

Perturbation theorems for upper and lower semi-Fredholm linear relations

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Abstract. In the setting of Banach spaces it follows from the classical perturbation results that a closed semi-Fredholm operator retains its index under strictly singular with strictly singular adjoint or small perturbations. In this paper we obtain generalisations to multivalued linear operators in normed spaces of the results of the type mentioned above. Examples are exhibited proving that these results are not valid in arbitrary normed spaces.

1. Introduction

Several authors [6], [9], [13]–[15], [18], etc. have studied the stability of the index of a closed upper or lower semi-Fredholm operator under either strictly singular or small perturbations. However, in many cases the perturbation theorems are given for the case when the normed spaces are complete. It is the purpose of this paper to extend this study to linear relations in normed spaces not necessarily complete.

Notations. We adhered to the notation and terminology of the book [5]: X, Y, \dots are normed spaces, B_X the closed unit ball of X , X' the dual space of X . If $M \subseteq X$ and $N \subseteq X'$ are subspaces, then $M^\perp = \{x' \in X' : x'(x) = 0 \text{ for all } x \in M\}$, $N^\top = \{x \in X : x'(x) = 0 \text{ for all } x' \in N\}$.

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A *multivalued linear operator* $T : X \rightarrow Y$ is a set valued map such that its graph $G(T) := \{(x, y) \in X \times Y : x \in D(T), y \in Tx\}$ is a subspace of $X \times Y$.

We use the term *linear relation* or simply *relation* to refer to such a multivalued linear operator denoted $T \in LR(X, Y)$. If T maps the points of its domain $D(T)$ to singletons, then T is said to be a *single valued linear operator* or simply *operator*.

Let M be a subspace of $D(T)$. Then the *restricti* $T|_M$ is defined by $G(T|_M) = \{(m, y) : m \in M, y \in Tm\}$. For any subspace M of X such that $D(T) \cap M \neq \emptyset$, we write $T|_{M \cap D(T)} = T|_M$. The *inverse* of T is the linear relation T^{-1} defined by $G(T^{-1}) = \{(y, x) \in Y \times X : (x, y) \in G(T)\}$. If T^{-1} is single valued, then T is called *injective*, that is, T is injective if and only if its *null space* $N(T) := T^{-1}(0) = \{0\}$, T is said to be *surjective* if its *range* $R(T) := TD(T) = Y$. The *completion* of T , denoted by \tilde{T} , is defined by $G(\tilde{T}) := \widetilde{G(T)} \subseteq \tilde{X} \times \tilde{Y}$, where \tilde{X} denotes the completion of X . For $T \in LR(X, Y)$ we define $\alpha(T) := \dim N(T)$, $\beta(T) := \dim Y/R(T)$, $\bar{\beta}(T) := \dim Y/\overline{R(T)}$. The *index* (respectively, *reduced index*) of T is defined as $i(T) = \alpha(T) - \beta(T)$ (respectively, $\bar{i}(T) = \alpha(T) - \bar{\beta}(T)$) provided $\alpha(T)$ and $\beta(T)$ (respectively, $\alpha(T)$ and $\bar{\beta}(T)$) are not both infinite. If $\alpha(T) = \beta(T) = \infty$ (respectively, $\alpha(T) = \bar{\beta}(T) = \infty$) then T is said to have no index (respectively, reduced index).

The *adjoint* or *conjugate* T' of T is defined by $G(T') := G(-T^{-1})^\perp \subseteq Y' \times X'$, where $\langle (y, x), (y', x') \rangle := \langle x, x' \rangle + \langle y, y' \rangle$. This means that $(y', x') \in G(T')$ if and only if $y'(y) - x'(x) = 0$ for all $(x, y) \in G(T)$.

For a given closed subspace E of X let Q_E^X or simply Q_E denote the natural quotient map from X onto X/E . We shall denote $Q_{T(0)}^Y$ by Q_T . Clearly $Q_T T$ is single valued. For $x \in D(T)$, $\|Tx\| := \|Q_T Tx\|$ and the *norm* of T is defined by $\|T\| := \|Q_T T\|$.

