$$x_1^{y^2} = x_1^{-1} x_2 t_1 t_3 z_1,\ (x_1^2)^{y^2} = x_1^{-2} z_1 z_2,\ (x_1^2)^y = x_2 t_3 z_2,\ x_2^{y^2} = x_2^{-1} z_2. \tag{4}$$

If $|G| = 2^n = 2^{2k+3}$, $n > 6$, that is $G \in \{G_{40}, G_{41}, G_{42}, G_{43}\}$, the groups of Fam 7 can be described in a little simpler way:

$$\begin{aligned}&\langle x_1, x_2, y : \\ &\quad x_1^{2^{k+1}} = x_2^{2^k} = 1,\ y^4 = t_1,\ x_1^y = y^2 x_1 x_2,\ x_2^y = x_1^{-2},\ x_2^{x_1} = x_2^{-1} t_4\rangle\end{aligned} \tag{5}$$

where $t_1, t_4 \in \{1, z\}$, $z = x_1^{2^k} x_2^{2^{k-1}}$. The values of t_1 and t_4 for particular groups are given in the following table:

	G_{40}	G_{41}	G_{42}	G_{43}
t_1	1	z	1	z
t_4	1	z	z	1

Table 6

The relations (3) are valid also for these groups.

The group G_{40} is the mainline group with the immediate descendant G_{28} which is mainline and seven immediate descendants G_{29}–G_{35} which are terminal. The groups G_{41} and G_{43} are terminal, G_{42} is capable with four immediate descendants G_{36}–G_{39} which in turn are terminal. All the groups G_{40}–G_{43} are immediate descendants of the mainline group G_{28}.

3. Groups with cyclic commutator subgroup

In this section we describe properties of the groups defined in Theorems 2.1, 2.2 and 2.3. All these groups are extensions of a noncyclic group of class at most two with a cyclic maximal subgroup by the group of order 2.

Let $G = G_m$, $1 \leqslant m \leqslant 16$. We let A be the subgroup of G generated by the elements x and t (for $m \in \{13, 14, 15, 16\}$ we put $t = y^2$). The following lemma is an easy observation obtained by a straightforward computation from the presentations of groups. We assume that all the groups G_m have order p^n, where $n \geqslant 5$.

Lemma 3.1. *Let* $G = G_m \in \mathrm{Fam}\,9 \cup \mathrm{Fam}\,50 \cup \mathrm{Fam}\,59$. *Then:*

(a) $\gamma_2(G) = \begin{cases} \langle x^2\rangle & \textit{if } G \in \mathrm{Fam}\,50 \cup \mathrm{Fam}\,59 \\ \langle x^2 t\rangle & \textit{if } G \in \mathrm{Fam}\,9 \end{cases}$

(b) *If* $i \geqslant 3$ *then* $\gamma_i(G) = \langle x^{2^{i-1}} \rangle$;

(c) $$\zeta(G_m) = \begin{cases} \langle x^{2^{n-3}}, t\rangle & \text{for } m \in \{1,2,3,7,8,13,14\},\\ \langle x^{2^{n-4}} t\rangle & \text{for } m \in \{4,9\},\\ \langle y^2\rangle & \text{for } m = 15,\\ \langle x^{2^{n-3}}\rangle & \text{for } m \in \{5,6,10,11,12,16\}; \end{cases}$$

(d) $|G : A| = 2$ *and* $\Omega_1(A) = \langle x^{2^{n-3}}, t\rangle$ *is an elementary abelian normal subgroup of* G *of order* 4*;*

(e) *If* A *is nonabelian then* $\gamma_2(A) = \langle x^{2^{n-3}}\rangle$, $\zeta(A) = \langle x^2\rangle$, *in particular the nilpotency class of* A *is not greater than 2.*

For counting the Roggenkamp and Quillen parameters of our groups we need information about the conjugacy classes of elements of order 2 lying outside A.

Lemma 3.2. *Let* $G = G_m \in \mathrm{Fam}\,9 \cup \mathrm{Fam}\,50 \cup \mathrm{Fam}\,59$.

(a) *The set* $G \setminus A$ *splits into the following four conjugacy classes:* C_y, C_{yx}, C_{yt}, C_{yxt}.

(b) *The orders of the elements* y, yx, yt, yxt *are given in the following tables.*

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9	G_{10}	G_{11}	G_{12}
y	2	2	4	4	2	4	2	4	2	2	2	4
yx	2	4	4	4	2	4	4	4	8	4	8	8
yt	2	2	4	2	2	4	2	4	4	4	2	4
yxt	2	4	4	2	4	2	4	4	8	4	8	8

Table 7

	G_{13}	G_{14}	G_{15}	G_{16}
y	4	4	8	4
yx	4	4	8	8
$yt = y^{-1}$	4	4	8	4
$yxt = y^{-1}x$	4	4	8	8

Table 8

Proof. (a) The unique subgroup of order 4 contained in $\langle x\rangle$ is equal to $\langle x^{2^{n-4}}\rangle$. Since $(x^{2^{n-4}})^y \neq x^{2^{n-4}}$ and $(x^{2^{n-3}})^y = x^{2^{n-3}}$, we have $C_{\langle x\rangle}(y) = \langle x^{2^{n-3}}\rangle$, that is $|C_{\langle x\rangle}(y)|=2$. But $\langle x^{2^{n-4}}\rangle \in \zeta(A)$, so for all $g \in \{y, yx, yt, yxt\}$ we have $|C_{\langle x\rangle}(g)| = 2$ and then $|C_{\langle x\rangle}(g)| \leqslant 8$, as $|G : \langle x\rangle| = 4$. Hence $|C_g| \geqslant 2^{n-3}$. On the other hand the size of the conjugacy class C_g does not exceed the size of the commutator subgroup because $C_g \subseteq gG'$. In our groups we have $|G'| = 2^{n-3}$. Consequently $|C_g| = 2^{n-3}$ and $C_g = gG'$. Finally, since the elements y, yx, yt, yxt lie in different cosets of G by G', we obtain $G \setminus A = yG' \cup yxG' \cup ytG' \cup yxtG' = C_y \cup C_{yx} \cup C_{yt} \cup C_{yxt}$.

(b) By the defining relations (Theorem 2.1) for the groups $G_m \in$ Fam 59 we have $y^2 = z_1$, $(yx)^2 = y^2x^yx = z_1z_2$, $(yt)^2 = y^2t^yt = z_1z_4$ and $(yxt)^2 = y^2x^yt^yxt = z_1z_2z_3z_4$. Using the values of z_i, $1 \leqslant i \leqslant 4$, we obtain the orders given in the table.

Similarly, for the groups $G_m \in$ Fam 9 we have $y^2 = z_1$, $(yx)^2 = y^2x^yx = z_1z_2z_3t$, $(yt)^2 = y^2t^yt = z_1z_4$ and $(yxt)^2 = y^2x^yt^yxt = z_1z_2z_4t$ and again using the values of z_i, $1 \leqslant i \leqslant 4$, we obtain the orders given in the table.

Finally, for the groups of Fam 50 we have $y^4 = (y^{-1})^4 = z_1$ and $(y^x)^4 = z_1z_2^2 = (y^{-1}x)^4$. The orders of the considered elements are easily seen from the defining relations of the groups of Fam 50. □

Proposition 3.1. *Let* $G = G_m \in$ Fam 9 $\cup$ Fam 50 $\cup$ Fam 59.

(a) *If A is abelian, that is* $m \in \{1, 2, 3, 4, 7, 8, 9, 13, 14, 15\}$, *then*

$$|Cl(G)| = 2^{n-2} + 6.$$

(b) *If A is nonabelian, that is* $m \in \{5, 6, 10, 11, 12, 16\}$, *then*

$$|Cl(G)| = 5 \cdot 2^{n-5} + 6.$$

Proof. (a) It follows immediately from the defining relations and Lemma 3.1 that A is abelian if and only if $|\zeta(G)| = 4$. Now, if A is abelian and $a \in A \setminus \zeta(G)$, then $C_G(a) = A$ that is $|C_a| = 2$. Thus $|Cl_G(A\setminus\zeta(G))| = \frac{|A|-|\zeta(G)|}{2} = \frac{2^{n-1}-2^2}{2} = 2^{n-2} - 2$. Now the four conjugacy classes contained in $G\setminus A$, the four 1-element classes and the classes counted above give the formula.

(b) For nonabelian A we have $\zeta(A) = \langle x^2\rangle$. The elements of $\zeta(A) \setminus \zeta(G)$ have centralizers equal to A. So $|Cl_G(\langle x^2\rangle)| = \frac{|\zeta(A)|-|\zeta(G)|}{2} + 2 =$

$\frac{2^{n-3}-2}{2}+2=2^{n-4}+1$. The subgroup $\langle x^2,t\rangle$ has index 2 in A. The set $t\langle x^2\rangle=\langle x^2,t\rangle\setminus\langle x^2\rangle$ contains exactly two elements of order 2 (they form a conjugacy class of the element t) and exactly two elements of order 4 (they form a conjugacy class of the element $tx^{2^{n-4}}$). All other elements of this set form $\frac{2^{n-3}-4}{4}=2^{n-5}-1$ four-element conjugacy classes. Every conjugacy class contained in $A\setminus\langle x^2,t\rangle$ has 4 elements. Therefore we obtain $|Cl(G)|=|Cl_G(G\setminus A)|+|Cl_G(A\setminus\langle x^2,t\rangle)|+|Cl_G(\langle x^2,t\rangle\setminus\langle x^2\rangle)|+|Cl_G(\langle x^2\rangle)|+|\zeta(G)|=4+2^{n-4}+(2^{n-5}+1)+(2^{n-4}-1)+2=2^{n-3}+2^{n-5}+6$. □

Proposition 3.2. *Let* $G=G_m\in \mathrm{Fam}\,9\cup\mathrm{Fam}\,50\cup\mathrm{Fam}\,59$.
a) *If A is abelian, that is $m\in\{1,2,3,4,7,8,9,13,14,15\}$, then*

$$R(G)=2^{n-1}+r_m,$$

where the values of r_m are given in the following table.

	G_1	G_2	G_3	G_4	G_7	G_8	G_9	G_{13}	G_{14}	G_{15}
r_m	20	18	16	16	14	12	10	12	12	8

Table 9

(b) *If A is nonabelian, that is $m\in\{5,6,10,11,12,16\}$, then*

$$R(G)=2^{n-2}+r_m,$$

where the values of r_m are given in the following table.

