Publ. Math. Debrecen
65/1-2 (2004), 193–203

On the paths of B-spline curves obtained by the modification of a knot

By MIKLÓS HOFFMANN (Eger) and IMRE JUHÁSZ (Miskolc)

Abstract. When a knot of a B-spline curve is modified its points move along curves called paths. We extend these paths and prove that points of these extended paths tend to control points. We also show that the family of paths possess an envelope which is a B-spline curve.

1. Introduction

B-spline curves and their rational generalizations are frequently applied curve description methods in approximation theory and in curve and surface design. These polynomial curves are defined piecewise over a closed interval of $\mathbb{R}$ and can be described as convex combinations of a sequence of predefined points in $\mathbb{R}^d$, $(d > 1)$ by the help of B-spline basis functions ([3], [13]). The basic definitions (see e.g. in [4]) are the following.

Definition 1. The recursive function $N_j^k(u)$ given by the equations

$$N_j^1(u) = \begin{cases} 1 & \text{if } u_j \leq u < u_{j+1}, \\ 0 & \text{otherwise} \end{cases}$$
$$N_j^k(u) = \frac{u - u_j}{u_{j+k-1} - u_j} N_j^{k-1}(u) + \frac{u_{j+k} - u}{u_{j+k} - u_{j+1}} N_{j+1}^{k-1}(u) \tag{1}$$

Mathematics Subject Classification: 68U05.
Key words and phrases: knot modification, B-spline curve, envelope.

is called normalized B-spline basis function of order $k > 1$ (degree $k - 1$). The numbers $u_j \le u_{j+1} \in \mathbb{R}$ are called knot values or simply knots, and $0/0 \doteq 0$ by definition. The $[u_j, u_{j+1})$ interval is called the j^{th} span.

Definition 2. The curve $\mathbf{s}(u)$ defined by

$$\mathbf{s}(u) = \sum_{l=0}^{n} N_l^k(u)\mathbf{d}_l, \quad u \in [u_{k-1}, u_{n+1}] \tag{2}$$

is called B-spline curve of order k (degree $k - 1$), $(1 < k \le n + 1)$, where $N_l^k(u)$ is the l^{th} normalized B-spline basis function of order k. For the evaluation of these basis functions the knots $u_0, u_1, \ldots, u_{n+k}$ are necessary. Points $\mathbf{d}_l$ are called control points or de Boor points, while the polygon formed by these points is called the control polygon.

Definition 3. The curve $\mathbf{s}(u)$ in $\mathbb{R}^d$, $(d > 1)$ defined by the formula

$$\mathbf{s}(u) = \sum_{l=0}^{n} w_l \mathbf{d}_l \frac{N_l^k(u)}{\sum_{j=0}^{n} w_j N_j^k(u)}, \quad u \in [u_{k-1}, u_{n+1}]$$

is called rational B-spline (or NURBS) curve of order k (degree $k - 1$), $(1 < k \le n + 1)$, where $N_l^k(u)$ is the l^{th} normalized B-spline basis function of order k, for the evaluation of which the knots $u_0, u_1, \ldots, u_{n+k}$ are needed. Points $\mathbf{d}_l \in \mathbb{R}^d$ are called control or de Boor points, and the scalars $w_j \in \mathbb{R}$, $(w_j \ge 0)$ are called weights.

Thus the B-spline curve is uniquely given by its order, control points and knot values, while in terms of rational B-spline curves, weights of the control points have to be specified in addition. An existing curve can be modified by altering these data. By changing the order one can obtain discrete positions of the curve, while the alteration of the other defining data yields a one-parameter family of B-spline curves, hence only the latter methods can be applied in practice for constrained shape modification.

If a control point, a weight or a knot value of a B-spline curve is modified, points of the curve move along curves which are called *paths*. The description of these paths is a key issue for practical curve modification techniques and has been done for control point and weight alteration by Piegl [9]. When a control point of a B-spline curve is translated, these paths will become line segments that are parallel to the translation vector and the length of which varies point by point, i.e., a functional translation of the curve is obtained (see [10]). Modifying the weight of a control point of the curve, the paths will also be line segments, an endpoint of which is the control point itself (perspective functional transformation, c.f. [9]). These results have been applied in several shape modification methods, such as in [9], [1], [12] and [5].