A linear relation $T \in LR(X, Y)$ is said to be *closed* if its graph is a closed subspace, *continuous* if for each neighbourhood V in $R(T)$, $T^{-1}(V)$ is a neighbourhood in $D(T)$ equivalently $\|T\| < \infty$, *open* if T^{-1} is continuous equivalently $\gamma(T) > 0$ where $\gamma(T)$ is the *minimum modulus* of T defined by $\gamma(T) := \sup\{\lambda \geq 0 : \lambda d(x, N(T)) \leq \|Tx\| \text{ for } x \in D(T)\}$, *compact* if $\overline{Q_T T B_{D(T)}}$ is compact, *precompact* if $Q_T T B_{D(T)}$ is totally bounded, *strictly singular* if there is no infinite dimensional subspace M of $D(T)$ for

which $T|_M$ is injective and open, F_+ if there exists a finite codimensional subspace E of X such that $T|_{E \cap D(T)}$ is injective and open and T is called F_- if its adjoint is F_+ .

Continuous everywhere defined linear operators are referred to as *bounded operators*.

Linear relations were introduced in functional analysis by J. VON NEUMANN [17], motivated by the need to consider non-densely defined linear differential operators which are considered by CODDINGTON [3], CODDINGTON and DIJKSMA [4], among others. Problems in optimisation and control also led to the study of set valued maps and differential inclusions (see, for example, AUBIN and CELLINA [1], CLARKE [2], among others). Studies of convex processes, tangent cones, . . . , form part of the theory of convex analysis developed to deal with non-smooth problems in viability and control theory, for example.

Other recent works on multivalued mappings include the treatise on partial differential relations by GROMOV [11], the application of multivalued methods to solving differential equations by FAVINI and YAGI [7], and the book “Multivalued Linear Operators” by CROSS [5]. This book has been essential to the development of this paper.

2. Perturbation theorems

Recall that a closed operator T is called *upper semi-Fredholm* if $\gamma(T) > 0$, $R(T)$ is closed and $\dim N(T) < \infty$; *lower semi-Fredholm* if $\gamma(T) > 0$ and $R(T)$ is a closed finite codimensional subspace and T is said to be *semi-Fredholm* if it is upper or lower semi-Fredholm.

These concepts can be generalised naturally to multivalued linear operators as follows

Definition 1. Let $T \in LR(X, Y)$ be closed. We say that T is upper semi-Fredholm if T is open, has closed range and finite dimensional null space; lower semi-Fredholm if it is open and its range is closed and finite codimensional. If T is upper or lower semi-Fredholm we say that T is semi-Fredholm.

The corresponding classes of linear relations will be abbreviated $USF(X, Y)$, $LSF(X, Y)$ and $SF(X, Y)$ respectively.

We first show that in the context of closed linear relations between Banach spaces the class of upper semi-Fredholm (respectively, lower semi-Fredholm) relations coincides with the class of F_+ (respectively, F_-) relations.

Proposition 2. *Let X and Y be complete and let $T \in LR(X, Y)$ be closed. Then*

- (i) T is F_+ if and only if T is upper semi-Fredholm.
- (ii) T is F_- if and only if T is lower semi-Fredholm.

PROOF. Notice that if X and Y are Banach spaces and $T \in LR(X, Y)$ is closed, then we have the following properties:

- (1) $\gamma(T) = \gamma(T')$, by [5, III. 5.3].
- (2) T is open if and only if $R(T)$ is closed, by the Open Mapping Theorem [5, III. 5.4]. Hence $R(T)$ is closed $\Leftrightarrow R(T')$ is closed.
- (3) T is F_+ if and only if T has closed range and finite dimensional null space, by [5, V. 1.7].
 - (i) The necessity is clear from (2) and (3), while the sufficiency follows noting that if $S \in LR(X, Y)$ satisfies $\alpha(T) < \infty$ and $\gamma(S) > 0$, then $S \in F_+$ by [5, V. 5.1].
 - (ii) Applying the above properties (1), (2) and (3) combined with the fact, $\alpha(T') = \overline{\beta}(T)$ (see [5, III. 1.4]), we have that $T \in F_- \Leftrightarrow T' \in F_+ \Leftrightarrow T'$ has closed range and $\dim N(T') < \infty \Leftrightarrow T$ is open and $R(T)$ is a closed finite codimensional subspace $\Leftrightarrow T$ is lower semi-Fredholm. \square

The following example due to LABUSCHAGNE [14, Ex. 21], shows that the assumption of completeness cannot be omitted from Proposition 2.

