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# On the divergence of partial sums of orthogonal series

By LÁSZLÓ LEINDLER (Szeged)

*Dedicated to Professor László Csernyák on our 50 years friendship*

**Abstract.** We slightly weaken the assumption of a theorem pertaining to the divergence of partial sums of orthogonal series from monotonicity to almost monotonicity.

## 1. Introduction

We take $(0,1)$ as the interval of orthogonality, and "almost everywhere" means simply in $(0,1)$ everywhere with the exception of at most a set of measure zero in the sense of Lebesgue.

Let $\{a_n\}$ be a given sequence of real numbers, and denote $\{m_n\}$ a fixed strictly increasing sequence of natural numbers. We put

$$A_n := \left\{a_{m_{n-1}}^2 + \cdots + a_{m_n}^2\right\}^{1/2} \quad (n = 1, 2, \ldots).$$

In a joint paper with L. Csernyák [1] we proved the following result, which is an improvement of a theorem due to K. Tandori [3].

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**Theorem A.** *If* $A_n \geq A_{n+1}$ *and*

$$\sum_{n=2}^{\infty} A_n^2 \log^2 n = \infty, \tag{1.1}$$

*then there exists a uniformly bounded orthonormal system* $\{\psi_n(x)\}$ *such that the* $m_n$*-th partial sums of the series*

$$\sum_{k=1}^{\infty} a_k \, \psi_k(x) \tag{1.2}$$

*are almost everywhere divergent.*

The aim of this note is to extend Theorem A such that, instead of the monotonicity of $\{A_n\}$, only its almost monotonicity is required.

A nonnegative sequence $\mathbf{c} := \{c_n\}$ is called almost monotone nonincreasing if there exists a constant $K := K(\mathbf{c})$, depending on the sequence $\mathbf{c}$ only, such that for all $n \geq m$

$$c_n \leq K \, c_m.$$

If a sequence $\mathbf{c}$ monotone nonincreasing, or almost monotone nonincreasing, we shall use the notations: $\mathbf{c} \in MS$ or $\mathbf{c} \in AMS$, respectively.

## 2. Results

**Theorem.** *Theorem A can be refined such that the condition* $\{A_n\} \in MS$ *is replaced by the assumption* $\{A_n\} \in AMS$.

With regard to a strictly increasing sequence $\mathbf{p} = \{p_n\}$ of natural numbers, we call a summability method $\mathbf{A}$ an $N(\mathbf{p})$ method if the following holds: In order that every orthogonal series $\sum_{n=1}^{\infty} c_n \varphi_n(x)$ with $\sum c_n^2 < \infty$ be summable $\mathbf{A}$ almost everywhere its $p_n$-th partial sums must converge almost everywhere. It is well known that every permanent Toeplitz summation process is an $N(\mathbf{p})$-summability with certain $\{p_n\}$. Our Theorem clearly implies the next result.

**Corollary.** *Let*

$$A_n^2(p) := \sum_{k=p_n+1}^{p_{n+1}} a_n^2.$$

*If* $\{A_n(p)\} \in AMS$ *and*

$$\sum_{n=2}^{\infty} A_n^2(p) \log^2 n = \infty,$$

*then there exists a uniformly bounded orthonormal system* $\{\psi_n(x)\}$ *such that the series* (1.2) *is not summable almost everywhere by some* $N(\mathbf{p})$ *method.*

We mention that, by virtue of a former theorem of the author ([2], Satz II) Theorem and Corollary can be improved both such that the system $\{\psi_n(x)\}$ is replaced by a uniformly bounded polynom system $\{P_n(x)\}$.

## 3. Lemmas

We shall use the following two lemmas. The first lemma is proved implicitly in [1].

