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On relatively equilateral polygons inscribed in a convex body

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Abstract. Let $C \subset E^2$ be a convex body. The C-length of a segment is the ratio of its length to the half of the length of a longest parallel chord of C. By a relatively equilateral polygon inscribed in C we mean an inscribed convex polygon all of whose sides are of equal C-length. We prove that for every boundary point x of C and every integer $k \geq 3$ there exists a relatively equilateral k-gon with vertex x inscribed in C. We discuss the C-length of sides of relatively equilateral k-gons inscribed in C and we reformulate this question in terms of packing C by k homothetical copies which touch the boundary of C.

Let C be a convex body in Euclidean n-space E^n. If pq is a longest chord of C in a direction l, we say that points p and q are *opposite* and we call pq *a diametral chord of* C *in direction* l. By the C*-distance* $\mathrm{dist}_C(a,b)$ of a and b we mean the ratio of the Euclidean distance $|ab|$ of a and b to the half of the Euclidean distance of end-points of a diametral chord of C parallel to ab (comp. [7]). We use here the term *relative distance* if there is no doubt about C. By the C-length of the segment ab we mean $\mathrm{dist}_C(a,b)$. If $C \subset E^2$, we define a C*-equilateral* k*-gon* as a convex k-gon all of whose sides have equal C-lengths. We also use the name *relatively equilateral* k*-gon* when C is fixed.

Section 1 is of an auxiliary nature. It presents properties of the C-distance, and especially properties of the C-distance of boundary points

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of C. Section 2 discusses the possibility of inscribing relatively equilateral k-gons in a planar convex body C. Section 3 shows that such a possibility can be equivalently expressed in terms of finding k positive homothetical copies of C which touch the boundary of C from inside (and also in terms of finding k negative homothetical copies of C which touch C) in such a way that every two consecutive copies touch.

1. Properties of relative distance and diametral chords

Lemmas 3–7 in this section speak about the relative distance of some boundary points of a planar strictly convex body. They are needed for the proofs of Theorems 1 and 2, in which we first deal with a strictly convex body, and later apply the well known fact that every convex body in E^n is a limit of a sequence of strictly and smooth convex bodies containing this body (for a proof see [4], pp. 69–71). This explains why we present Lemma 2 about the continuity of the relative distance. In fact, Lemma 2 is also applied for the proofs of Lemmas 3–7. We start with Lemma 1 which collects some known or easy-to-show properties of diametral chords which are applied in this paper.

Lemma 1. *Let $C \subset E^2$ be a convex body. If ab is a diametral chord of direction l, then there exist parallel supporting lines of C at a and b. The Euclidean length of the diametral chord changes continuously as l rotates. If C is strictly convex, the diametral chord in each direction is unique. If C is smooth, the supporting lines at the end-points of each C-diametral chord are unique. If C is strictly convex and smooth and if the direction of the diametral chord rotates counterclockwise, then the direction of the supporting lines at the end-points of this segment also change counterclockwise. If C is strictly convex, then every two C-diametral chords have a common point.*

Observe that the C-distance of any two points is equal to their $\left(\frac{1}{2}C + \frac{1}{2}(-C)\right)$-distance. The body $\frac{1}{2}C + \frac{1}{2}(-C)$ is centrally symmetric. Thus it is the unit ball of a normed space. So for each convex body $C \subset E^n$ the Euclidean length of diametral chords of C in direction l is a continuous function of l and thus the relative distance $\mathrm{dist}_C(x, y)$ is a continuous function of variables x and y. We need the following more general fact.

Lemma 2. *The relative distance* $\mathrm{dist}_C(x, y)$ *is a continuous function of* x, y *and* C.

In order to prove Lemma 2, we reformulate it in a convenient form: *if a sequence of convex bodies* $C_1, C_2, \dots$ *tends to a convex body* C*, if a sequence* $a_1, a_2, \dots$ *of points tends to* a*, and if a sequence* $b_1, b_2, \dots$ *of points tends to* b*, then* $\lim \mathrm{dist}_{C_i}(a_i, b_i) = \mathrm{dist}_C(a, b)$. In order to show this, in C_i we find a C_i-diametral chord $a'_i b'_i$ such that $\overrightarrow{a_i b_i}$ and $\overrightarrow{a'_i b'_i}$ have the same orientation. Then in $D_i = \frac{1}{2}C_i + \frac{1}{2}(-C_i)$ we find its translation $a^*_i b^*_i$ centered at the center of D_i. Since the sequence $\overrightarrow{a_1 b_1}, \overrightarrow{a_2 b_2}, \dots$ tends to $\overrightarrow{ab}$ and since $D_1, D_2, \dots$ tends to $D = \frac{1}{2}C + \frac{1}{2}(-C)$, we conclude that $a^*_1, a^*_2, \dots$ tends to a boundary point a^* of D, that $b^*_1, b^*_2, \dots$ tends to a boundary point b^* of D, and that $a^* b^*$ is parallel to ab. Of course, $\lim |a_i b_i| = |ab|$ and $\lim |a'_i b'_i| = \lim |a^*_i b^*_i| = |a^* b^*|$. Moreover, a translate of $a^* b^*$ is a diametral chord of C parallel to ab. Thus $\lim \mathrm{dist}_{C_i}(a_i, b_i) = \mathrm{dist}_C(a, b)$.