Example 3. There exists a closed operator $T \in F_+ \cap F_-$ such that $T \notin USF \cup LSF$.

Let $X = Y = c_{oo}$ be the space of all scalar sequences with at most finitely many non-zero coordinates normed by the norm $\|(\alpha_n)\| = \sup\{|\alpha_n|\}$:

$n \in N\}$ and define K as follows:

$$Kx = (-\alpha_1, \alpha_1, \alpha_2/2, \dots, \alpha_n/n, \dots) \quad \text{for every } x = (\alpha_n) \in c_{oo}.$$

Then, by an argument similar to that used in [9, III. 1.7] it may be verified that K is a precompact operator but not compact. Thus $I - K \in F_+ \cap F_-$ by [5, V. 3.2 and V. 5.12]. Moreover, LABUSCHAGNE [14, Ex. 21] proves that $I - K$ is injective, $R(I - K)$ is closed and finite codimensional in Y but $\gamma(I - K) = 0$. In consequence, $I - K \notin USF \cup LSF$.

We are interested in the following question: Let $K, T \in LR(X, Y)$ such that T is semi-Fredholm. Under that conditions do we have $T + K$ is semi-Fredholm with $i(T) = i(T + K)$? It is well known (see, for example, [9, V. 1.6, V. 2.1, V. 2.2]), that if T is a semi-Fredholm operator between Banach spaces, then $T + K$ has the same property and $i(T) = i(T + K)$ whenever K is either bounded, strictly singular having strictly singular adjoint or bounded with sufficiently small norm. Theorems 9, 10 and 18 contain both these perturbation properties in more general form.

We start proving some results that we shall need to obtain the main theorems.

Lemma 4. *Let M be a closed subspace of X , and let $N \subset X$ be a subspace such that $M \subset N$. Then N is closed in X if and only if N/M is closed in X/M .*

PROOF. Follows immediately from the definitions. □

Lemma 5. *Let $T \in LR(X, Y)$ be closed. Then*

- (i) $Q_T T$ is a closed operator and $T(0)$ and $N(T)$ are closed subspaces.
- (ii) $R(T)$ is closed if and only if so is $R(Q_T T)$.
- (iii) $N(T) = N(Q_T T)$, $\gamma(T) = \gamma(Q_T T)$, and $\beta(T) = \beta(Q_T T)$.
- (iv) $R(T') = R((Q_T T)'), \alpha(T') = \bar{\beta}(T)$ and $\alpha(T) \leq \bar{\beta}(T')$ with equality if T is open.

PROOF. (i) According to [5, II. 5.3], T is closed if and only if $Q_T T$ is closed and $T(0)$ is closed. Moreover, it is clear that T is closed if and only if so is its inverse and hence $N(T) := T^{-1}(0)$ is a closed subspace if T is a closed linear relation.

(ii) Combine Lemma 4 with (i).

(iii) That $N(T) = N(Q_T T)$ and $\gamma(T) = \gamma(Q_T T)$ follows from [5, II. 3.4] and (i), while the property $\beta(T) = \beta(Q_T T)$ is a simple consequence of the property $T(0)$ closed and [5, I. 6.10].

(iv) The equality $R(T') = R((Q_T T)')$ holds trivially by [5, III. 1.10]. Finally, as $N(T') = R(T)^\perp$ and $N(T) = R(T')^\top$ by [5, III. 1.4], we obtain that $\alpha(T') = \dim N(T') = \dim R(T)^\perp = \dim(Y/\overline{R(T)'}) = \dim Y/\overline{R(T)} = \overline{\beta(T)}$ and $\alpha(T) = \dim N(T) = \dim N(T') = \dim X'/N(T)^\perp \leq \dim X'/\overline{R(T')} = \overline{\beta(T')}$. (Note that here $N(T)$ is closed since T is closed.) But if T is open, then we have that $N(T)^\perp = R(T')$ by [5, III. 4.6] and thus $\alpha(T) = \beta(T')$. \square

Throughout this paper $P(X)$ denotes the family of all closed finite codimensional subspaces of X .

The next Proposition was proved in [20, 5. 1.8]. We include a proof for the convenience of the reader.