	G_5	G_6	G_{10}	G_{11}	G_{12}	G_{16}
r_m	18	16	13	13	11	10

Table 10

PROOF. (a) It is seen from Lemma 3.1 that A is abelian if and only if $|\zeta(G)|=4$. But $\zeta(G)\leqslant A$, so for $a\in A\setminus\zeta(G)$ we have $C_G(a)=A$, which is 2-generated. For $a\in\zeta(G)$ we have obviously $C_G(a)=G$. Therefore by the proof of Lemma 3.1(a) we obtain

$$\begin{aligned} R_G(A)&=2\cdot|Cl_G(A\setminus\zeta(G))|+4\cdot d(G)\\ &=2^{n-1}+\begin{cases}8 & \text{if } m\in\{1,2,3,4\}\\ 4 & \text{if } m\in\{7,8,9,13,14,15\}.\end{cases}\end{aligned}\tag{6}$$

As it was already noted in the proof of Proposition 3.2 for $g \in G \setminus A$ we have $|C_g| = 2^{n-3}$, so $|C_G(g)| = 2^3$ and $C_G(g) = \langle g, \zeta(G)\rangle$. If g has order 2 and $\zeta(G)$ is elementary abelian, then $d(C_G(g)) = 3$. If $\zeta(G)$ is cyclic of order 4 (i.e. $m \in \{4, 9\}$) or g has order 4, then obviously $d(C_G(g)) = 2$. Finally $d(C_G(g)) = 1$, when g has order 8. Now using the formula (6) and the information from the tables of Proposition 3.2 we get the assertion.

(b) It is obvious that for $a \in \zeta(A) \setminus \zeta(G)$ we have $C_G(a) = A$ that is $d(C_G(a)) = 2$. Hence

$$R_G(\langle x^2\rangle) = 2 \cdot \frac{2^{n-3}-2}{2} + 2 \cdot d(G) = 2^{n-3} + \begin{cases} 4 & \text{if } m \in \{5, 6\} \\ 2 & \text{if } m \in \{10, 11, 12, 16\}. \end{cases}$$

If the conjugacy class of an element $a \in t\langle x^2\rangle$ has four elements, then obviously its centralizer is equal to $\langle x^2, a\rangle = \langle x^2, t\rangle$, which is 2-generated. For the representatives t and $x^{2^{n-4}}t$ of the 2-element classes (see the proof of Proposition 3.1b) we have

$$C_G(t) = \begin{cases} \langle x^2, y, t\rangle & \text{if } m \in \{5, 6, 11, 12\}, \\ \langle x^2, xy\rangle & \text{if } m \in \{10, 16\} \end{cases}$$

and

$$C_G(tx^{2^{n-4}}) = \begin{cases} \langle x^2, y, t\rangle & \text{if } m \in \{5, 6, 10\}, \\ \langle x^2, xy\rangle & \text{if } m \in \{11, 12, 16\}. \end{cases}$$

Thus

$$R_G(t\langle x^2\rangle) = 2 \cdot \frac{2^{n-3}-4}{4} + \begin{cases} 6 & \text{if } m \in \{5, 6\} \\ 5 & \text{if } m \in \{10, 11, 12\} \\ 4 & \text{if } m = 16. \end{cases}$$

It is clear that all the elements $a \in A \setminus \langle x^2, t\rangle$ have order 2^{n-2} and by Proposition 3.1b we have $|C_a| = 4$. Thus $C_G(a) = \langle a\rangle$ and, consequently, $R_G(A \setminus \langle x^2, t\rangle) = |Cl_G(A \setminus \langle x^2, t\rangle)| = 2^{n-4}$. Finally,

$$R(G) = R_G(G \setminus A) + 2^{n-2} + \begin{cases} 8 & \text{if } m \in \{5, 6\}, \\ 5 & \text{if } m \in \{10, 11, 12\} \\ 4 & \text{if } m = 16, \end{cases}$$

so after counting the centralizers of the elements $g \in \{y, yx, yt, yxt\}$ in a similar way as in the proof of part (a) we get the assertion. $\square$

We finish this section with counting the Quillen parameters. It was noted in Lemma 3.1 that $\Omega_1(A)$ is an elementary abelian normal subgroup of G of order 4. Hence, if there are no elements of order 2 outside A, $\Omega_1(A)$ is the unique maximal elementary abelian subgroup of G, i.e. the Quillen parameter of G is equal to $(0, 1, 0, 0)$. Since $|G : A| = 2$, there is not an elementary abelian subgroup of order 4 with trivial intersection with A. Hence, in all other cases the parameter has type $(0, q_2, q_3, 0)$. The values of q_2 and q_3 can be easily counted by studying the centralizers of the elements of order 2 lying outside A and the intersections of these centralizers with $\Omega_1(A)$. We leave the details to the reader.

Proposition 3.3. *If* $G \in \mathrm{Fam}\,9 \cup \mathrm{Fam}\,50 \cup \mathrm{Fam}\,59$ *then the Quillen parameters are given in Table 11.*

4. Groups with 2-generated commutator subgroup

In this section we describe the groups defined in Theorem 2.4. All these groups are extensions of a subgroup of nilpotency class $\leqslant 2$ by the cyclic group of order 4.

Let $G = G_m$, $18 \leqslant m \leqslant 27$. In this section we let A be the subgroup of G generated by the elements x_1 and x_2. The following lemmas are easy observations obtained by a straightforward computation from the presentations of groups. We assume that all the groups G_m have order p^n, where $n \geqslant 7$.

Lemma 4.1. *Let* $G \in \mathrm{Fam}\,8$. *Then* $\gamma_2(G) = \langle x_1^2, x_2\rangle$, $\gamma_3(G) = \langle x_1^2, x_2^2\rangle$, *and in general,* $\gamma_{2i}(G) = \langle x_1^{2^i}, x_2^{2^{i-1}}\rangle$, $\gamma_{2i+1}(G) = \langle x_1^{2^i}, x_2^{2^i}\rangle$ *for* $i \geqslant 1$.

Lemma 4.2. *Let* $G \in \mathrm{Fam}\,8$. *Then:*

(a) *If* $G = G_m$, *where* $m \in \{21, 22, 26, 27\}$, *then* $A' = \{z_1\}$ *and* $\zeta(A) = \gamma_3(G)$; *in all other cases* A *is abelian.*

(b) *If* $g \in \{y, yx_1, y^{-1}, y^{-1}x_1, y^2x_1\}$, *then* $(G, g) = \gamma_2(G)$; *if* $g \in \{y^2, y^2x_2\}$, *then* $(G, g) = \gamma_3(G)$.

Group	Isomorphism type of $\zeta(G)$	Quillen parameter	Representatives of the conjugacy classes of maximal elementary abelian subgroups
G_1	$C_2 \times C_2$	$(0,0,2,0)$	$\langle x^{2^{n-3}}, y, t\rangle$, $\langle x^{2^{n-3}}, yx, t\rangle$
G_2	$C_2 \times C_2$	$(0,0,1,0)$	$\langle x^{2^{n-3}}, y, t\rangle$
G_3	$C_2 \times C_2$	$(0,1,0,0)$	$\langle x^{2^{n-3}}, t\rangle$
G_4	C_4	$(0,3,0,0)$	$\langle x^{2^{n-3}}, y\rangle, \langle x^{2^{n-3}}, yx\rangle, \langle x^{2^{n-3}}, x^{2^{n-4}}t\rangle$
G_5	C_2	$(0,1,1,0)$	$\langle x^{2^{n-3}}, yx\rangle$, $\langle x^{2^{n-3}}, y, t\rangle$
G_6	C_2	$(0,2,0,0)$	$\langle x^{2^{n-3}}, t\rangle$, $\langle x^{2^{n-3}}, xyt\rangle$
G_7	$C_2 \times C_2$	$(0,0,1,0)$	$\langle x^{2^{n-3}}, y, t\rangle$
G_8	$C_2 \times C_2$	$(0,1,0,0)$	$\langle x^{2^{n-3}}, t\rangle$
G_9	C_4	$(0,2,0,0)$	$\langle x^{2^{n-3}}, t\rangle$, $\langle x^{2^{n-3}}, y\rangle$
G_{10}	C_2	$(0,2,0,0)$	$\langle x^{2^{n-3}}, t\rangle$, $\langle x^{2^{n-3}}, y\rangle$
G_{11}	C_2	$(0,0,1,0)$	$\langle x^{2^{n-3}}, y, t\rangle$
G_{12}	C_2	$(0,1,0,0)$	$\langle x^{2^{n-3}}, t\rangle$
G_{13}	$C_2 \times C_2$	$(0,1,0,0)$	$\langle x^{2^{n-3}}, y^2\rangle$,
G_{14}	$C_2 \times C_2$	$(0,1,0,0)$	$\langle x^{2^{n-3}}, y^2\rangle$,
G_{15}	C_4	$(0,1,0,0)$	$\langle x^{2^{n-3}}, y^2x^{2^{n-4}}\rangle$
G_{16}	C_2	$(0,1,0,0)$	$\langle x^{2^{n-3}}, y^2\rangle$

Table 11

(c) $\Omega_1(A)$ *is an elementary abelian subgroup of order* 4 *and* $\Omega_1(A) \leqslant \zeta(\langle A, y^2\rangle)$.

Lemma 4.3. *The set $G \setminus A$ splits into 7 conjugacy classes, which are $y\gamma_2(G)$, $yx_1\gamma_2(G)$, ${}^{-1}\gamma_2(G)$, $y^{-1}x_1\gamma_2(G)$, $y^2x_1\gamma_2(G)$, $y^2\gamma_3(G)$, $y^2x_2\gamma_3(G)$. Moreover, first four classes do not contain elements of order 2.*

Proof. It is obvious that A is a normal subgroup of index 4 with the cyclic factor group $G/A = \langle yA\rangle$. We have also by Lemma 4.1 that $A = x_1\gamma_2(G) \cup x_2\gamma_3(G) \cup \gamma_3(G)$. Hence

$$\begin{aligned} G \setminus A &= yA \cup y^{-1}A \cup y^2A \\ &= y\gamma_2(G) \cup yx_1\gamma_2(G) \cup y^{-1}\gamma_2(G) \cup y^{-1}x_1\gamma_2(G) \end{aligned}$$

$$\cup\, y^2x_1\gamma_2(G) \cup y^2\gamma_3(G) \cup y^2x_2\gamma_3(G).$$

By Lemma 4.2(b) each of the sets from the last two lines is a conjugacy class. Since yA and $y^{-1}A$ are elements of order 4 in G/A, none of these cosets contains elements of order 2 and the lemma is proved. □

Lemma 4.4. *Let $G \in \mathrm{Fam}\,8$, and let C be one of the conjugacy classes $y^2x_1\gamma_2(G)$, $y^2\gamma_3(G)$, $y^2x_2\gamma_3(G)$. If C consists of elements of order 2, then the family $\Theta = \{\langle g, x_1^{2^{k+\epsilon-1}}, x_2^{2^{k-1}}\rangle : g \in C\}$ is a conjugacy class of maximal elementary abelian subgroups of G. Moreover $|\Theta| = |C|/4$.*

Proof. Notice first that any two elements from different conjugacy classes listed in the lemma are not commuting. So they cannot lie together in an abelian subgroup. Further, if C is one of these classes and $g \in C$, then as it follows from Lemma 4.2(c), $\langle g, \Omega_1(G)\rangle$ is abelian. So by the above, if g has order 2, it is maximal elementary abelian. It is clear that one such subgroup contains 4 elements belonging to C and different subgroups determine disjoint such four-element subsets. Thus $|\Theta| = \frac{|C|}{4}$. □

Corollary 4.1. *If $G \in \mathrm{Fam}\,8$ and $|G| \geq 2^6$, then maximal elementary abelian subgroups of G have order not greater than 2^3. If $|G| > 2^6$, and B is a maximal elementary abelian subgroup of G of order 2^3, then B is not normal in G.*

The proof of the following lemma needs standard and not difficult calculations.