A number of properties in the forthcoming lemmas are formulated for the counterclockwise order. Clearly, analogous properties hold true for the clockwise order.

Two points of the boundary of a convex body C are called *opposite* if they are in opposite parallel supporting lines of C. For every boundary point x of a planar convex body C denote by x^+ the first boundary point of C counted in the counterclockwise order such that x and x^+ are opposite. We call x^+ the *first opposite point* to the point x.

Lemmas 3–6, below, generalize analogous results proved in [3] under the assumption of central symmetry of C. This special case of Lemma 3 is also given in [9].

Lemma 3. *Let* x *be a boundary point of a strictly convex body* $C \subset E^2$*. If a point* y *moves counterclockwise in the boundary of* C *from* x *to* x^+*, then its relative distance from* x *increases all the time and it accepts all values from the interval* $[0, 2]$.

Proof. In order to prove the first statement, it is sufficient to show that if y_1, y_2 are different points in the boundary of C such that x, y_1,

y_2, x^+ are counterclockwise ordered points in the boundary of C, then $\mathrm{dist}_C(x, y_1) < \mathrm{dist}_C(x, y_2)$.

Denote by ab_2 the C-diametral chord parallel to xy_2. The notation is chosen such that $\overrightarrow{ab_2}$ and $\overrightarrow{xy_2}$ have the same orientation. Since the C-diametral chords xx^+ and ab_2 intersect, the three boundary points x, y_1, y_2 are on one side of the segment ab_2. From the strict convexity of C and since ab_2 is a C-diametral chord we see that the straight lines containing the segments ax and b_2y_2 intersect at a point h. Denote by b_1 the boundary point of C such that the segments ab_1 and xy_1 are parallel.

Case 1, when y_2, b_1, b_2 are in counterclockwise order in the boundary of C (we include the case $b_1 = y_2$).

The assumption of this case implies that the segments y_2b_2 and ab_1 intersect at a point t (see Figure 1). Moreover, the straight line through x and y_1 intersects the segment y_2h at a point w. Thanks to the strict convexity of C, the point y_1 is strictly between x and w. Thus we have $\mathrm{dist}_C(x, y_1) < \mathrm{dist}_C(x, w) \leq 2 \cdot \frac{|xw|}{|ab_1|} \leq 2 \cdot \frac{|xw|}{|at|} = 2 \cdot \frac{|xh|}{|ah|} = 2 \cdot \frac{|xy_2|}{|ab_2|} = \mathrm{dist}_C(x, y_2)$.

Case 2, when y_1, b_1, y_2 are in counterclockwise order in the boundary of C (this time we assume that $b_1 \neq y_2$).

Provide the straight line containing b_1y_2 (see Figure 2). From the assumptions of Case 2 and from the strict convexity of C we conclude that this line intersects the segment xh at a point $g \neq h$ and that it intersects the line containing xy_1 at a point u such that y_1 is strictly between x and u. Consequently, $\mathrm{dist}_C(x, y_1) \leq 2 \cdot \frac{|xy_1|}{|ab_1|} < 2 \cdot \frac{|xu|}{|ab_1|} = 2 \cdot \frac{|xg|}{|ag|} < 2 \cdot \frac{|xg|+|gh|}{|ag|+|gh|} = 2 \cdot \frac{|xh|}{|ah|} = 2 \cdot \frac{|xy_2|}{|ab_2|} = \mathrm{dist}_C(x, y_2)$. This ends the consideration of Case 2.

Since in both cases $\mathrm{dist}_C(x, y_1) < \mathrm{dist}_C(x, y_2)$, we see that the relative distance $\mathrm{dist}_C(x, y)$ increases all the time when y moves counterclockwise from x to x^+.

From Lemma 2 we see that $\mathrm{dist}_C(x, y)$ is a continuous function of y. So the relative distance of x and y attains all values from $[0, 2]$ when y moves from x to x^+. □

Lemma 3 implies Lemma 4. Note that Lemma 4 defines a point $f_d(x)$.