Proposition 6. *Let $T \in LR(X, Y)$ with finite dimensional null space. Then the following properties are equivalent:*

- (i) T is open.
- (ii) For every $M \in P(D(T))$, TM is closed in $R(T)$ and $T|_M$ is open.
- (iii) There exists $M \in P(D(T))$ such that TM is closed in $R(T)$ and $T|_M$ is open.

PROOF. (i) \implies (ii) Let $M \in P(D(T))$. Then $M + N(T)$ is closed since $\alpha(T) < \infty$. Now suppose $Tx_k \rightarrow Tx$ for $(x_k) \subset M$. Since T is open, we have $d(x - x_k, N(T)) \rightarrow 0$. Thus there exists $(n_k) \subset N(T)$ such that $x_k + n_k \rightarrow x$. Since $M + N(T)$ is closed, $x \in M + N(T)$ and hence, $Tx \in T(M)$, that is, $R(T|_M)$ is closed in $R(T)$.

Without loss of generality we may suppose $M \cap N(T) = \{0\}$. Then if P is a continuous single valued projection defined on $M + N(T)$ with null space $N(T)$, we have $Px_k = P(x_k + n_k) \rightarrow Px$. Furthermore, $TPx = Tx$ since $(I - P)x \in N(T)$, and hence, $(T|_M)^{-1}Tx_k = x_k \rightarrow x$. Since $(Tx_k) \subset TM$ was arbitrary, it follows that $T|_M$ is open.

The implication (ii) \implies (iii) is obvious.

(iii) \implies (i) Suppose that (iii) is true and $\alpha(T) < \infty$. Then $N(T) + M \in P(D(T))$ and hence there exists a finite dimensional subspace F of $D(T)$ such that $M + F + N(T) = D(T)$ and $F \cap (N(T) + M) = \{0\}$. Furthermore, we have $\dim R(T)/TM \leq \dim D(T)/M < \infty$. Thus, $R(T) = TM + F_2$, where F_2 is finite dimensional and $(T|_{M+F})^{-1}$ is single valued with domain $TM + F_2$. Now, we observe that $((T|_{M+F})^{-1})|_{F_2}$ is continuous since $\dim F_2 < \infty$, and $((T|_{M+F})^{-1})|_{TM} = (T|_M)^{-1}$ is continuous since $T|_M$ is open. Thus, $(T|_{M+F})^{-1}$ is continuous equivalently $T|_{M+F}$ is open. Hence $0 < \gamma(T|_{M+F}) = \inf\{\|Tx\| / \|x\| : x \in M + F\} \leq \inf\{\|Tx\| / d(x, N(T)) : x \in M + F\} = \inf\{\|Tx\| / d(x, N(T)) : x \in D(T) \setminus N(T)\} = \gamma(T)$. \square

Let us recall some perturbation results for linear relations.

Proposition 7. *Let $T \in F_+(X, Y)$ and let $S \in LR(X, Y)$ be strictly singular with $S(0) \subset \overline{T(0)}$ (for example, if S is a compact single valued map). Then $T + S \in F_+(X, Y)$.*

PROOF. See [5, V. 3.2]. \square

Proposition 8. *Let $\gamma(T) > 0$ and let S satisfy $S(0) \subset \overline{T(0)}$, $D(S) \supset D(T)$ and $\|S\| < \gamma(T)$. Then we have.*

- (i) $\alpha(T + S) \leq \alpha(T)$ and $\overline{\beta}(T + S) \leq \overline{\beta}(T)$.
- (ii) If T is injective, then $T + S$ is open and $\overline{\beta}(T + S) \leq \overline{\beta}(T)$.

PROOF. See [5, III. 7.4, III. 7.6]. \square

Our next Theorems 9 and 10 show that the property, upper semi-Fredholm, is stable under compact perturbation.

Theorem 9. *Let $T \in USF(X, Y)$. If $K \in LR(X, Y)$ is compact single valued with $\overline{D(T)} \subset D(K)$, then $T + K \in USF(X, Y)$ and $i(T) = i(T + K)$.*

PROOF. We first prove that $T + K$ is closed. Assume that T is single valued. Let (x_n) be a sequence in $D(T + K) = D(T) \cap D(K) = D(T)$ (as $\overline{D(T)} \subset D(K)$) such that $x_n \rightarrow x$ and $(T + K)x_n \rightarrow y \in Y$. Then $x \in \overline{D(T)} \subset D(K)$ and $Kx_n \rightarrow Kx$ since K is continuous. Thus $Tx_n \rightarrow Kx - y$ and, since T is closed, $x \in D(T)$ and $Tx = Kx - y$, that is, $T + K$ is closed. For the general case, if T is a closed linear relation then by Lemma 5 so is the operator $Q_T T$. Hence, from what has been proved for

the single valued case, $Q_{T+K}(T+K) = Q_T T + Q_T K$ is closed and again by Lemma 5 we deduce that $T+K$ is closed, as desired.