Lemma 4.5. *Let $G \in \mathrm{Fam}\,8$. Then:*

(a) *for $g \in \zeta(G)$, $C_G(g) = G$;*

(b) *for $g \in \langle z_1, z_2\rangle \setminus \zeta(G)$, $C_G(g) = \langle y^2, x_1, x_2\rangle$;*

(c) *for $g \in \gamma_3(G) \setminus \langle z_1, z_2\rangle$, $C_G(g) = A$;*

(d) *for $g \in A \setminus \gamma_3(G)$,* $C_G(g) = \begin{cases} A, & \text{if } A \text{ is abelian},\\ \langle g, \gamma_3(G)\rangle, & \text{if } A \text{ is nonabelian};\end{cases}$

(e) *for $g \in y^2x_1\gamma_2(G)$, $C_G(g) = \langle g, z_1, z_2\rangle$;*

(f) *for $g \in y^2\gamma_3(G) \cup y^2x_2\gamma_3(G)$, $C_G(g) = \langle h, z_1, z_2\rangle$, where $h^2 = g$;*

(g) *for $g \notin \langle y^2, A\rangle$, $C_G(g) = \langle g, \zeta(G)\rangle$.*

Proposition 4.1. *Let $G \in \mathrm{Fam}\,8$.*

(a) *If A is abelian, that is $m \in \{18, 19, 20, 23, 24, 25\}$, then*

$$|Cl(G)| = 9 + 2^{2k+\epsilon-2}.$$

(b) *If A is nonabelian, that is $m \in \{21, 22, 26, 27\}$, then*

$$|Cl(G)| = 9 + 5 \cdot 2^{2k+\epsilon-5}.$$

Proof. Assume first that A is abelian. We split A into the set theoretic sum of 3 disjoint subsets: $A = \zeta(G) \cup (\langle z_1, z_2\rangle \setminus \zeta(G)) \cup (A \setminus \langle z_1, z_2\rangle)$. In $\zeta(G)$ we have two one-element classes, the set $\langle z_1, z_2\rangle \setminus \zeta(G)$ forms one two-element class and finally in the last subset we have $\frac{|A|-4}{4} = 2^{2k-2+\epsilon} - 1$ four-element classes. All this classes together with the seven classes contained in $G \setminus A$ give $9 + 2^{2k+\epsilon-2}$ conjugacy classes.

Now let A be nonabelian and let us split it into the set theoretic sum of 4 disjoint sets $A = \zeta(G) \cup (\langle z_1, z_2\rangle \setminus \zeta(G)) \cup (\gamma_3(G) \setminus \langle z_1, z_2\rangle) \cup (A \setminus \gamma_3(G))$. As in the previous case, conjugacy classes contained in the three first sets have respectively 1, 2 and 4 elements. Each conjugacy class contained in $A \setminus \gamma_3(G)$ has 8 elements, by Lemma 4.5(d). Therefore $|Cl(G)| = 2 + 1 + \frac{2^{2k+\epsilon-2}-4}{4} + \frac{2^{2k+\epsilon}-2^{2k+\epsilon-2}}{8} + 7 = 9 + 5 \cdot 2^{2k+\epsilon-5}$. $\square$

Lemma 4.6. *Let $G \in \mathrm{Fam}\,8$. The conjugacy classes of G not contained in A have elements of order as listed in Table 12.*

Groups	G_{18}	G_{19}	G_{20}	G_{21}	G_{22}	G_{23}	G_{24}	G_{25}	G_{26}	G_{27}
A abelian	+	+	+	−	−	+	+	+	−	−
Representatives of conjugacy classes										
y, y^{-1}	4	8	8	4	8	4	8	4	4	8
yx_1, $y^{-1}x_1$	4	8	8	8	4	4	4	8	4	8
y^2x_1	2	4	4	2	4	2	4	2	2	4
y^2	2	4	4	4	2	2	2	4	2	4
y^2x_2	2	4	2	2	4	4	4	4	4	4

Table 12

Proof. The order of y, y^{-1} and y^2 can be easily fixed by Table 4. For yx_1 and $y^{-1}x_1$ we have $(yx_1)^2 = y^2x_1^2x_2^{x_1} \in y^2x_2\gamma_3(G)$ and similarly

$(y^{-1}x_1)^2 \in y^2x_2\gamma_3(G)$. Further $(y^2x_2)^2 = t_1t_2t_2^yt_3$. If $t_2 = z_1$, then obviously $t_2^y = t_2$. If $t_2 = z_2$, then $t_2^y = z_1z_2$. Now it is an easy task to fill the table. $\square$

Lemma 4.7. *Let $G \in \mathrm{Fam}\,8$.*

(a) *If A is abelian, that is $m \in \{18, 19, 20, 23, 24, 25\}$, then*

$$R_G(A) = 5 + 2^{2k+\epsilon-1}.$$

(b) *If A is nonabelian, that is $m \in \{21, 22, 26, 27\}$, then*

$$R_G(A) = 5 + 5 \cdot 2^{2k+\epsilon-4}.$$

Proof. The only conjugacy class contained in A whose representatives have 3-generated centralizers is $C_{z_2} = \{z_2, z_1z_2\}$. Representatives of all other conjugacy classes contained in A have 2-generated centralizers. $\square$

Now we are ready to count the Roggenkamp and the Quillen parameters of all the groups of Fam 8.

Proposition 4.2. *Let $G = G_m \in \mathrm{Fam}\,8$, $|G| > 2^6$. Then*

$$R(G) = \begin{cases} 2^{2k+\epsilon-1} + r_m, & \text{if } A \text{ is abelian, that is } m \in \{18, 19, 20, 23, 24, 25\}, \\ 5 \cdot 2^{2k+\epsilon-4} + r_m, & \text{if } A \text{ is nonabelian, that is } m \in \{21, 22, 26, 27\}, \end{cases}$$

where the values of r_m are given in the following table.

	G_{18}	G_{19}	G_{20}	G_{23}	G_{24}	G_{25}	G_{21}	G_{22}	G_{26}	G_{27}
r_m	20	15	16	19	17	17	18	17	19	15

Table 13

Proof. By Lemma 4.7 in order to find the Roggenkamp parameters we need to count minimal numbers of generators of the centralizers of the elements listed in Table 12. The elements $y, y^{-1}, yx_1, y^{-1}x_1$ have centralizers of order 8, so if any of these elements has order 4, then it must have a 2-generated centralizer. If it has order 8, then its centralizer is cyclic. The elements y^2, y^2x_2 are 2-powers and have centralizers of order 16 which are 2-generated. The centralizer of y^2x_1 has 8 elements and depending on its order it is 3-generated when its order is equal 2 and 2-generated, when its order is equal 4. $\square$

Proposition 4.3. *If $G = G_m \in \mathrm{Fam}\,8$, $|G| > 2^6$, then the Quillen parameter of G is such as it is listed in the following table.*

Group	Quillen parameter	The representatives of the conjugacy classes of elementary abelian maximal subgroup
G_{18}	$(0,0,3,0)$	$\langle y^2x_1, \Omega_1(A)\rangle$, $\langle y^2, \Omega_1(A)\rangle$, $\langle y^2x_2, \Omega_1(A)\rangle$
G_{19}	$(0,1,0,0)$	$\Omega_1(A)$
G_{20}	$(0,0,1,0)$	$\langle y^2x_2, \Omega_1(A)\rangle$
G_{21}	$(0,0,2,0)$	$\langle y^2x_1, \Omega_1(A)\rangle$, $\langle y^2x_2, \Omega_1(A)\rangle$
G_{22}	$(0,0,1,0)$	$\langle y^2, \Omega_1(A)\rangle$
G_{23}	$(0,0,2,0)$	$\langle y^2x_1, \Omega_1(A)\rangle$, $\langle y^2, \Omega_1(A)\rangle$
G_{24}	$(0,0,1,0)$	$\langle y^2, \Omega_1(A)\rangle$
G_{25}	$(0,0,1,0)$	$\langle y^2x_1, \Omega_1(A)\rangle$
G_{26}	$(0,0,2,0)$	$\langle y^2x_1, \Omega_1(A)\rangle$, $\langle y^2, \Omega_1(A)\rangle$
G_{27}	$(0,1,0,0)$	$\Omega_1(A)$

Table 14

Proof. It was mentioned earlier that if the elements y^2, y^2x_1, y^2x_2 have order 4 then $\Omega_1(A)$ is the unique maximal elementary abelian subgroup of G and then $(0,1,0,0)$ is the Quillen parameter of G. If among these elements there are i elements of order 2, $i \in \{1,2,3\}$, then the Quillen parameter of G has the form $(0,0,i,0)$. So using the information from Table 12 one can easily fill Table 13. □

5. Groups with 3-generated commutator subgroup

In this section we describe the groups defined in Theorem 2.5.

Let $G = G_m$, $28 \leqslant m \leqslant 43$. In this section we let A be the subgroup of G generated by the elements x_1^2 and x_2. This subgroup will play a key role in our computations.

The following lemmas are easy observations obtained by a straightforward computation from the presentations of the groups. We assume that all the groups G_m have order p^n, where $n \geqslant 6$.