The j^{th} arc of the B-spline curve of Definition 2 is of the form

$$\mathbf{s}_j(u) = \sum_{l=j-k+1}^{j} \mathbf{d}_l N_l^k(u), \quad u \in [u_j, u_{j+1}), \ (j = k-1, \ldots, n).$$

The modification of the knot value u_i (between its two neighboring knots) of the B-spline curve $\mathbf{s}(u)$ alters the shape of the arcs $\mathbf{s}_j(u)$, $(j = i - k + 1, i - k + 2, \ldots, i + k - 2$, since this knot modification affects only the basis functions $N_{i-k}^k(u), N_{i-k+1}^k(u), \ldots, N_i^k(u)$. In a recent paper ([6]) the authors proved that by altering a knot value of the curve the paths become rational curves, the order of which varies span by span. The point $\mathbf{s}_j(\widetilde{u})$, $\widetilde{u} \in [u_j, u_{j+1})$ moves along the path

$$\mathbf{s}_j(\widetilde{u}, u_i) = \sum_{l=j-k+1}^{j} \mathbf{d}_l N_l^k(\widetilde{u}, u_i), \quad u_i \in [u_{i-1}, u_{i+1}]. \tag{3}$$

For these paths the following property holds.

Theorem 1. *Modifying the single multiplicity knot u_i (in $[u_{i-1}, u_{i+1}]$) of the k^{th} order B-spline curve $\mathbf{s}(u)$, the points of the arcs $\mathbf{s}_{i-k+1}(u), \ldots, \mathbf{s}_{i+k-2}(u)$ move along rational curves, the order of which decreases symmetrically from k to 2 as the indices of the arcs get farther from i, i.e., the paths of $\mathbf{s}_{i-l}(u)$ and $\mathbf{s}_{i+l-1}(u)$ are rational curves of order $k - l + 1$ in u_i, $(l = 1, \ldots, k-1)$.*

If the multiplicity of u_i is $m, (0 < m < k)$, i.e., $u_i = u_{i+1} = \cdots = u_{i+m-1}$ then the modified arcs are $\mathbf{s}_{i-k+1}(u), \ldots, \mathbf{s}_i(u), \ldots, \mathbf{s}_{i+k-2}(u), \ldots, \mathbf{s}_{i+k+m-3}(u)$ and the order of paths of $\mathbf{s}_{i-l}(u)$ and $\mathbf{s}_{i+m+l-2}(u)$ is $k-l+1$, $(l = 1, \ldots, k-1)$.

Consider the following B-spline curve of order $k-1$

$$\mathbf{b}(v) = \sum_{l=i-k+1}^{i-1} \mathbf{d}_l N_l^{k-1}(v), \quad v \in [v_{i-1}, v_i], \tag{4}$$

where the control points $\mathbf{d}_l$ coincide with the control points of the original B-spline curve (2), and where its knot values are $v_j = \begin{cases} u_j, & \text{if } j < i \\ u_{j+1}, & \text{if } j \geq i, \end{cases}$ i.e., we omit the i^{th} knot from the original knots u_j.

In [6] the authors proved that the curve (4) is an envelope of the family of curves

$$\mathbf{s}(u, u_i) = \sum_{l=0}^{n} \mathbf{d}_l N_l^k(u, u_i), \quad u \in [u_{k-1}, u_{n+1}], \ k > 2 \tag{5}$$

with the family parameter $u_i \in [u_{i-1}, u_{i+1})$. We use the term envelope in a generalized meaning namely, we call a curve envelope if it has a G^1 contact with each element of a one parameter family of curves (either planar or spatial). In [7] it is also shown that arbitrary order derivatives of (4) and (5) with respect to u differs only in a constant multiplier at $u = u_i$.

In [8] shape modification methods of cubic B-spline curves are presented, based on the simultaneous modification of three consecutive knots. These methods enable the user to specify modifications by intuitive geometric constraints, such as points and tangent lines.

Certainly, paths (3) and the family of curves (5) can also be interpreted as the two sets of isoparametric curves of the surface patch

$$\mathbf{s}(u, u_i) = \sum_{l=0}^{n} \mathbf{d}_l N_l^k(u, u_i), \quad u, u_i \in [u_{i-1}, u_{i+1}]. \tag{6}$$

In [7] we have shown, that curve (4) is not only a singularity of parametrization but it is actually a singular curve of the surface (6) itself.

In this paper we prove further properties of this surface. In Section 2 the extension of paths is discussed, and in Section 3 it is proved that the curve (4) is an envelope of the paths as well. The results are generalized for the modification of knots of higher multiplicity.

2. Extension of paths

Paths obtained by the modification of the knot u_i are relatively short arcs. In order to get more information about their characteristics we extend their domain, i.e., we let u_i be smaller than u_{i-1} and larger than u_{i+1}.

For these extended paths the following holds.