Since $\alpha(T) < \infty$ and $\gamma(T) > 0$ it follows from [5, V. 5.1] that T is F_+ and so by [5, V. 1.6] there exists $M \in P(D(T))$ for which $T|_M$ is injective and open. Now, the Proposition 6 assures that TM is closed in $R(T)$ and also in Y since $R(T)$ is closed. As $T \in F_+$ and K is strictly singular, $T+K \in F_+$ by Proposition 7.

Clearly $K|_M$ is precompact or equivalently $\inf\{\|K|_Z\| : Z \in P(M)\} = 0$ by [5, V. 2.2]. Thus, there exists $N \in P(M)$ such that $\|K|_N\| < \gamma(T|_M)$. But since the relations $T|_M$ and $T|_{N(T|_M)+N}$ have the same null space, we obtain that $\gamma(T|_M) \leq \gamma(T|_N)$. Again by Proposition 6, TN is in fact closed and as $T|_N$ is injective and open, by virtue of Proposition 8 we obtain that $\alpha((T+K)|_N) = 0$, $\gamma((T+K)|_N) > 0$ and $\bar{\beta}((T+K)|_N) = \bar{\beta}(T|_N)$. If we can now show that $(T+K)N$ is closed, $T+K$ would be open by Proposition 6. Since $T|_N(0) = (T+K)|_N(0)$ we have trivially that $Q_{T|_N} = Q_{(T+K)|_N}$ and $Q_{T|_N}K|_N$ is a compact operator. Moreover, from Lemma 5 it follows that $N(T|_N) = N(Q_{T|_N}T|_N) = N(Q_{T|_N}(T+K)|_N) = \{0\}$, $\gamma(T|_N) = \gamma(Q_{T|_N}T|_N) > 0$, $\gamma((T+K)|_N) = \gamma(Q_{T|_N}(T+K)|_N) > 0$. Hence, the operator $Q_{T|_N}(T+K)|_N$ has a continuous inverse. In consequence, if $(T+K)n_k \rightarrow y$ for $(n_k) \subset N$ we have that (n_k) is a Cauchy sequence. By the compactness of $Q_{T|_N}K|_N$, $(Q_{T|_N}K|_N n_k)$ has a convergent subsequence assumed to be itself. Suppose $Q_{T|_N}K|_N n_k \rightarrow z$. Then $Q_{T|_N}T|_N n_k \rightarrow Q_{T|_N}y - z$. As $T|_N N$ and $Q_{T|_N}T|_N N$ are closed, it follows that there exists an element n in N for which $Q_{T|_N}y - z = Q_{T|_N}T|_N n$. From the continuity of the operator $(Q_{T|_N}T|_N)^{-1}$ we infer that $Q_{T|_N}T|_N n_k \rightarrow Q_{T|_N}T|_N n$ implies $n_k \rightarrow n$. Consequently $Q_{T|_N}y = (Q_{T|_N}y - z) + z \in Q_{T|_N}(T+K)|_N N$. Thus $Q_{T|_N}(T+K)|_N N$ is closed and by Lemma 5, $(T+K)|_N N$ is closed, thereby establishing the openness of $T+K$.

It only remains to verify that $T+K$ has closed range and $i(T) = i(T+K)$. For this, we note that it is easy to see that $\dim R(T+K)/R((T+K)|_N) \leq \dim D(T)/N < \infty$ and hence $R(T+K) = (T+K)N + D$ where D is finite dimensional and $(T+K)N$ is closed. Therefore $R(T+K)$ is closed. In [5, V. 15.5], it is shown that if $U, V \in LR(X, Y)$ such that U is an extension of V with $\dim D(U)/D(V) = m < \infty$, then $i(U) = m + i(V)$ if V has an index. Combining this property with $i(T|_N) = i((T+K)|_N)$

and $\dim D(T+K)/D((T+K)|_N) = \dim D(T)/N := m < \infty$ yields $i(T) = m + i(T|_N) = m + i((T+K)|_N) = i(T+K)$, as required. \square

We do not know if the same Theorem 9 is true in the case where $T \in LSF(X, Y)$ with infinite dimensional null space.