Lemma 5.1. *Let $G \in \mathrm{Fam}\,7$. Then $\gamma_2(G) = \langle y^2x_1^2, x_1^2x_2, x_2^2\rangle$, $\gamma_3(G) = \langle x_1^2x_2, x_2^2\rangle$, and, in general for $m \geqslant 2$, $\gamma_{2m}(G) = \langle x_1^{2^m}, x_2^{2^{m-1}}\rangle$, $\gamma_{2m+1}(G) = \langle x_1^{2^m}x_2^{2^{m-1}}, x_2^{2^m}\rangle$.*

Lemma 5.2. *Let $G \in \mathrm{Fam}\,7$. Then:*

(a) $\gamma_3(G) \le A$ *and* $|A : \gamma_3(G)| = 2$;

(b) *A is normal in G and the factor group $\overline{G} = G/A$ is isomorphic to the dihedral group of order* 8;

(c) *If $G = G_m$, with $36 \le m \le 39$, then $[A, A] = \langle z_1 \rangle$; in all other cases A is abelian;*

(d) *The subgroup $H = \langle y^2, A\rangle$ is the unique normal subgroup of G of index 4 containing A;*

(e) *There exist exactly 4 non-normal subgroups of G of index 4 containing A: $H_1 = \langle x_1, A\rangle$, $H_2 = \langle y^2x_1, A\rangle$, $H_3 = \langle yx_1^{-1}, A\rangle$, $H_4 = \langle y^3x_1, A\rangle$. Moreover, $H_1^y = H_2$ and $H_3^{x_1} = H_4$.*

Let $H = \Phi(G) = \langle y^2, A\rangle$. Since $d(G) = 2$, G has exactly three maximal subgroups. These are $M_1 = \langle y, H\rangle$, $M_2 = \langle x_1, H\rangle$ and $M_3 = \langle yx_1, H\rangle$. It can be easily seen that $d(M_1) = d(M_2) = 2$ and $d(M_3) = 3$. In further considerations we will use the splitting of G into the following set-theoretic sum of pairwise disjoint subsets:

$$G = A \cup (H \setminus A) \cup (M_1 \setminus H) \cup (M_2 \setminus H) \cup (M_3 \setminus H).$$

Lemma 5.3. *Let $G \in \mathrm{Fam}\,7$. Then*

(a) $Cl_G(H \setminus A) = \{y^2\gamma_3(G), y^2x_1^{-2}\gamma_3(G)\}$,

(b) $Cl_G(M_1 \setminus H) = \{y\gamma_2(G),\ y^3\gamma_2(G)\}$.

Proof. (a) Since $H \setminus A = y^2A$ and $A = \gamma_3(G) \cup x_1^{-2}\gamma_3(G)$, we have $H \setminus A = y^2\gamma_3(G) \cup y^2x_1^{-2}\gamma_3(G)$. Now the relations 3 and 4 give $[G, y^2] = [G, y^2x_1^{-2}] = \gamma_3(G)$. Hence for $g \in \{y^2, y^2x_1^{-2}\}$, $C_g = g\gamma_3(G)$.

(b) By Lemma 5.1, $H = \gamma_2(G) \cup x_1^2\gamma_2(G)$, so $M_1 \setminus H = yH = y\gamma_2(G) \cup yx_1^2\gamma_2(G)$. Now $C_G(y) = \langle y, \zeta(G)\rangle$ and $C_G(yx_1^2) = \langle yx_1^2, \zeta(G)\rangle$, that is for $g \in \{y, yx_1^2\}$, $|C_x| = |\gamma_2(G)|$ and the assertion follows. $\square$

The following lemma will allow us to count the Quillen parameters of all groups of Fam 7.

Lemma 5.4. *Let $G = G_m$ be a group of* Fam 7. *Then the set of all elements of order 2 of G is contained in*

$$\Omega_1(A) \cup \bigcup_{g \in X} C_g$$

where

$$X = \begin{cases} \{y^2, y^2x_1^{-2}, yx_1^{-1}, yx_1^{-1}x_2^{2^{k-1}}\} & \text{if } A \text{ is abelian,} \\ \{y^2, yx_1^{-1}x_2^{2^{k-2}}\} & \text{if } A \text{ is nonabelian.} \end{cases}$$

Proof. It suffices to show, that conjugacy classes which are not represented by elements from X, do not consist of elements of order 2. By Lemma 5.3 elements belonging to $M_1 \setminus H$ are conjugated either to y or to y^3, so their order is equal either 4 or 8. It can be also easily verified that elements of the set $M_2 \setminus H$ have order equal to $o(x_1) = 2^{k+1}$. So let us consider elements of $M_3 \setminus H$. Since $M_3 \setminus H = yx_1^{-1}A \cup y^3x_1A$ and $(yx_1^{-1}A)^y = y^3x_1A$, we consider only elements from $yx_1^{-1}A$. Let $g = yx_1^{-1}(x_1^{2r}x_2^s)$ be an arbitrary element of this set. First let us assume that $|G| = 2^{2k+2}$ (in this case $0 \le r < 2^k$, $0 \le s < 2^{k-1}$). If A is abelian, then $g^2 = t_1t_2t_4(x_1^{-2}x_2)^{s-r}$. Since $t_1, t_2, t_4 \in \{1, x_1^{2^k} = x_2^{2^{k-1}}\}$, g has order 2 if and only if $t_1t_2t_4 = 1 = (x_1^{-2}x_2)^{s-r}$. It follows from Table 5 that for $m \in \{29, 31, 33, 35\}$ $t_1t_2t_4 = z_1$ so the subset $M_3 \setminus H$ of G_m does not contain elements of order 2. If $m \in \{28, 30, 32, 34\}$ then $t_1t_2t_4 = 1$ and then g is of order 2, when $r - s \equiv 0 \pmod{2^{k-1}}$. Since this congruence has exactly 2^k solutions, there exist exactly 2^{k+1} elements of order 2 in $M_3 \setminus H$ (half of them lie in $yx_1^{-1}A$ and the second half in y^3x_1A). It can be easily checked that for $m \in \{28, 30\}$, $C_G(yx_1^{-1}) = \langle yx_1^{-1}, y^2x_1^{-2}, z_1, x_1^{-1}x_2\rangle$, so $|C_{yx_1^{-1}}| = 2^k$. The second conjugacy class consisting of elements of order 2 is the class represented by $yx_1^{-1}z_1$. If $m \in \{32, 34\}$, then $C_G(yx_1^{-1}) = \langle yx_1^{-1}, z_1, x_1^{-1}x_2\rangle$ and then all elements of order 2 of $M_3 \setminus H$ lie in one conjugacy class.

Now let us assume that A is nonabelian, that is $m \in \{36, 37, 38, 39\}$. For $g = yx_1^{-1}(x_1^{2r}x_2^s)$ we have

$$g^2 = t_1t_2t_3^{1+r+s}z_1^{1+r(s-r)}z_2x_1^{-2(s-r)}x_2^{s-r}. \tag{7}$$

Therefore, if $g^2 = 1$ we have $x_1^{-2(s-r)}x_2^{s-r} \in \Omega_1(A)$, which says that $x_1^{-2(s-r)}x_2^{s-r} \in \{1, x_1^{-2^{k-1}}x_2^{2^{k-2}} = z_1z_2\}$. In particular s and r have to be of the same parity, and because of that $g^2 = t_1t_2t_3z_1z_2x_1^{-2(s-r)}x_2^{s-r}$. Thus, as it follows from Table 5, there is no element of order 2 in $M_3 \setminus H$ for G_{37} and G_{39}. If $G = G_{36}$ or $G = G_{38}$ then $g^2 = 1$ if and only if $r - s \equiv 2^{k-2} \pmod{2^k}$. This congruence has 2^k solutions which means, in particular, that in $M_3 \setminus H$

there exist exactly 2^{k+1} elements of order 2 and they all belong to the class of the element $yx^{-1}x_2^{2^{k-2}}$ as $C_G(yx^{-1}x_2^{2^{k-2}}) = \langle yx^{-1}x_2^{2^{k-2}}, z_1, x_1^{-2}x_2 \rangle$, i.e. $|C_{yx^{-1}x_2^{2^{k-2}}}| = 2^{k+1}$.

Finally, let $|G| = 2^{2k+3}$. Then $g^2 = t_1 t_4^{1+r}(x_1^{-2}x_2)^{s-r}$ and so g has order 2 if and only if $t_1 t_4^{1+r}(x_1^{-2}x_2)^{s-r} = 1$, $0 \leq s, r \leq 2^k - 1$. For arbitrary values of t_1 and t_4 the last equality is valid for exactly 2^k different pairs (s, r). Now notice that $(yx_1^{-1})^{y^2x_1^{-2}} = (yx_1^{-1})t_1t_4$ and $(yx_1^{-1}x_2^{2^{k-1}})^{y^2x_1^{-2}x_2^{2^{k-1}}} = (yx_1^{-1}x_2^{2^{k-1}})zt_1t_4$. Hence $t_1 = t_4$ implies that yx_1^{-1} has order 2, $C_G(yx_1^{-1}) = \langle yx_1^{-1},\ y^2x_1^{-2},\ x_1^{-2}x_2 \rangle$ and $|C_{yx_1^{-1}}| = 2^{k+1}$. Therefore all elements of order 2 of the set $M_3 \setminus H$ lie in $C_{yx_1^{-1}}$. If $t_1 \neq t_4$, then $yx_1^{-1}x_2^{2^{k-1}}$ has order 2 and similarly as in the previous case, all elements of order 2 of the set $M_3 \setminus H$ lie in the conjugacy class represented by $yx_1^{-1}x_2^{2^{k-1}}$ since $C_G(yx_1^{-1}x_2^{2^{k-1}}) = \langle yx_1^{-1}x_2^{2^{k-1}},\ y^2x_1^{-2}x_2^{2^{k-1}},\ x_1^{-2}x_2 \rangle$. □

Corollary 5.1. *Tables 15, 16 and 17 contain full information about orders of elements of X in particular groups.*

g	G_{28}	G_{29}	G_{30}	G_{31}	G_{32}	G_{33}	G_{34}	G_{35}
y^2	2	2	4	4	2	2	4	4
$y^2x_1^{-2}$	2	2	2	2	4	4	4	4
yx_1^{-1}	2	4	2	4	2	4	2	4
$yx_1^{-1}x_2^{2^{k-1}}$	2	4	2	4	2	4	2	4

Table 15

g	G_{36}	G_{37}	G_{38}	G_{39}
y^2	2	2	4	4
$yx_1^{-1}x_2^{2^{k-2}}$	2	4	2	4

Table 16

g	G_{40}	G_{41}	G_{42}	G_{43}
y^2	2	4	2	4
$y^2x_1^{-2}$	2	2	4	4
yx_1^{-1}	2	2	4	4
$yx_1^{-1}x_2^{2^{k-1}}$	4	4	2	2

Table 17

Now, similarly as in the case of Fam 8 we can use it for building maximal elementary abelian subgroups and counting the Quillen parameters for our groups.