Lemma 4. *Let x be a boundary point of a strictly convex body $C \subset E^2$. For every $d \in [0, 2]$ there is exactly one point $f_d(x)$ in the boundary*

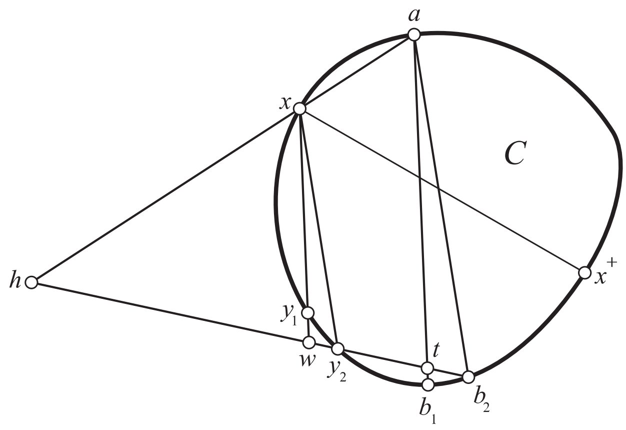

Figure 1

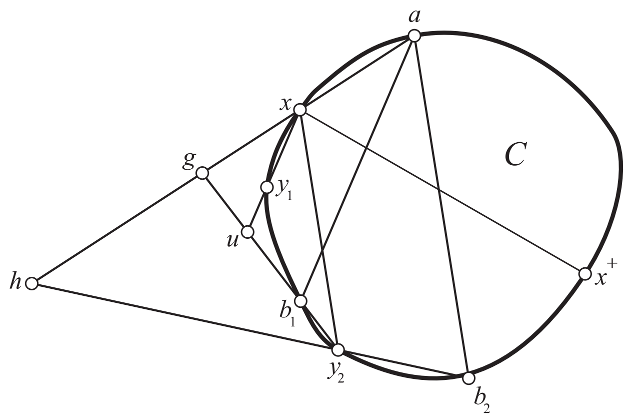

Figure 2

of C such that $\mathrm{dist}_C(x, f_d(x)) = d$ *and such that* x, $f_d(x)$, x^+ *are in the counterclockwise order in the boundary of* C.

From the second part of Lemma 3 and from Lemma 4 we obtain Lemma 5.

Lemma 5. *Let C be a strictly convex body in the plane. For every fixed boundary point x of C the function $f_d(x)$ is a continuous function of the variable d. When d grows from 0 to 2, then the point $f_d(x)$ moves counterclockwise and continuously without any stop in the boundary of C from x to x^+.*

Lemma 6. *Let x be a boundary point of a strictly convex body C in the plane. If x, u_1, w_1, w_2, u_2, x^+ are counterclockwise ordered points in the boundary of C such that u_1 and u_2 are not opposite, and such that the segments u_1u_2 and w_1w_2 do not coincide, then* $\mathrm{dist}_C(w_1, w_2) < \mathrm{dist}_C(u_1, u_2)$.

Proof. If u_1u_2 and w_1w_2 are parallel, by the strict convexity of C and since the segments do not coincide, we have $|w_1w_2| < |u_1u_2|$, which gives $\mathrm{dist}_C(w_1, w_2) < \mathrm{dist}_C(u_1, u_2)$. If u_1u_2 and w_1w_2 are not parallel, we provide the segment $w_1'w_2'$ parallel to u_1u_2 which separates u_1u_2 from w_1w_2, and such that $w_1' = w_1$ or $w_2' = w_2$. We have $\mathrm{dist}_C(w_1', w_2') \leq \mathrm{dist}_C(u_1, u_2)$. Moreover, by Lemma 3 (or by its clockwise version together with the assumption that u_1 and u_2 are not opposite), we get $\mathrm{dist}_C(w_1, w_2) < \mathrm{dist}_C(w_1', w_2')$. Both the inequalities give the inequality promised in Lemma 6. $\square$

Modifications of Lemmas 3 and 6 hold true when C is a convex body, and not necessarily strictly convex. Then in Lemma 3 we can claim that $\mathrm{dist}_C(x, y)$ is nondecreasing. In Lemma 6 we have $\mathrm{dist}_C(w_1, w_2) \leq \mathrm{dist}_C(u_1, u_2)$.

By recursion, for every strictly convex body $C \subset E^2$ we put $f_d^0(x) = x$ and $f_d^m(x) = f_d\big(f_d^{m-1}(x)\big)$ for $m = 1, 2, \dots$. By Lemma 4, for any fixed strictly convex body $C \subset E^2$, for every boundary point x of C, for every integer $k \geq 0$, and for every $d \in [0, 2]$, the point $f_d^k(x)$ is uniquely defined. From Lemma 5 we obtain the following Lemma 7.