Theorem 10. *Let $T \in USF(X, Y)$ and let Y be complete. If $K \in LR(X, Y)$ is compact such that $K(0) \subset T(0)$ and $D(K) \supset \overline{D(T)}$, then $T + K \in USF(X, Y)$ with $i(T) = i(T + K)$.*

PROOF. $T + K$ is closed. Indeed, suppose that T and K are single valued. Let (x_n) be a sequence in $D(T+K) = D(T)$ such that $x_n \rightarrow x$ and $(T+K)x_n \rightarrow y$. Then $\|T(x_n - x_m)\| \leq \|(T+K)(x_n - x_m)\| + \|K\| \|x_n - x_m\|$. Thus (Tx_n) is a Cauchy sequence in Y and, since Y is complete $Tx_n \rightarrow z$ for some $z \in Y$. Since T is closed and K is continuous we obtain that $x \in D(T)$ and $(T + K)x = y$, that is, $T + K$ is closed. Turning to the general case, it follows from by hypothesis and Lemma 5 that $(T+K)(0) = T(0)$ (so $Q_{T+K} = Q_T$), $Q_T T$ is a closed single valued and $Q_T K$ is a compact single valued. Applying the first part of the proof, we have that $Q_{T+K}(T + K) = Q_T T + Q_T K$ is closed and again by Lemma 5 it follows that $T + K$ is closed.

The rest of the proof is now somewhat similar to that of Theorem 9. \square

The next example illustrates that the condition, $K(0) \subset T(0)$, is necessary in Theorem 10.

Example 11. Let X be an infinite dimensional normed space and let $K \in LR(X)$ be defined by $G(K) = X \times X$. Then K is compact, the identity operator I_X is clearly upper semi-Fredholm, while $I_X + K = K$ is not upper semi-Fredholm.

The Example 3 shows that the property, semi-Fredholm in arbitrary normed spaces, need not be preserved under strictly singular perturbations.

The completion \tilde{T} of $T \in LR(X, Y)$ is also a linear relation between Banach spaces and we obtain

Proposition 12. *Let $T \in SF(X, Y)$. Then $\tilde{T} \in SF(\tilde{X}, \tilde{Y})$ with $i(T) = i(\tilde{T})$.*

PROOF. The result is a simple consequence of Lemma 5 combined with the equivalences

$T \in F_+ \Leftrightarrow R(\tilde{T})$ is closed and $\alpha(\tilde{T}) < \infty$ [5; V. 7.6]; $T \in F_- \Leftrightarrow \tilde{T}$ has closed range and $\beta(\tilde{T}) < \infty$ [5, V. 5.2]. \square

We note that the proof of Proposition 12 assures that the completion of a closed $T \in F_+ \cup F_-$ is a semi-Fredholm linear relation. But, in this case $i(T)$ and $i(\tilde{T})$ need not be equal, as we see from the next example.

Example 13. There exists a bounded operator $T \in F_+ \cap F_-$ such that $i(T) \neq i(\tilde{T})$.

Let C denote the Cesàro matrix. This is a lower triangular matrix whose non-zero entries in the n -th row are equal to n^{-1} , and it defines a bounded operator C_p in l_p for $1 < p < \infty$. If $1/p + 1/q = 1$ then by [19] we have that the spectrum of C_p is given by

$$\sigma(C_p) = \{\lambda \in \mathbb{C} : |\lambda - q/2| \leq q/2\}. \quad (*)$$

Now, we denote $X_n := (l_{n/n-1}, \|\cdot\|_2)$ for every $n > 1$, where $\|\cdot\|_2$ is the usual norm in l_2 . If we denote by $C^{(n)}$ the operator induced by C in X_n , then $C^{(n)}$ is a bounded operator in X_n and $(*)$ tell us that, for $n > 1$, we have $\sigma(C^{(n)}) = \{\lambda \in \mathbb{C} : |\lambda - n/2| \leq n/2\}$ and it is proved in [10] that if $|\lambda - n/2| < n/2$ then $\lambda I - C^{(n)}$ is injective with $\beta(\lambda I - C^{(n)}) = 1$.