Proposition 5.1. *Let $G = G_m \in$ Fam 7. Then the Quillen parameter of G is such as in the following table*

Group	Quillen parameter	Representatives of the conjugacy classes of maximal elementary abelian subgroup
G_{28}	$(0,0,1,1)$	$\langle y^2, \Omega_1(A)\rangle$, $\langle yx_1^{-1}, y^2x_1^{-2}, \Omega(A)\rangle$
G_{29}	$(0,0,2,0)$	$\langle y^2, \Omega_1(A)\rangle$, $\langle y^2x_1^{-2}, \Omega_1(A)\rangle$
G_{30}	$(0,0,0,1)$	$\langle yx_1^{-1}, y^2x_1^{-2}, \Omega_1(A)\rangle$
G_{31}	$(0,0,1,0)$	$\langle y^2x_1^{-2}, \Omega_1(A)\rangle$
G_{32}	$(0,0,2,0)$	$\langle y^2, \Omega_1(A)\rangle$, $\langle yx_1^{-1}, \Omega_1(A)\rangle$
G_{33}	$(0,0,1,0)$	$\langle y^2, \Omega_1(A)\rangle$
G_{34}	$(0,0,1,0)$	$\langle yx_1^{-1}, \Omega_1(A)\rangle$
G_{35}	$(0,1,0,0)$	$\Omega_1(A)$
G_{36}	$(0,0,2,0)$	$\langle y^2, \Omega_1(A)\rangle$, $\langle yx_1^{-1}x_2^{2^{k-2}}, \Omega_1(A)\rangle$
G_{37}	$(0,0,1,0)$	$\langle y^2, \Omega_1(A)\rangle$
G_{38}	$(0,0,1,0)$	$\langle yx_1^{-1}x_2^{k-2}, \Omega_1(A)\rangle$
G_{39}	$(0,1,0,0)$	$\Omega_1(A)$
G_{40}	$(0,0,3,0)$	$\langle y^2, \Omega_1(A)\rangle, \langle y^2x_1^{-2}, \Omega_1(A)\rangle, \langle yx_1^{-1}, y^2x_1^{-2}, z\rangle$
G_{41}	$(0,0,2,0)$	$\langle y^2x_1^{-2}, \Omega_1(A)\rangle, \langle yx_1^{-1}, y^2x_1^{-2}, z\rangle$
G_{42}	$(0,1,1,0)$	$\langle y^2, \Omega_1(A)\rangle, \langle yx_1^{-1}x_2^{2^{k-1}}, z\rangle$
G_{43}	$(0,2,0,0)$	$\Omega_1(A), \langle yx_1^{-1}x_2^{2^{k-1}}, z\rangle$

Table 18

For counting the numbers of conjugacy classes and the Roggenkamp parameters we need much more detailed considerations than in the previous families.

Lemma 5.5. *Let $G \in$ Fam 7.*

(a) *If $|G| = 2^{2k+3}$, then $|Cl_G(A)| = 2^{2k-3} + 2^{k-1} + 2^{k-2} + 1$;*

(b) *If $|G| = 2^{2k+2}$ and A is abelian, then $|Cl_G(A)| = 2^{2k-4} + 2^{k-1} + 1$;*

(c) *If $|G| = 2^{2k+2}$ and A is nonabelian, then*

$$|Cl_G(A)| = 2^{2k-5} + 2^{2k-7} + 2^{k-2} + 2^{k-3} + 2^{k-4} + 1.$$

Moreover,

$$R_G(A) = \begin{cases} 2|Cl_G(A)| & \text{if } |G| = 2^{2k+3}, \\ 2|Cl_G(A)| + 1 & \text{if } |G| = 2^{2k+2}. \end{cases}$$

Proof. (a) Since A is abelian and $|G : A| = 8$, each conjugacy class of G contained in A has at most 8 elements. For a noncentral element g of $\Omega_1(A) = \langle (x_1^2)^{2^{k-1}}, x_2^{2^{k-1}} \rangle$ we have $C_G(g) = \langle y^2, x_1, x_2\rangle = \langle y^2, x_1\rangle$ and then $|C_g| = 2$. For $g \in A \setminus \Omega_1(A)$, $g^{y^2} = g^{-1}h \neq g$, where $h \in \{1, z\}$. Hence $y^2 \notin C_G(g)$ and because of that $|G : C_G(g)| \geqslant 4$, which says that either $C_G(g) = A$ or $C_G(g) = H_i$ for certain $i \in \{1, 2, 3, 4\}$. Thus for $g \in A \setminus \Omega_1(A)$, $|C_g| \leq 4$ if and only if $g \in T = C_A(x_1) \cup C_A(y^2x_1) \cup C_A(yx_1) \cup C_G(y^3x_1)$. Straightforward computations show that $C_A(x_1) = \langle x_1^2, \Omega_1(A)\rangle$, $C_A(y^2x_1) = \langle x_2, \Omega_1(A)\rangle$, $C_A(yx_1) = \langle x_1^2x_2^{-1}\rangle$, $C_A(y^3x_1) = \langle x_1^2x_2\rangle$. It is clear that $C_A(x_1) \cap C_A(y^2x_1) = \Omega_1(A)$ and $C_A(yx_1) \cap C_A(y^3x_1) = \langle (x_1^2x_2)^{2^{k-1}}\rangle \leq \Omega_1(A)$. Hence $|C_A(x_1) \cup C_A(y^2x_1)| = 2^{k+2} - 4$, $|C_A(yx_1) \cup C_A(y^3x_1)| = 2^{k+1} - 2$ and so $|T| = 2^{k+2} + 2^{k+1} - 8$. The set T is of course a normal subset of G and it contains three conjugacy classes having less than 4 elements, namely $\{e\}$, $\{(x_1^2x_2)^{2^{k-1}}\}$ and $\{(x_1^2)^{2^{k-1}}, x_2^{2^{k-1}}\}$. Therefore $|Cl_G(T)| = 3 + \frac{2^{k+2}+2^{k+1}-12}{4} = 2^k + 2^{k-1}$. All other elements of A lie in 8-element conjugacy classes. So we have $\frac{2^{2k}-2^{k+2}-2^{k+1}+8}{8} = 2^{2k-3} - 2^{k-1} - 2^{k-2} + 1$ such classes. Consequently $|Cl_G(A)| = 2^{2k-3} + 2^k - 2^{k-2} + 1 = 2^{2k-3} + 2^{k-1} + 2^{k-2} + 1$.

(b) Similarly as in the proof of the first case for $a \in A$, $|C_a| = 4$ if and only if $C_G(a)$ is one of the five subgroups of index 4 containing A. If $a \notin \Omega_1(A)$, then $a^{y^2} = a^{-1}t$, where $t \in \Omega_1(A)$, so H is not a centralizer of an element from $A \setminus \Omega_1(A)$. If $C_G(a) = H_1$, then $a \in \langle x_1^2\rangle = A_1$. Since $(x_1^2)^y = x_2t_3t_4$, $C_G(a) = H_2$ for $a \in \langle x_2t_3t_4\rangle = A_2$. Further, if $C_G(a) = H_3$, then $a \in \langle x_1^{-2}x_2, \Omega_1(A)\rangle = B_1$ and similarly $C_G(a) = H_4$, if $a \in \langle x_1x_2, \Omega_1(A)\rangle = B_2$. Since $A_1A_2 = A$ and $|A_1| = |A_2| = 2^k$ we obtain $|A_1 \cap A_2| = 2$. Analogously, $|A : B_1B_2| = 2$ and $|B_1| = |B_2| = 2^k$, so $|B_1 \cap B_2| = 4$. Hence $|A_1 \cup A_2| = 2^{k+1} - 2$, $|B_1 \cup B_2| = 2^{k+1} - 4$ and from $|(A_1 \cup A_2) \cap (B_1 \cup B_2)| = 2$ we obtain $|A_1 \cup A_2 \cup B_1 \cup B_2| = 2^{k+2} - 8$. Since $A_1 \cup A_2 \cup B_1 \cup B_2$ contains all elements, whose conjugacy classes have no more than 4 elements, we obtain $3 + \frac{2^{k+2}-12}{4} = 2^k$ 4-element classes. Each of all other classes contained in A has 8 elements. So we have $\frac{2^{2k-1}-2^{k+2}+8}{8} = 2^{2k-4} - 2^{k-1} + 1$ such classes. Finally we obtain $|Cl_G(A)| = 2^{2k-4} + 2^k - 2^{k-1} + 1 = 2^{2k-4} + 2^{k-1} + 1$.

(c) It is clear that A is nonabelian only when $x_2^{x_1} = x_2^{-1} z_2$, that is when $G \in \{G_{36},\ G_{37},\ G_{38},\ G_{39}\}$. In this case the action of G on A^2 is similar to the action of G on A in the previous case, that is when A is abelian and $|A| = 2^{2k-3}$. So $|Cl_G(A^2)| = 2^{2k-6}+2^{k-2}+1$. Conjugacy classes which are contained in $A \setminus A^2$ have either 8 or 16 elements. Since for every $g \in A \setminus A^2$, $C_g = C_{gz_1}$ and $\zeta(G) = \langle z_1 \rangle$, images of these classes in $\widetilde{G} = G/\zeta(G)$ have twice less elements and images of different classes are different. By (a) $|Cl_{\widetilde{G}}(\widetilde{A} \setminus \widetilde{A}^2)| = (2^{2k-5} + 2^{k-2} + 2^{k-3} + 1) - (2^{2k-7} + 2^{k-3} + 2^{k-4} + 1) = 2^{2k-5} - 2^{2k-7} + 2^{k-3} + 2^{k-4}$. Therefore $|Cl_G(A \setminus A^2)| = 2^{2k-5} - 2^{2k-7} + 2^{k-3} + 2^{k-4} + 2^{2k-6} + 2^{k-2} + 1 = 2^{2k-5} + 2^{2k-7} + 2^{k-2} + 2^{k-3} + 2^{k-4} + 1$.