Lemma 7. *Let $C \subset E^2$ be a strictly convex body, let x be a boundary point of C and let $k \geq 1$ be an integer. When d grows continuously from 0 to 2, the point $f_d^k(x)$ moves counterclockwise and continuously without any stop in the boundary of C.*

2. Relatively equilateral polygons inscribed in a convex body

The following Theorem 1 generalizes the analogous result which is proved in [3] under the assumption of central symmetry of C.

Theorem 1. *Let $C \subset E^2$ be a convex body and let $k \geq 3$ be an integer. For every boundary point x of C there exists a relatively equilateral k-gon inscribed in C with vertex x. If C is strictly convex, such an inscribed C-equilateral k-gon is unique.*

Proof. Case 1, when C is strictly convex. Put $x_m = f_d^{m-1}(x)$ for $m = 1, 2, \dots$. From Lemma 7 we see that for $m = 2, \dots, k$ the positions of the points x_m depend continuously on $d \in [0, 2]$. The smallest value of $d \in (0, 2)$ such that $x_{k+1} = x_1$ gives the relatively equilateral k-gon $x_1 x_2 \dots x_k$ that we are looking for. It remains to show that such a smallest value of d exists, i.e. that $x_{k+1} = x_1$ for a $d < 2$. It is sufficient to consider the case when $k = 3$. If $f_2^2(x) = x$, it is obvious by Lemma 7. In the opposite case, Lemma 1 implies that C contains the non-degenerate triangle $x f_2^1(x) f_2^2(x)$, and $f_2^3(x)$ is in the arc of C which starts at x and ends at $f_2^1(x)$. By Lemma 7 there is a $d < 2$ such that $f_d^3(x) = x$.

This construction gives exactly one relatively equilateral k-gon with vertex x inscribed in C. An analogous procedure for the clockwise order also gives exactly one inscribed relatively equilateral k-gon. Since each of those k-gons is unique, we see that they coincide.

Case 2, when C is an arbitrary convex body. We present C as a limit of a sequence of strictly convex bodies $C_1, C_2, \dots$. Let $p_{i\,1}$ be a boundary point of C_i in the smallest Euclidean distance from x. Clearly, the sequence $p_{1\,1}, p_{2\,1}, \dots$ tends to x. By Case 1 for every $i \in \{1, 2, \dots\}$ there exists a relatively equilateral k-gon $P_i = p_{i\,1} p_{i\,2} \dots p_{i\,k}$ inscribed in C_i. By compactness arguments, from the sequence of indices $i = 1, 2, \dots$ it is possible to select a subsequence for which the sequence of the second vertices of the corresponding k-gons converges to a point p_2. Clearly, p_2 is a boundary point of C. We repeat $k-2$ times the procedure of selecting a convergent subsequence from the previously obtained subsequence; during the m-th selection we deal with m-th vertices of the previously selected k-gons. As a result, we obtain points $p_3, \dots, p_k$ which are consecutive vertices of a convex k-gon $P = x p_2 \dots p_k$ inscribed in C. Since P is the limit of a subsequence of the sequence $P_1, P_2, \dots$ and since P_i is a C_i-equilateral k-gon inscribed in C_i, from Lemma 2 we conclude that P is a C-equilateral k-gon inscribed in C. $\square$

In Theorem 1 we do not claim that the relatively equilateral k-gon with a given vertex is unique when C is not strictly convex. Also the length of the sides may vary. For instance, when x is any boundary point of a triangle T. If x is a vertex of T, the C-length of sides of inscribed T-equilateral triangles varies from 0 to 2. If x is strictly between vertices v_1 and v_2 of T, we have infinitely many relatively equilateral triangles of the T-length of sides equal to $\mathrm{dist}_C(x, v_1)$ (equal to $\mathrm{dist}_C(x, v_2)$) inscribed in T. Applying Lemma 6 and the comment after it one can show that *all relatively equilateral k-gons with $k \geq 4$ inscribed in any planar convex body C which have a common vertex are of equal C-length of sides.*

Theorem 2. *Assume that there exists a convex k-gon contained in a convex body $C \subset E^2$ all of whose sides have C-lengths at least d. Then we can inscribe in C a relatively equilateral k-gon whose sides have C-length at least d.*

Proof. Case 1, when C is smooth and strictly convex. Consider a side ab of a convex polygon P contained in C. We provide the diametral chord a_1b_1 in C parallel to the segment ab. Let the notation be chosen such that the segments a_1b and b_1a intersect (see Figure 3). Of course, a_1, b_1 are boundary points of C. Denote by A_1, B_1 the supporting straight lines of C through a_1 and b_1, respectively. By Lemma 1 they are parallel.