In particular, we have that $3I - C^{(4)}$ is injective, $\beta(3I - C^{(4)}) = 1$ and its completion $3I - C^{(2)}$ is invertible. Combining this fact with Proposition 12 we obtain that $3I - C^{(4)}$ is not semi-Fredholm, but $3I - C^{(4)} \in F_+ \cap F_-$ by [5, V. 5.2 and V. 7.6].

According to Harte [12] a bounded operator T is called *almost upper semi-Fredholm* if T is open and $\dim N(T) < \infty$ and T is called *almost lower semi-Fredholm* if T is almost open (that is, $\overline{TB_X} \supset \lambda B_{R(T)}$ for some $\lambda > 0$) and $\overline{\beta}(T) < \infty$. These notions can be generalised naturally to arbitrary linear relations.

Definition 14. Let $T \in LR(X, Y)$. We say that T is almost upper semi-Fredholm if $\alpha(T) < \infty$ and $\gamma(T) > 0$ and T is called almost lower semi-Fredholm if it is almost open and $\overline{\beta}(T) < \infty$.

Here we characterise these classes in terms of the completions of the linear relations.

Recall that a linear relation T is said to be *completely closable* if $\tilde{T}x = Tx$ for all $x \in D(T)$ [5, III. 4.1].

Proposition 15. *Let $T \in LR(X, Y)$. Then*

- (i) *If T is completely closable, then T is almost upper semi-Fredholm if and only if \tilde{T} has closed range, $\dim N(\tilde{T}) < \infty$ and $N(T) = N(\tilde{T})$.*
- (ii) *T is almost lower semi-Fredholm if and only if \tilde{T} has closed finite codimensional range.*

PROOF. (i) It is enough to combine the following properties:

- (a) $\gamma(T) \leq \gamma(\tilde{T})$ with equality holding if $N(T)$ is dense in $N(\tilde{T})$ [5, II. 5.9].
- (b) $N(T)$ is dense in $N(\tilde{T})$ whenever $\gamma(T) > 0$. Indeed, suppose that T is open. Then, since $0 < \gamma(T) \leq \gamma(\tilde{T})$ by (a), it follows from [5, III. 1.3, III. 4.6] that $N(T)^\perp = R(T') = R(\tilde{T}') = N(\tilde{T})^\perp$ and so $\overline{N(T)} = N(\tilde{T})$.
- (c) $\gamma(T) > 0$ and $\alpha(T) < \infty \implies T \in F_+$ ([5, V. 5.1]) $\Leftrightarrow \tilde{T}$ has closed range and finite dimensional null space ([5, V. 1.7]).

(ii) Follows immediately from the equivalences

T almost open $\Leftrightarrow T'$ open ([5, III. 5.2]); $T \in F_- \Leftrightarrow \gamma(T') > 0$ and $\overline{\beta}(T) < \infty \Leftrightarrow \tilde{T}$ has closed finite codimensional range ([5, V. 5.2]) \square

The example 3 shows that the condition, $N(T) = N(\tilde{T})$, in Proposition 15 is not superfluous.

Finally, we investigate the stability of semi-Fredholm relations under small perturbation, as well as the behaviour of the index under perturbation. The following example due to MENNICKEN and SAGRALOFF [15] shows that the class of semi-Fredholm operators in arbitrary normed spaces is not stable under small perturbation.

Example 16. Let $T : c_{oo} \rightarrow c_{oo}$ be the left-shift operator. Then T is semi-Fredholm and for any $0 < |\lambda| < \gamma(T)$, $\lambda I + T$ is not open and $\bar{i}(T) \neq \bar{i}(\lambda I + T)$.

Indeed, notice that $\alpha(T) = \gamma(T) = 1$, $\beta(T) = 0$ and perturbing T by any λI with $0 < |\lambda| < 1$, it follows that $\gamma(\lambda I + T) = 0$, $\alpha(\lambda I + T) = \overline{\beta}(\lambda I + T) = 0$.