For the proof of the last assertion it is enough to notice that $d(M_2) = 2$, $d(M_3) = 3$ and if $g \in \Omega_1(A) \setminus \zeta(G)$, then

$$C_G(g) = \begin{cases} M_2 & \text{if } |G| = 2^{2k+3}, \\ M_3 & \text{if } |G| = 2^{2k+2}. \end{cases}$$

All other elements of A have centralizers equal to H_i for some i, $1 \leqslant i \leqslant 4$, or to G. All these groups are 2-generated. □

Lemma 5.6. *Let $G \in \mathrm{Fam}\,7$ and let $x \in M_2 \setminus H$.*

(a) *If $|G| = 2^{2k+3}$, then $C_G(x) = \langle x, \Omega_1(A) \rangle$ and $|C_x| = 2^{k+1}$. Moreover,*

$$|Cl_G(M_2 \setminus H)| = 2^k \quad \textit{and} \quad R_G(M_2 \setminus H) = 2^{k+1}.$$

(b) *If $|G| = 2^{2k+2}$, then $C_G(x) = \langle x \rangle$ and $|C_x| = 2^{k+1}$. Moreover,*

$$|Cl_G(M_2 \setminus H)| = 2^{k-1} = R_G(M_2 \setminus H).$$

Proof. Since $M_2 \setminus H = x_1 A \cup y^2 x_1 A$ and $(x_1 A)^y = y^2 x_1 A$, it suffices to count $|C_x|$ for $x \in x_1 A$. Put $x = x_1 x_1^{2r} x_2^s \in x_1 A$. If $g \in G$ centralizes x then $\overline{g} = gA$ centralizes $\overline{x} = xA$ in $\overline{G} = G/A$ that is $\overline{g} \in \langle \overline{y}^2, \overline{x}_1 \rangle$. Now, it follows from properties of $\overline{G}$ that $g = y^2 a$ or $g = y^2 x_1 a$ or $g \in \langle x_1, A \rangle$ for suitable $a = x_1^{2u} x_2^v \in A$. None of the elements of the form either $g = y^2 a$ or $g = y^2 x_1 a$ centralizes x. In fact,

$$x^{y^2 a} = (x_1 x_1^{2r} x_2^s)^{y^2 x_1^{2u} x_2^v} = x_1 x_1^{-2r+2} x_2^{-s+1+2v} a_1,$$

where $a_1 \in \Omega_1(A)$. Similarly,

$$x^{y^2x_1a} = (x_1x_1^{2r}x_2^s)^{y^2x_1x_1^{2u}x_2^v} = x_1x_1^{-2r+2}x_2^{s-1+2v}a_1.$$

So in both cases we obtain $x \neq x^g$. Therefore we assume that $g \in \langle x_1, A\rangle$. In this case we need only to describe $C_A(x)$. For $x = x_1$ it was done in the proof of Lemma 5.5. For the general case the calculations and conclusions are similar. So for a) we obtain $C_G(x) = \langle x, \Omega_1(A)\rangle$ and $d(C_G(x)) = 2$. Hence $|C_x| = \frac{2^{2k+3}}{2^{k+2}} = 2^{k+1}$. This means that $|Cl_G(M_2 \setminus H)| = \frac{|M_2\setminus H|}{2^{k+1}} = 2^k$ and $R_G(M_2 \setminus H) = 2|Cl_G(M_2 \setminus H)| = 2^{k+1}$.

For b) the proof of Lemma 5.5 yields $C_G(x) = \langle x\rangle$ (and of course $d(C_G(x)) = 1$). Thus $|Cl_G(M_2 \setminus H)| = \frac{|M_2\setminus H|}{2^{k+1}} = 2^{k-1} = R_G(M_2 \setminus H)$. $\square$

For counting the centralizers of elements of $M_3\setminus H$ notice that $M_3\setminus H = yx_1^{-1}A\cup y^3x_1A$, $(yx_1^{-1}A)^{x_1} = y^3x_1A$ and $(y^3x_1A)^{x_1} = yx_1^{-1}A$. So it suffices to study only these conjugacy classes which are represented by elements of the form $g = yx_1^{-1}a$, where $a = x_1^{2r}x_2^s \in A$.

Lemma 5.7. *Let $G \in \mathrm{Fam}\,7$ and let A be abelian. If $g \in yx_1^{-1}A$, then*

$$C_A(yx_1^{-1}a) = \begin{cases} \langle x_1^2x_2^{-1}\rangle & \text{if } |G| = 2^{2k+3}, \\ \langle x_1^2x_2^{-1}, z_1\rangle & \text{if } |G| = 2^{2k+2}. \end{cases}$$

Proof. For every $h = x_1^{2u}x_2^w \in A$ we obtain from the relations of the group and the relations (3) that

$$y^h = (x_2^{-w}x_1^{-2u})y(x_1^{2u}x_2^w) = yx_1^{2w}x_2^{-u}t_4^{-u}x_1^{2u}x_2^w = yx_1^{2(u+w)}x_2^{-u+w}t_4^{-u}$$

$$(x_1^{-1})^h = x_2^{-w}x_1^{-1}x_2^w = x_1^{-1}x_2^{2w}t_4^{-w}.$$

Therefore

$$\begin{aligned}(yx_1^{-1})^h &= (yx_1^{2(u+w)}x_2^{-u+w}t_4^{-u})(x_1^{-1}x_2^{2w}t_4^{-w}) \\ &= yx_1^{-1}x_1^{2(u+w)}x_2^{u-w}t_4^{-2u+w}x_2^{2w}t_4^{-w}) \\ &= yx_1^{-1}(x_1^2x_2)^{u+w}.\end{aligned} \tag{8}$$

Hence for every $a = x_1^{2r}x_2^s \in A$, $h = x_1^{2u}x_2^w$ centralizes $yx_1^{-1}a$ if and only if $(x_1^2x_2)^{u+w} = 1$. This means for the case $|G| = 2^{2k+3}$ that $u + w = 0$ (mod 2^k) Consequently $h = (x_1^{-2}x_2)^w$ and then $C_A(yx_1^{-1}a) = \langle(x_1^{-2}x_2)\rangle$. If

$|G| = 2^{2k+2}$, then we get $u + w = 0 \pmod{2^{k-1}}$ or equivalently $u + w = 0$ or $2^{k-1} \pmod{2^k}$. Hence $h = (x_1^{-2}x_2)^w$ or $h = (x_1^{-2}x_2)^w z_1$. Finally $C_A(yx_1^{-1}a) = \langle x_1^{-2}x_2, z_1\rangle$. □

The centralizer of the image of $g = yx_1^{-1}a$ in the factor group $\overline{G} = G/A$ is equal to $\langle \overline{g}, \overline{y}^2\rangle$. So we need also to check for which $g \in yx_1^{-1}A$ there exist elements in y^2A centralizing g.

Lemma 5.8. *Let $G = G_n \in \mathrm{Fam}\,7$ and let A be abelian.*

(a) *If $n \in \{28, 29, 30, 31\}$ and $g \in \{yx_1^{-1}, yx_1^{-1}x_2^{2^{k-1}}\}$ or $n \in \{32, 33, 34, 35\}$ and $g \in \{yx_1^{-1}x_2^{2^{k-2}}, yx_1^{-1}x_2^{-2^{k-2}}\}$, then $C_G(g) = \langle g, y^2x_2^{-1}, x_1^{-2}x_2, x_2^{2^{k-1}}\rangle$.*

(b) *If $n \in \{40, 41, 42, 43\}$ and $g \in \{yx_1^{-1}, yx_1^{-1}x_2^{2^{k-1}}\}$, then $C_G(g) = \langle g, y^2x_2^{-1}, x_1^{-2}x_2\rangle$.*

(c) *If $g \in yx_1^{-1}A$ is not conjugated to elements distinguished in* (a) *and* (b) *then $C_G(g) = \langle g, x_1^{-2}x_2, \zeta(G)\rangle$.*

(d) *$Cl_G(M_3 \setminus A) = 2^{k-1} + 1$.*

Proof. It follows from Lemma 5.7 that in all the cases $|C_A(g)| = 2^k$ for all $g \in yx_1^{-1}A$. It is also easily seen that $g^2 \in C_A(g)$ and for every $h \in A$, $(y^2h)^2 \in C_A(g)$. Thus if $C_G(g) = \langle g, y^2h, C_A(g)\rangle$, then $|C_G(g)| = 2^{k+2}$ and so

$$|C_g| = \begin{cases} 2^k & \text{for } |G| = 2^{2k+2}, \\ 2^{k+1} & \text{for } |G| = 2^{2k+3}. \end{cases}$$

All other conjugacy classes have 2^{k+1} and 2^{k+2} elements in the cases $|G| = 2^{2k+2}$ and $|G| = 2^{2k+3}$ respectively.

Now let $h = x_1^{2u}x_2^w$ and $a = x_1^{2r}x_2^s$ be arbitrary elements of A. As it was seen in the proof of Lemma 5.7

$$y^{y^2h} = y^h = yx_1^{2(u+w)}x_2^{-u+w}t_4^{-u}.$$

Moreover

$$\begin{aligned}(x_1^{-1})^{y^2h} &= (x_1^{-1})^{y^2x_1^{2u}x_2^w} = (t_1x_2^{-1}x_1)^{x_1^{2u}x_2^w} = (t_1x_2^{-1}x_1)^{x_2^w} \\ &= x_2^{-w}(t_1x_2^{-1}x_1)x_2^w = x_1t_4^{-w}x_2^w(t_1t_4^{-1}x_2)x_2^w \\ &= x_1x_2^{2w+1}(t_1t_4^{-1-w})\end{aligned}$$

and

$$a^{y^2h} = (x_1^{2r}x_2^s)^{y^2} = x_1^{-2r}x_2^{-s}t_4^{r+s} = a^{-1}t_4^{r+s}.$$

Hence

$$\begin{aligned}(yx_1^{-1}a)^{y^2x_1^{2u}x_2^w} &= (yx_1^{2(u+w)}x_2^{-u+w}t_4^{-u})(x_1x_2^{2w+1}(t_1t_4^{-1-w}))(a^{-1}t_4^{r+s}) \\ &= yx_1^{-1}(x_1^2x_2)^{u+w+1}t_4^{-1}(a^{-1}t_4^{r+s}) \\ &= yx_1^{-1}a(x_1^2x_2)^{u+w+1}a^{-2}(t_1t_4^{r+s-1}).\end{aligned} \tag{9}$$

Since $t_1, t_4 \in \zeta(G)$, we obtain that if $y^2h \in C_G(yx_1^{-1}a)$, then $a^2 \in \langle x_1^2x_2, \zeta(G)\rangle$. If $|G| = 2^{2k+3}$, then the subgroup $\langle x_1^2x_2, \zeta(G)\rangle$ is cyclic of order 2^k. Therefore for every $a \in A$ satisfying $a^2 \in \langle x_1^2x_2, \zeta(G)\rangle$ we can find u and w such that $y^2(x_1^{2u}x_2^w) \in C_G(yx_1^{-1}a)$. There exists exactly 2^{k+1} possible values for a, because a can be an arbitrary element of $\langle x_1^2x_2, \Omega_1(A)\rangle$. It is an easy task to show that for $g = yx_1^{-1}$,

$$\begin{aligned}C_g = \{&yx_1^{-1}(x_1^2x_2)^r : r = 0, 1, \ldots, 2^k - 1\} \\ &\cup \{y^3x_1^{-1}(x_1^{-2}x_2)^r : r = 0, 1, \ldots, 2^k - 1\}\end{aligned}$$

and for $g = yx_1^{-1}x_2^{2^{k-1}}$,

$$\begin{aligned}C_g = \{&yx_1^{-1}(x_1^2x_2)^r x_2^{2^{k-1}} : r = 0, 1, \ldots, 2^k - 1\} \\ &\cup \{y^3x_1^{-1}(x_1^{-2}x_2)^r x_2^{2^{k-1}} : r = 0, 1, \ldots, 2^k - 1\}.\end{aligned}$$

Both classes contain obviously all elements of $yx_1^{-1}A$ having centralizer of order 2^{k+2}. All other elements $g \in M_3 \setminus H$ have centralizers of the form $\langle g, x_1^{-2}x_2\rangle$, because no element from the coset y^2A centralizes g.