Through a and b we provide straight lines A and B parallel to A_1 and B_1. There is a boundary point a_2 of C in A and a boundary point b_2 of C in B such that the interior of the quadrangle abb_2a_2 is disjoint with P. Thanks to the strict convexity of C the segment a_2b_2 is determined uniquely. We call it *the C-projection of the side ab of P on the boundary of C*. Since ab is parallel to a_1b_1, and since there is no segment in C of greater C-length connecting the lines A_1 and B_1, we see that $\mathrm{dist}_C(a_2, b_2) \geq \mathrm{dist}_C(a, b)$.

From the above construction and from the part of Lemma 1 which says that the direction of the supporting lines of C at the endpoints of the C-diametral chord changes counterclockwise we conclude that the C-projections of different sides of P on the boundary of C do not have common interior points. Consider the k-gon $w_1w_2 \dots w_k$ inscribed in C whose vertices are the first points of the segments being the C-projections of the sides of P when we go counterclockwise. By Lemma 3 all its sides are of C-lengths at least d. Take into account points $f_d^m(w_1)$ for $m = 2, \dots, k$.

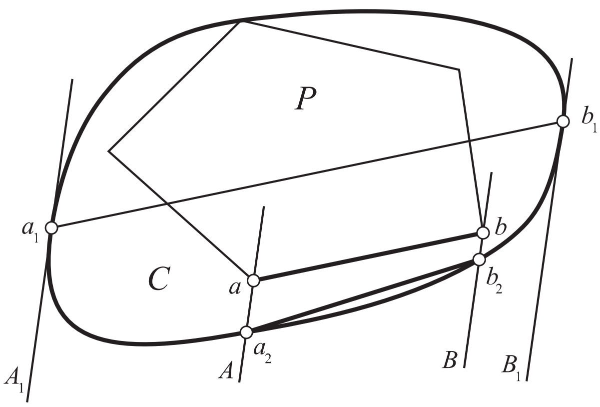

Figure 3

By Lemmas 3 and 6, for every $m \in \{2, \ldots, k\}$, points $w_1, f_d^m(w_1), w_m$ are in the counterclockwise order in the boundary of C. Thus $f_d^k(w_1) = w_1$ or $f_d^k(w_1)$ "has not arrived yet" to w_1. The first possibility means that $w_1 f_d^1(w_1) \ldots f_d^{k-1}(w_1)$ is the promised inscribed k-gon, and for the second possibility we apply Lemma 7 increasing d until we find the smallest $c \in (d, 2)$ for which $f_c^k(w_1) = w_1$.

Case 2, when C is an arbitrary convex body. We present C as a limit of a sequence $C_1, C_2, \ldots$ of strictly and smooth convex bodies containing C. For every body C_i we apply Case 1; we inscribe in C_i a C_i-equilateral k-gon $P_i = p_{i\,1} p_{i\,2} \ldots p_{i\,k}$, whose sides have C_i-lengths at least d where $i = 1, 2, \ldots$. Then, successively k times, we select subsequences of polygons. During the j-th selection, where $j = 1, 2, \ldots, k$, we take care in order to get the convergence of the j-th vertices to a point v_j. Finally we obtain a convex k-gon $V = v_1 v_2 \ldots v_k$. Since the sequence $C_1, C_2, \ldots$ tends to C, the k-gon V is inscribed in C. Lemma 2 implies that V is a relatively equilateral k-gon inscribed in C. $\square$

Lemma 3 with its clockwise version and the comment after Lemma 6 imply Claim 1.

Claim 1. *Let P be a relatively equilateral polygon of sides of C-length d inscribed in a convex body $C \subset E^2$. Then every pair of vertices of P is in C-distance at least d. If C is strictly convex, then the C-distance of every non-consecutive pair of vertices is over d.*

Inscribed relatively equilateral triangles. BEZDEK, FODOR and TALATA [1] show that the boundary of every convex body C contains three points in pairwise C-distances at least $\frac{4}{3}$, and at least 1.546 provided C is a centrally symmetric convex body. LÁNGI [5] improves the first estimate up to $\frac{1}{5}(2+2\sqrt{6}) \approx 1.38$. The second estimate is improved up to $1+\frac{1}{3}\sqrt{3} \approx 1.577$ in [8]. So from Theorem 2 we conclude that in every convex body C we can inscribe a relatively equilateral triangle of sides C-length at least $\frac{1}{5}(2+2\sqrt{6}) \approx 1.38$ and at least $1+\frac{1}{3}\sqrt{3} \approx 1.577$ when C is centrally symmetric. A conjecture from [7] is that in every convex body C we can inscribe a relatively equilateral triangle of sides of C-length at least $\frac{1}{2}(1+\sqrt{5}) \approx 1.618$. The case when C is a regular pentagon shows that this value cannot be increased. Another conjecture says that if C is centrally symmetric, then in C we can inscribe a relatively equilateral triangle of sides of C-length at least $1+\frac{1}{2}\sqrt{2} \approx 1.707$ (see [3] and [7]). This value is attained for the regular octagon.