Proposition 17. *Let $T \in F_+ \cup F_-$ and suppose that $K \in LR(X, Y)$ satisfy $D(K) \supset \overline{D(T)}$, $K(0) \subset \overline{T(0)}$ and $\|K\| < \gamma(\tilde{T})$. Then $T + K \in F_+ \cup F_-$ and $i(\tilde{T}) = i(\tilde{T} + K)$.*

PROOF. This result was proved by WILCOX [20, 6. 1.1]. The proof is along the lines of the proof of the analogous result provided in [5, V. 15.6] for the case when K is single valued. \square

Theorem 18. *Let X and Y be complete and $T \in SF(X, Y)$. Then for any $K \in LR(X, Y)$ with $K(0) \subset T(0)$, $D(K) \supset \overline{D(T)}$ and $\|K\| < 1/\gamma(T)$ we have that $T + K \in SF(X, Y)$ and $i(T) = i(T + K)$.*

PROOF. It is enough to observe that $T + K$ is closed (proceeding as in Theorem 10) and apply Propositions 2 and 17. \square

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References

- [1] J. P. AUBIN and A. CELLINA, *Differential Inclusions*, Springer-Verlag, New York, 1984.
- [2] F. H. CLARKE, *Optimization and Nonsmooth Analysis*, Wiley-Interscience Publication, Wiley and Sons, Toronto, 1983.
- [3] E. A. CODDINGTON, *Multivalued operators and boundary value problems*, Lecture Notes in Math., Vol. **183**, Springer-Verlag, Berlin, 1971.
- [4] E. A. CODDINGTON and A. DIJKSMA, *Selfadjoint subspaces and eigenfunction expansions for ordinary differential subspaces*, *J. Differential Equations* **20** (2) (1976), 473–526.
- [5] R. W. CROSS, *Multivalued Linear Operators*, Marcel Dekker, New York, 1998.
- [6] D. VAN DULTS, *Perturbation theory and strictly singular operators in locally convex spaces*, *Studia Math.* **38** (1970), 341–372.
- [7] A. FAVINI and A. YAGI, *Multivalued linear operators and degenerate evolution equations*, *Ann. Mat., Pura. Appl. (4)* **163** (1993), 353–384.
- [8] C. FOIAS, *Invariant para-Closed Subspaces*, *Indiana Univ. Math. J.* **21** (1972), 887–906.
- [9] S. GOLDBERG, *Unbounded Linear Operators*, McGraw-Hill, New York, 1966.
- [10] M. GONZALEZ, *The fine spectrum of the Cesàro operator in l_p ($1 < p < \infty$)*, *Archiv Math.* **46** (1985), 355–358.
- [11] M. GROMOV, *Partial Differential Relations*, Springer-Verlag, Berlin, 1986.

- [12] R. HARTE, Invertibility and Singularity for Bounded Linear Operators, *Marcel Dekker, New York*, 1988.
- [13] T. KATO, Perturbation theory for nullity, deficiency and other quantities of linear operators, *J. Anal. Math.* **6** (1958), 273–322.
- [14] L. E. LABUSCHAGNE, The Perturbation of Relatively Open operators with Reduced Index, *Math. Proc. Cambridge Phil. Soc.* **112** (1992), 385–402.
- [15] R. MENNICKEN and B. SAGRALOFF, Störungstheoretische Untersuchungen Über Semi-Fredholmpaare und-operatoren in lokalkonvexen Vektorräumen, I, *J. Reine Angew. Math.* **280** (1976), 1–36.
- [16] R. MENNICKEN and B. SAGRALOFF, Störungstheoretische Untersuchungen Über Semi-Fredholmpaare und-operatoren in lokalkonvexen Vektorräumen, II, *J. Reine Angew. Math.* **303** (1978), 389–436.
- [17] J. VON NEUMANN, Functional Operators, Vol. 2: The Geometry of Orthogonal Spaces, Ann. Math. Stud., Vol. 22, *Princeton U. P.*, 1950.
- [18] A. PIETSCH, Homomorphismen in lokalkonvexen Vektorräumen, *Math. Nach.* **22** (1960), 162–174.
- [19] B. E. RHOADES, Generalized Hausdorff matrices bounded on l_p and c , *Acta Sci. Math.* **43** (1981), 333–345.
- [20] D. WILCOX, Multivalued Semi-Fredholm Operators in Normed Linear Spaces, Ph.D. Thesis, *Univ. Cape Town*, 2001.

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