Now assume that $|G| = 2^{2k+2}$. Then $\langle x_1^2x_2, \zeta(G)\rangle = \langle x_1^2x_2, z_1\rangle$ is not cyclic and if for $a = x_1^{2r}x_2^s$ we have $a^2 \in \langle x_1^2x_2, z_1\rangle$, then $a \in \langle x_1^2x_2, \Omega_2(A)\rangle$. Since we assumed in the beginning of this section that $|G| \geqslant 2^7$ we have $k > 2$ and then in (9) we obtain $r + s \equiv 0 \pmod 2$. If $t_1t_4 = 1$ then $y^2h \in C_G(g)$, $g = yx_1^{-1}a$, if and only if a belongs to $\langle x_1^2x_2, \Omega_1(A)\rangle$ which is of order 2^k. So there exist in $yx_1^{-1}A$ exactly 2^k elements with the centralizers of order 2^{k+2}. It is clear that they are divided into two conjugacy classes as for such g we have $|C_g \cap yx_1^{-1}A| = 2^{k-1}$. It is also clear that the elements yx_1^{-1} and $yx_1^{-1}z_1$ are not conjugated and if g is one of them then

$C_G(g) = \langle g, y^2x_2^{-1}, \Omega_1(A)\rangle$. All other elements have the centralizers of the form $\langle g, C_A(g)\rangle$ that is of order 2^{k+1}.

If $t_1t_4 = z_1$, then $y^2h \in C_G(g)$, $g = yx_1^{-1}a$, if and only if a belongs to $\langle x_1^2x_2, \Omega_2(A)\rangle$ and $a^2 = z_1$. So there exist again 2^k possible values for a. Let $a = x_2^{\pm 2^{k-1}}$, $u = 0$ and $w = -1$. Then from (9) we obtain

$$(yx_1^{-1}x^{\pm 2^{k-2}})^{y^2x_2^{-1}} = yx_1^{-1}x_2^{\pm 2^{k-2}}x_2^{-2^{k-1}}z_1 = yx_1^{-1}x_2^{\pm 2^{k-2}}.$$

Since the elements $yx_1^{-1}x^{2^{k-2}}$ and $yx_1^{-1}x^{-2^{k-2}}$ are not conjugated the conjugacy classes of them contain all the elements with the centralizers of order 2^{k+2}. All other elements have the centralizers of the form $\langle g, C_A(g)\rangle$ that is of order 2^{k+1}.

For the proof of (d) let us notice that if $|G| = 2^{2k+3}$ then we have 2 conjugacy classes having 2^{k+1} elements. All other classes have twice more elements. Since $|M_3 \setminus H| = 2^{2k+1}$ we have $2 + \frac{2^{2k+1}-2^{k+2}}{2^{k+2}} = 2^{k-1} + 1$ conjugacy classes. The calculation for the case $|G| = 2^{2k+2}$ is similar. □

Lemma 5.9. *Let $G = G_n \in \mathrm{Fam}\,7$, $|G| = 2^{2k+2+\epsilon}$, where $\epsilon \in \{0,1\}$. If A is abelian, then*

$$R_G(M_3 \setminus H) = \begin{cases} 2^k + 2^{k-2} + 5 & \text{if } n \in \{28, 30\}, \\ 2^k + 2^{k-2} + 3 & \text{if } n \in \{32, 34\}, \\ 2^k + 4 & \text{if } n \in \{29, 31, 33, 35\}, \\ 2^{k-1} + 2^{k-2} + 4 & \text{if } n \in \{40, 41, 42, 43\}. \end{cases}$$

Proof. First assume that $|G| = 2^{2k+3}$ i.e. $n \in \{40, 41, 42, 43\}$. If $g = yx_1^{-1}$ or $g = yx_1^{-1}x_2^{2^{k-1}}$ then $C_G(g) = \langle g, y^2x_1^{-2}, x_1^{-2}x_2\rangle$ by Lemma 5.8(b). Thus $d(C_G(g)) = 3$. Hence it remains to consider conjugacy classes contained in $M_3 \setminus H$ containing neither yx_1^{-1} nor $yx_1^{-1}x_2^{2^{k-1}}$. Let g be a representative of such a class. It follows from Lemma 5.8(c) that $C_G(g) = \langle g, x_1^2x_2\rangle$. Thus this centralizer is either 2-generated or cyclic. Put $g = yx_1^{-1}(x_1^{2r}x_2^s)$. So $g^2 = (x_1^2x_2^{-1})^{r-s}t_1t_4$ and then $C_G(g)$ is cyclic if and only if $r - s$ is invertible in $\mathbb{Z}_{2^k}$. There are exactly 2^{2k-1} such elements. Since they split into conjugacy classes, each having 2^{k+1} elements in $yx_1^{-1}A$, we have 2^{k-2} such classes that their representatives have cyclic centralizers. Representatives of other classes contained in $M_3 \setminus H$ have 2-generated centralizers. Therefore by Lemma 5.8(d)

$$R_G(M_3 \setminus H) = 6 + 2^{k-2} + 2 \cdot (2^{k-2} - 1) = 2^{k-1} + 2^{k-2} + 4.$$

Now assume that $|G| = 2^{2k+2}$. If g is one of the elements distinguished in Lemma 5.8(a) then $C_G(g) = \langle g, y^2 x_2^{-1}, x_1^2 x_2, z_1\rangle$ and then $d(C_G(g)) = 4$ if and only if $o(g) = o(y^2 x_2^{-1}) = 2$. It follows from Table 12 that this happens for the groups G_{28} and G_{30} only. In all other cases we have $d(C_G(g)) = 3$. So it remains to consider conjugacy classes contained in $M_3 \setminus H$ not containing elements distinguished in Lemma 5.8(a). It follows from Lemma 5.8(c) that $C_G(g) = \langle g, x_1^{-2} x_2, z_1\rangle$. Thus this centralizer is either 2-generated or 3-generated. Put $g = yx_1^{-1}(x_1^{2r} x_2^s)$. So $g^2 = t_1 t_2 t_4 (x_1^{-2} x_2)^{s-r}$, and then $C_G(g)$ is 2-generated if and only if $t_1 t_2 t_4 = z_1$ or $t_1 t_2 t_4 = 1$ and $s - r$ is invertible in $\mathbb{Z}_{2^{k-1}}$. By Table 5 $t_1 t_2 t_4 = z_1$ in G_m if $m \in \{29, 31, 33, 35\}$ only. So in these groups $d(C_G(g)) = 2$. If $m \in \{28, 30, 32, 34\}$, then $t_1 t_2 t_4 = 1$ and we have exactly 2^{2k-2} such elements g that $r - s$ is invertible. Since they split into conjugacy classes having 2^k elements from $yx_1^{-1}A$ we have 2^{k-2} such classes that their representatives have 2-generated centralizers. Representatives of other classes contained in $M_3 \setminus H$ have 3-generated centralizers. Therefore the Roggenkamp number of $M_3 \setminus H$ is equal to

$$R_G(M_3 \setminus H) = \begin{cases} 2 \cdot 2^{k-2} + 3 \cdot (2^{k-2} - 1) + 8 \\ 2 \cdot 2^{k-2} + 3 \cdot (2^{k-2} - 1) + 6 \\ 2 \cdot (2^{k-1} - 1) + 6 \end{cases}$$

$$= \begin{cases} 2^k + 2^{k-2} + 5 & \text{if } \ m \in \{28, 30\}, \\ 2^k + 2^{k-2} + 3 & \text{if } \ m \in \{32, 34\}, \\ 2^k + 4 & \text{if } \ m \in \{29, 31, 33, 35\}. \end{cases} \qquad \square$$

Proposition 5.2. *If G is a group of* Fam 7 *with A abelian then*

$$|Cl(G)| = \begin{cases} 2^{2k-4} + 3 \cdot 2^{k-1} + 6 & \text{if } |G| = 2^{2k+2}, \\ 2^{2k-3} + 9 \cdot 2^{k-2} + 6 & \text{if } |G| = 2^{2k+3}. \end{cases}$$

Proof. For the proof it is enough to count the conjugacy classes described in Lemmas 5.3–5.8. We gather this information in Table 19.