Inscribed relatively equilateral quadrangles and pentagons. From the proof of Theorem in [7] we see that every planar convex body contains four points in relative distances at least 1. DOLIWKA [2] improved this by showing that the boundary of every planar convex body contains five points in relative distances at least 1. Thus from Theorem 2 we see that in every convex body we can inscribe a relatively equilateral quadrangle and a relatively equilateral pentagon of sides of relative length at least 1. Both the estimates cannot be improved as it follows from the example of a triangle. The second estimate cannot be improved also under the assumption of central-symmetry as it results from the example of a parallelogram. Every centrally symmetric convex body C permits to inscribe a relatively equilateral quadrangle of sides of C-length at least $\sqrt{2}$, and this value cannot be improved for the usual disk as C (see [3] and [7]).

Inscribed relatively equilateral hexagons. Every lower estimate of the C-length of sides of a triangle which can be inscribed in C induces an estimate half as big for the C-length of the sides of a relatively equilateral hexagon which can be inscribed in C. We just take the vertices and the midpoints of sides of the triangle and we apply our Theorem 2. So from the above estimates for large triangles we obtain that every convex body C permits to inscribe a relatively equilateral hexagon of sides of C-length at least $\frac{2}{3}$ (respectively, at least $\frac{1}{5}(1+\sqrt{6}\,) \approx 0.69$). We conjecture that the worst convex body for inscribing a large relatively equilateral hexagon is the regular pentagon (see also Figure 6 and the comment to it). In every centrally symmetric convex body C we can inscribe a relatively equilateral hexagon of sides of C-length 1; of course, as the hexagon we can take any affine-regular hexagon inscribed in C (cf. [7] and [3]). What is more, this estimate is sharp [3].

Inscribed relatively equilateral heptagons. Lángi [5] proves that each convex body C permits to inscribe a relatively equilateral heptagon of sides of C-length at least $\frac{2}{3}$. This cannot be improved for triangles.

Inscribed relatively equilateral octagons, nonagons and decagons. As previously for triangles, from the estimates about the C-length of sides of C-equilateral quadrangles and pentagons which can be inscribed in a convex body C we obtain two times smaller estimates about the C-length of sides of relatively C-equilateral octagons, nonagons and decagons. Again we take vertices and midpoints of sides. So we can inscribe in C a relatively equilateral octagon, nonagon and decagon of sides of C-length at least $\frac{1}{2}$. The last estimate cannot be improved as we see from the example of a triangle.

Probably, besides some exceptional values of k, triangles are the worst convex bodies for inscribing relatively equilateral k-gons of sides of large relative length. Such exponential values are for instance $k = 3$ and $k = 6$. In both cases the regular pentagon is a counterexample.

3. Homothetical copies touching the boundary of the body

We say that a convex body *touches* another convex body if the two bodies have non-empty intersection and empty intersection of their interiors. We say that a convex body A *touches* the boundary of a convex body C *from inside* if $A \subset C$ and if the intersection of A with the boundary of C is non-empty.

Clearly, when we speak about two *consecutive vertices* of a convex k-gon $p_1 \dots p_k$, we mean any pair of vertices p_i, p_{i+1} for $i \in \{1, \dots, k-1\}$, and also the pair p_k, p_1. Below we consider homothetical copies of a convex body C whose homothety centers are at such vertices $p_1, \dots, p_k$. If two homothety centers are consecutive vertices, then the corresponding homothetical copies of C are called *consecutive*.

Claim 2. *If the homothety center is in the boundary of a convex body $C \subset E^n$, then every homothetical copy of C with a positive ratio at most 1 touches the boundary of C from inside, and every homothetical copy of C with a negative ratio touches C.*

Since the homothety center x is in the boundary of C, Claim 2 results from the following obvious facts. If the ratio is between 0 and 1, the copy is a subset of C. Every negative homothetic copy of C is on the opposite side of any supporting line of C at x.

Lemma 8. *Let $C \subset E^n$ be a convex body and let $x, y \in E^n$. The following conditions* (i)–(iii) *are equivalent for every $d \in (0, 2)$. Conditions* (i) *and* (ii) *are equivalent also for $d = 2$.*

(i) $\mathrm{dist}_C(x, y) = d$,

(ii) *homothetical copies of C with centers x, y and ratio $\frac{d}{2+d}$ touch,*

(iii) *homothetical copies of C with centers x, y and ratio $-\frac{d}{2-d}$ touch.*