**Lemma 1.** *Under the assumptions of Theorem A there exist an index-sequence* $N_0 < N_1 \cdots < N_m < \dots$, *a uniformly bounded orthonormal system* $\{\psi_n(x)\}$ *and a sequence of simple sets* $H_k$ *(*$H_k$ *is the union of finite intervals) such that*

(i) *for every* $x \in H_k$ *there is some* $n_k(x) \in \mathbb{N}$ *such that*

$$\left| \sum_{i=N_k}^{N_k+n_k(x)} \sum_{\ell=m_{i-1}+1}^{m_i} a_\ell \, \psi_\ell(x) \right| \geq D, \quad (k = 1, 2, \dots), \tag{3.1}$$

*where* $D$ *is a positive constant, and the sums*

$$s_i(x) := \sum_{\ell=m_{i-1}+1}^{m_i} a_\ell \, \psi_\ell(x) \quad (i = N_k, \dots, N_k + n_k(x)) \tag{3.2}$$

*have equal signs,*

(ii) *the sets* $H_k$ $(k = 0, 1, \dots)$ *are stochastically independent and*

$$\sum_{k=0}^{\infty} \mu(H_k) = \infty, \tag{3.3}$$

*where* $\mu(H)$ *denotes the Lebesgue measure of* $H$*.*

**Lemma 2.** *If* $\mathbf{c} := \{c_n\} \in AMS$ *and* $\gamma_n := \sup_{k\geq n} c_k$*, then*

$$c_n \leq \gamma_n \leq K(\mathbf{c})c_n \tag{3.4}$$

*holds.*

Proof. By $\mathbf{c} \in AMS$

$$\gamma_n \leq \sup_{k\geq n} K(\mathbf{c})c_n = K(\mathbf{c})c_n,$$

this and the definition of $\gamma_n$ clearly yield (3.4).

## 4. Proof of Theorem

Denote $\alpha := \{A_n\}$ and let $A_n^* := \sup_{k\geq n} A_k$. Then clearly $\{A_n^*\} \in MS$, and by Lemma 2

$$A_n \leq A_n^* \leq K(\alpha)A_n \tag{4.1}$$

holds. Thus, for $\rho_n := \frac{A_n^*}{A_n}$ we have

$$1 \leq \rho_n \leq K(\alpha). \tag{4.2}$$

Moreover by (1.1) and (4.1)

$$\sum_{n=2}^{\infty} (A_n^*)^2 \log^2 n = \infty. \tag{4.3}$$

Next let us define a new sequence $\{a_k^*\}$ as follows:

$$a_k^* := \rho_n a_k \quad \text{if } m_{n-1} < k \leq m_n, \quad n = 1, 2, \dots$$

Then

$$\sum_{k=m_{n-1}+1}^{m_n} (a_k^*)^2 = \rho_n^2 \sum_{k=m_{n-1}+1}^{m_n} a_k^2 = \rho_n^2 A_n^2 = (A_n^*)^2.$$

Therefore we can apply Lemma 1 with the sequence $\{a_k^*\}$ and obtain that (3.1) holds with $\{a_k^*\}$ in place of $\{a_k\}$, furthermore the sums in (3.2) with $\{a_\ell^*\}$ have equal signs for all $i$. Using these facts we get that

$$\left|\sum_{i=N_k}^{N_k+n_k(x)} \sum_{\ell=m_{i-1}+1}^{m_i} a_\ell^*\, \psi_\ell(x)\right| = \left|\sum_{i=N_k}^{N_k+n_k(x)} \rho_i \sum_{\ell=m_{i-1}+1}^{m_i} a_\ell\, \psi_\ell(x)\right| \geq D,$$

whence, by (4.2),

$$\left|\sum_{i=N_k}^{N_k+n_k(x)} \sum_{\ell=m_{i-1}+1}^{m_i} a_\ell\, \psi_\ell(x)\right| \geq \frac{D}{K(\alpha)} > 0 \tag{4.4}$$

follows.

In virtue of (3.3) and the Borel–Cantelli lemma we get that

$$\mu\Big(\overline{\lim_{k\to\infty}} H_k\Big) = 1,$$

that is, almost every $x \in (0,1)$ belongs to $\overline{\lim}_{k\to\infty} H_k$. Thus (4.3) holds almost everywhere for infinite $k$.

Consequently the $m_n$-th partial sums of the series (1.2) are almost everywhere divergent.

This completes the proof. □

## References

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LÁSZLÓ LEINDLER
BOLYAI INSTITUTE
UNIVERSITY OF SZEGED
ARADI VÉRTANÚK TERE 1
H-6720 SZEGED
HUNGARY

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