Subset	$\lvert G\rvert = 2^{2k+2}$	$\lvert G\rvert = 2^{2k+3}$
A	$2^{2k-4} + 2^{k-1} + 1$	$2^{2k-3} + 2^{k-1} + 2^{k-2} + 1$
$H \setminus A$	2	2
$M_1 \setminus H$	2	2
$M_2 \setminus H$	2^{k-1}	2^k
$M_3 \setminus H$	$2^{k-1} + 1$	$2^{k-1} + 1$
$\sum$	$2^{2k-4} + 2^k + 2^{k-1} + 6$	$2^{2k-3} + 2^{k+1} + 2^{k-2} + 6$

Table 19 □

Proposition 5.3. *Let $G = G_n$ be a group of* Fam 7 *and let A be abelian. Then*

$$R(G) = \begin{cases} 2^{2k-3} + 11 \cdot 2^{k-2} + r_n & \text{if } n \in \{28, 30, 32, 34\}, \\ 2^{2k-3} + 5 \cdot 2^{k-1} + r_n & \text{if } n \in \{29, 31, 33, 35\}, \\ 2^{2k-2} + 17 \cdot 2^{k-2} + r_n & \text{if } n \in \{40, 41, 42, 43\}, \end{cases}$$

where r_n for all the groups is given in the following table:

G_{28}	G_{29}	G_{30}	G_{31}	G_{32}	G_{33}	G_{34}	G_{35}	G_{40}	G_{41}	G_{42}	G_{43}
18	16	16	14	14	15	12	13	15	13	15	13

Table 20

Proof. For centralizers of representatives of conjugacy classes we need to count numbers of generators in minimal generating sets. First we list centralizers of representatives of conjugacy classes contained in $M_2 \setminus A = (M_2 \setminus H) \cup (H \setminus A)$. The element g appearing in the centralizer of $y^2 x_2^{-1}$ is one of the elements defined in Lemma 5.8

g	$C_G(g)$	G_{28}	G_{29}	G_{30}	G_{31}	G_{32}	G_{33}
y^2	$\langle y, \Omega_1(A)\rangle$	2	2	2	2	2	2
$y^2 x_2^{-1}$	$\langle y^2 x_2^{-1}, g, \Omega_1(A)\rangle$	4	3	4	3	3	2
y	$\langle y, z_1\rangle$	2	2	1	1	2	2
y^3	$\langle y, z_1\rangle$	2	2	1	1	2	2
	$\sum$	10	9	8	7	9	8

Table 21

g	$C_G(g)$	G_{33}, G_{34}	G_{40}, G_{42}	G_{41}, G_{43}
y^2	$\langle y, \Omega_1(A)\rangle$	2	2	2
$y^2x_2^{-1}$	$\langle y^2x_2^{-1}, g, \Omega_1(A)\rangle$	2	3	3
y	$\langle y, z_1\rangle$	1	2	1
y^3	$\langle y, z_1\rangle$	1	2	1
	$\sum$	6	9	7

Table 22

Now the sum of numbers from the appropriate columns of the above tables and formulas counted for $R_G(A)$, $R_G(M_2 \setminus H)$ and $R_G(M_3 \setminus H)$ in Lemmas 5.5, 5.6 and 5.9 gives the formulas of the proposition. □

Lemma 5.10. *Let $|G| = 2^{2k+2}$ and let A be nonabelian.*

(a) *If either $g = yx_1^{-1}$ or $g = yx_1^{-1}x_2^{2^{k-1}}$ then $C_G(g) = \langle g, x_1^2x_2^{-1}, y^2x_1^{-2}, \Omega_1(A)\rangle$ and $|C_g| = 2^k$.*

(b) *If $g \in M_3 \setminus H$ is not conjugated neither to yx_1^{-1} nor to $yx_1^{-1}x_2^{2^{k-1}}$ then $C_G(g) = \langle g, x_1^2x_2^{-1}, z_1\rangle$ and $|C_g| = 2^{k+1}$.*

(c) $|Cl_G(M_3 \setminus H)| = 2^{k-2} + 2^{k-3} + 1$.

Proof. Because of similar reasons as in the proof of Lemma 5.8 we study classes represented by elements of the form $g = yx_1^{-1}x_1^{2r}x_2^s$ only. Since $C_G(g)/\zeta(G) \le C_{G/\zeta(G)}(g\zeta(G))$ by Lemma 5.8(b) there are no elements centralizing g in $(M_1 \setminus H) \cup (M_2 \setminus H)$. Let $h = x_1^{2u}x_2^w \in A$ be an arbitrary element. Then by straightforward computations

$$g^h = g(x_1^{2(u+w)}x_2^{u+w}z_1^{ur+ws+u}t_3^{u+w}). \tag{10}$$

Hence $h \in C_G(g)$ if and only if

$$u + w \equiv 0 \pmod{2^k} \quad \text{or} \quad u + w \equiv 2^{k-1} \pmod{2^k} \tag{11}$$

and

$$ur + ws + u \equiv 0 \pmod 2. \tag{12}$$

If r and s are of different parity then each solution of (11) is also a solution of (12). In this case there are no elements centralizing g in $H\setminus A$. Otherwise the image of g in $\widetilde{G} = G/\zeta(G)$ would have such a centralizer which is not possible by Lemma 5.8(b). The number of elements of $yx_1^{-1}A$ of the form $g = yx_1^{-1}x_1^{2r}x_2^s$ such that r and s are of different parity is equal to 2^{2k-2}. The centralizer of such g has order 2^{k+1}, so $|C_g| = 2^{k+1}$. Half of elements of C_g lie in $yx_1^{-1}A$. Thus we have just $\frac{2^{2k-2}}{2^k} = 2^{k-2}$ of such classes. Note that in this case $C_g \neq C_{gz_1}$ (g and gz_1 are not conjugated), the images of C_g and C_{gz_1} in $\widetilde{G}$ are equal and have the same size as C_g. Hence there are 2^{k-3} conjugacy classes in $\widetilde{G}$ which are images of classes just considered. In $\widetilde{M_3} \setminus \widetilde{A}$ there are $2^{k-2} + 1$ classes. So we have to fix images of which classes of $M_3 \setminus A$ are the remaining classes. If r and s are of the same parity, then odd solutions of (11) does not satisfy (12). Moreover in this case g is conjugated to gz_1 ($g^{x_1^2 x_2} = gz_1$). So images in $\widetilde{G}$ of different such classes are different but they have twice less elements. Hence we have just $2^{k-2} + 1 - 2^{k-3} = 2^{k-3} + 1$ of such classes. Therefore $Cl_G(M_3 \setminus A) = 2^{k-2} + 2^{k-3} + 1$. □

Lemma 5.11. *Let $|G| = 2^{2k+2}$ and let A be nonabelian. If $k > 3$, then*

$$R_G(M_3 \setminus H) = \begin{cases} 2^{k-1} + 2^{k-2} + 2^{k-4} + 4 & \text{if } G \in \{G_{36}, G_{38}\}, \\ 2^{k-1} + 2^{k-2} + 4 & \text{if } G \in \{G_{37}, G_{39}\}. \end{cases}$$

If $k = 3$, then for $G \in \{G_{36}, G_{38}\}$, $R_G(M_3 \setminus H) = 10$ and for $G \in \{G_{37}, G_{39}\}$, $R_G(M_3 \setminus H) = 8$.

Proof. We consider only the case $k > 3$. We give only sketch arguments because detail calculations are similar as in previous proofs. First let g be one of the elements yx_1^{-1} and $yx_1^{-1}x_2^{2^{k-2}}$. The images of the classes represented by these elements in $\widetilde{G} = G/\zeta(G)$ are different by Lemma 5.8(b) and have centralizers of the form $\langle \widetilde{g}, C_{\widetilde{A}}(\widetilde{g}), \widetilde{y}^2\widetilde{h} \rangle$, where $h \in A$. So the centralizer of g is also of this form. If $n \in \{37, 39\}$, then both elements have 3-generated centralizers. If $n \in \{36, 38\}$, then one element have 3-generated centralizer and the other one 4-generated centralizer.

Now let $g = yx^{-1}x_1^{2r}x_2^s \in yx^{-1}A$ be an arbitrary element with r, s of different parity. Accordingly to the proof of Lemma 5.5 there exist 2^{k-2} classes represented by elements of this form. By (10), $C_G(g) = \langle g, x_1^{-2u}x_2^u, z_1 \rangle$, so by (7) it is 2-generated, since $x_1^{-2u}x_2^u \in \langle g, z_1 \rangle$.

We have still to consider the remaining $2^{k-3}-1$ classes. If $n \in \{37, 39\}$ then all representatives of these conjugacy classes are 2-generated. So assume that $n \in \{36, 38\}$. Let $g = yx^{-1}x_1^{2r}x_2^s \in yx^{-1}A$ be an arbitrary element with r, s of the same parity. If $s-r$ is divisible by 4 then $C_G(g) = \langle g, x_1^{-4}x_2^2, z_1\rangle$ is 3-generated. There exist $2^{k-4}-1$ such classes. If $s-r$ is not divisible by 4 then $C_G(g)$ is 2-generated and there exist 2^{k-4} such classes.

Summarizing, if $n \in \{37, 39\}$ then $R_G(M_3 \setminus H) = 6 + 2 \cdot 2^{k-2} + 2 \cdot (2^{k-3}-1) = 2^{k-1} + 2^{k-2} + 4$. If $n \in \{36, 38\}$ then $R_G(M_3 \setminus H) = 7 + 2 \cdot 2^{k-2} + 3 \cdot (2^{k-4}-1) + 2 \cdot 2^{k-4} = 2^{k-1} + 2^{k-2} + 2^{k-4} + 4$. □

Proposition 5.4. *Let G be a group of* Fam 7, $|G| = 2^{2k+2}$. *If A is nonabelian then $|Cl(G)| = 5 \cdot 2^{2k-7} + 21 \cdot 2^{k-4} + 6$.*

Proof. It follows from Lemmas 5.3, 5.5, 5.6 and 5.10 that

$$\begin{aligned}|Cl(G)| &= |Cl_G(A)| + |Cl_G(H \setminus A)| + \sum_{i=1}^{3} |Cl_G(M_i \setminus H)| \\ &= (2^{2k-5} + 2^{2k-7} + 2^{k-1} + 2^{k-2} + 2^{k-3} + 2^{k-4} + 1) \\ &\quad + 2 + (2 + 2^{k-1} + 2^{k-2} + 2^{k-3} + 1)\end{aligned}$$

which is equal to the formula given in the proposition. □

Proposition 5.5. *Let $G = G_m$ be a group of* Fam 7, $|G| = 2^{2k+2}$. *If A is nonabelian and $k > 3$, then*

$$R(G_m) = \begin{cases} 5 \cdot 2^{2k-6} + 35 \cdot 2^{k-4} + r_m & \text{if } m \in \{36, 38\}, \\ 5 \cdot 2^{2k-6} + 17 \cdot 2^{k-3} + r_m & \text{if } m \in \{37, 39\}, \end{cases}$$

where $r_{36} = 16$, $r_{37} = 15$, $r_{38} = 14$ and $r_{39} = 13$. For $k = 3$ we have $R(G_{36}) = 38$, $R(G_{37}) = 35$, $R(G_{38}) = 36$, $R(G_{39}) = 33$.

Proof. Note that if $n \in \{36, 38\}$, then $R_G(H \setminus A) = 5$. If $n \in \{37, 39\}$ then $R_G(H \setminus A) = 4$. We have also $R_G(M_1 \setminus H) = 4$ for $n \in \{36, 37\}$ and $R_G(M_1 \setminus H) = 2$ for $n \in \{38, 39\}$. Now using the formulas obtained in Lemmas 5.5, 5.6 and 5.9 we get the formulas of the proposition. □

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