Proof. We can easily show the following generalization of Lemma 2 from [6]: if xy and ab are parallel segments in E^n and if $d = 2(|xy| \,/\, |ab|) < 2$, then the two segments S_1 and S_2 being homothetical copies of the segment ab with homothety centers x and y have exactly one common point if and only if the homothety ratio is $\frac{d}{2+d}$ or $-\frac{d}{2-d}$. For ab we take a diametral chord parallel to xy such that the vectors $\overrightarrow{ab}$ and $\overrightarrow{xy}$ have the same orientation. Although our consideration is general, in Figure 4 we see the special situation (like in the forthcoming Theorem 3) when x and y are boundary points of C. In the case of the positive homothety ratio $\frac{d}{2+d}$ the segments S_1 and S_2 are gw and wh as in Figure 4. Here w is the homothetic image (with ratio $\frac{d}{2+d}$) of a and homothety center y, and also of b and the homothety center x. In the case of the negative homothety ratio $-\frac{d}{2-d}$ the segments S_1 and S_2 are du and uc as in Figure 4. Now u is the homothetic image (with ratio $-\frac{d}{2-d}$) of a and homothety center x, and also it is the homothety image of b with center y. Moreover, since ab is a C-diametral chord of C, from Lemma 1 we see that at a and b we can provide parallel

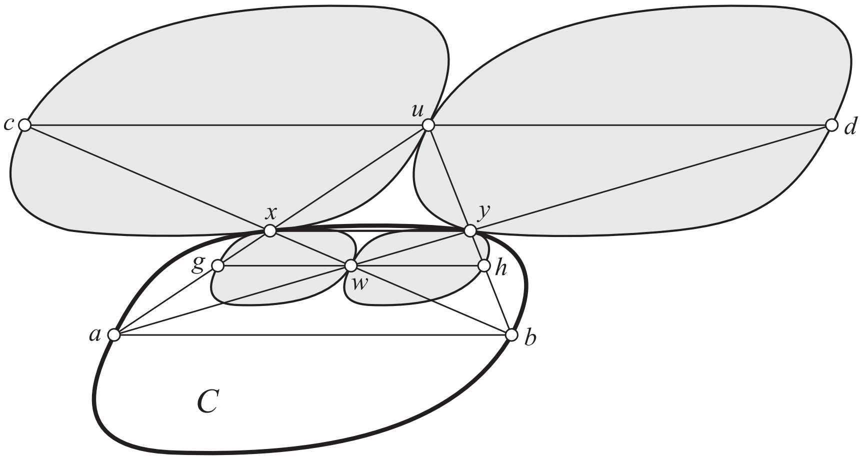

Figure 4

supporting lines of C. Consequently, the pair of homothetic copies of C with centers x, y and the ratio $\frac{d}{2+d}$ (respectively, with the ratio $-\frac{d}{2-d}$) is separated by a straight line through w (respectively, through u).

From the above consideration we see that the condition (i) implies conditions (ii) and (iii). If (i) does not hold true, then $\mathrm{dist}_C(x, y) \neq d$ which implies that the considered pair of homothetical copies with ratio $\frac{d}{2+d}$ (with ratio $-\frac{d}{2-d}$) has empty intersection or that the intersection of their interiors is non-empty, i.e. that (ii) and that (iii) do not hold true.

The equivalence of (i) and (ii) for $d = 2$ is obvious. $\square$

From Lemma 8 we obtain Theorem 3.

Theorem 3. *Let C be a convex body in E^2, let $p_1p_2 \ldots p_k$ be a convex polygon inscribed in C and let $d \in (0, 2)$. The following conditions are equivalent. The equivalence of* (i) *and* (ii) *is true also for $d = 2$.*

(i) *$p_1p_2 \ldots p_k$ is a relatively equilateral k-gon of side of C-length d,*

(ii) *every two consecutive from the homothetical copies of C of ratio $\frac{d}{2+d}$ and homothety centers $p_1, p_2, \ldots, p_k$ touch,*

(iii) *every two consecutive from the homothetical copies of C of ratio $-\frac{d}{2-d}$ and homothety centers $p_1, p_2, \ldots, p_k$ touch.*

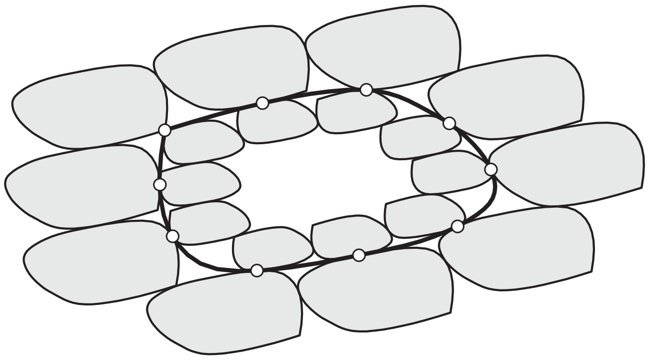

Figure 5

Having in mind Claim 2, we illustrate Theorem 3 in Figure 5. Similarly we show that for boundary points $p_1, p_2, \dots, p_k$ of any convex body $C \subset E^n$ the following conditions are equivalent: (i) *pairwise relative distances of points* $p_1, p_2 \dots, p_k$ *are at least (at most)* d, (ii) *homothetical copies of* C *of ratio* $\frac{d}{2+d}$ *and homothety centers* $p_1, p_2, \dots, p_k$ *are packed in* C *(respectively: they pairwise intersect),* (iii) *homothetical copies of* C *of ratio* $-\frac{d}{2-d}$ *and homothety centers* $p_1, p_2, \dots, p_k$ *have pairwise disjoint interiors (respectively: they pairwise intersect).*

This equivalence explains why we introduce the following three number characteristics for any integer $k \geq 2$ and every convex body $C \subset E^n$, and why we present Claim 3. We define $d_C(k)$ as the greatest value v such that there exist k boundary points of C in pairwise C-distances at least v. By $r_C(k)$ we mean the greatest ratio of k homothetical copies of C which can be packed in C and which touch the boundary of C from inside. Let $s_C(k)$ denote the smallest possible negative ratio of k homothetical copies of C which touch C and which have pairwise disjoint interiors. If such a smallest ratio does not exist, we put $s_C(k) = -\infty$. Simple compactness arguments show that $d_C(k)$, $r_C(k)$ and $s_C(k)$ exist. We have $0 < d_C(k) \leq 2$ and $0 < r_C(k) \leq \frac{1}{2}$. Moreover, $-\infty < s_C(k) < 0$ or $r_C(k) = -\infty$.

By Theorem 2 and Claim 1, *for* $n = 2$ *the number* $d_C(k)$ *equals to the greatest possible* C*-length of sides of a relatively equilateral* k*-gon inscribed in* C*. Also the values of* $r_C(k)$ *and* $s_C(k)$ *do not change by adding the requirement that every two consecutive copies touch.*

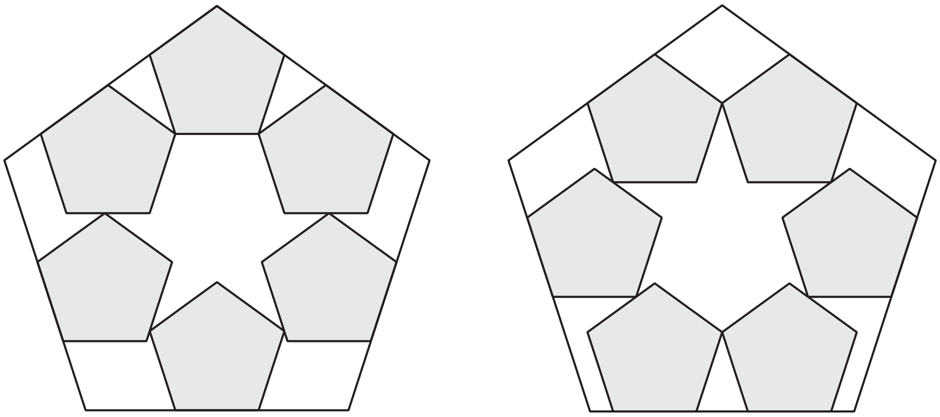

Figure 6

From the equivalence presented after Theorem 3 we obtain the following property.

Claim 3. *Let $C \subset E^n$ be a convex body and let $k \geq 2$ be an integer. We have $r_C(k) = \frac{d_C(k)}{2+d_C(k)}$. For $d_C(k) < 2$ we have $s_C(k) = -\frac{d_C(k)}{2-d_C(k)}$, and for $d_C(k) = 2$ we have $s_C(k) = -\infty$. Conversely: $d_C(k) = \frac{2r_C(k)}{1-r_C(k)}$, and $d_C(k) = \frac{2s_C(k)}{s_C(k)-1}$ for $s_C(k) \neq -\infty$ with $d_C(k) = 2$ provided $s_C(k) = -\infty$.*

Theorem 3 with the comment after it and also Claim 3 permit to reformulate facts and conjectures about configurations of points in some C-distances in terms of packing positive (or placing egative) homothetical copies of C. Just for example, the earlier mentioned conjecture that the regular pentagon is the worst planar convex body for inscribing in it a large relatively equilateral hexagon is illustrated in Figure 6 in terms of packing positive homothetical copies which touch from inside the boundary of the pentagon and such that every consecutive pair touches. The left figure gives probably the best such a packing of the regular pentagon by 6 copies. The ratio is about 0.32. Claim 3 implies that the relative length of the sides of the corresponding inscribed relatively equilateral hexagon is about 0.94. In the right figure the ratio is smaller than on the left figure.

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