Theorem 2. *Modifying the single multiplicity knot u_i of the B-spline curve $\mathbf{s}(u)$, points of the extended paths of the arcs $\mathbf{s}_{i-1}(u)$ and $\mathbf{s}_i(u)$ tend to the control points $\mathbf{d}_i$ and $\mathbf{d}_{i-k}$ as u_i tends to $-\infty$ and ∞, respectively, i.e.,*

$$\lim_{u_i\to-\infty}\mathbf{s}(u,u_i)=\mathbf{d}_i,\ \lim_{u_i\to\infty}\mathbf{s}(u,u_i)=\mathbf{d}_{i-k},\quad \forall u\in[u_{i-1},u_{i+1}).$$

Proof. We prove the statement for the arc $\mathbf{s}_i(u)$, for $\mathbf{s}_{i-1}(u)$ it can be proved analogously. Denote the original knot values by $\bar{u}_j$, $(j=0,1,\dots,n+k)$. For the description of extended paths we will use the knot values $u_j=\bar{u}_j$, $(j=0,1,\dots,n+k)$ which will differ from the original values only in u_i along the proof. Paths of the points $\mathbf{s}_i(u)$ can be written as

$$\mathbf{s}_i(u,u_i)=\sum_{l=i-k+1}^{i}\mathbf{d}_l N_l^k(u,u_i),\quad u_i\in[\bar{u}_{i-1},\bar{u}_{i+1}),\ u\in[\bar{u}_i,\bar{u}_{i+1}].\quad (7)$$

Limits of this summation are modified when we extend these paths, since if $u_i>\bar{u}_{i+1}$ (i.e., $u_i\to\infty$) then $N_i^k(u)\equiv 0$, $u\in[\bar{u}_i,\bar{u}_{i+1}]$ and $N_{i-k}^k(u)\neq 0$, $u\in[\bar{u}_i,\bar{u}_{i+1}]$, thus these arcs of the extended paths become

$$\mathbf{s}_i(u,u_i)=\sum_{l=i-k}^{i-1}\mathbf{d}_l N_l^k(u,u_i),\quad u_i>\bar{u}_{i+1},\ u\in[\bar{u}_i,\bar{u}_{i+1}].$$

It can easily be seen that

$$N_{i-k}^k(u,u_i)=\frac{(u_i-u)^{k-1}}{\prod_{j=1}^{k-1}(u_i-u_{i-j})}$$

where both the numerator and the denominator are polynomials of degree $k-1$ in u_i, and the main coefficient in both polynomials is 1. This yields $\lim_{u_i\to\infty}N_{i-k}^k(u,u_i)=1$. Now we prove by induction on k that $\lim_{u_i\to\infty}N_j^k(u,u_i)=0$, $(j=i-k+1,\dots,i-1)$.

i) for $k=3$

$$N_{i-1}^3(u,u_i)=\frac{(u-u_{i-1})^2}{(u_{i+1}-u_{i-1})(u_i-u_{i-1})},$$

$$N_{i-2}^{3}(u,u_i)=\frac{(u-u_{i-2})(u_i-u)}{(u_i-u_{i-2})(u_i-u_{i-1})}+\frac{(u_{i+1}-u)(u-u_{i-1})}{(u_{i+1}-u_{i-1})(u_i-u_{i-1})}$$

for which the statement holds.

ii) $k-1\to k$

By Definition 1

$$N_{i-1}^{k}(u,u_i)=\frac{u-u_{i-1}}{u_{i+k-2}-u_{i-1}}N_{i-1}^{k-1}(u,u_i)+\frac{u_{i+k-1}-u}{u_{i+k-1}-u_i}N_{i}^{k-1}(u,u_i)$$

$$N_{i-2}^{k}(u,u_i)=\frac{u-u_{i-2}}{u_{i+k-3}-u_{i-2}}N_{i-2}^{k-1}(u,u_i)+\frac{u_{i+k-2}-u}{u_{i+k-2}-u_{i-1}}N_{i-1}^{k-1}(u,u_i)$$

$$\vdots$$

$$N_{i-k+1}^{k}(u,u_i)=\frac{u-u_{i-k+1}}{u_i-u_{i-k+1}}N_{i-k+1}^{k-1}(u,u_i)+\frac{u_{i+1}-u}{u_{i+1}-u_{i-k+2}}N_{i-k+2}^{k-1}(u,u_i).$$

Therefore, the k^{th} order functions are linear combinations of functions of order $k-1$ where the numerator is independent of u_i and the denominator is linear at most in u_i, i.e., the order of the numerator can not be greater than that of the denominator. Thus, from the assumption for $k-1$, the case of k results too.

If $u_i<\bar{u}_{i-1}$ $(u_i\to-\infty)$ then the limits of the summation (7) are not modified. It is easy to show that in this case

$$N_i^k(u,u_i)=\frac{(u-u_i)^{k-1}}{\prod_{j=1}^{k-1}(u_{i+j}-u_i)}$$

which immediately yields $\lim\limits_{u_i\to-\infty}N_i^k(u,u_i)=1$.

Equalities $\lim\limits_{u_i\to-\infty}N_{i-j}^k(u,u_i)=0$, $(j=1,\ldots,k-1)$ can be proved by induction on k. □

This property of the extended paths is illustrated in Figure 1.

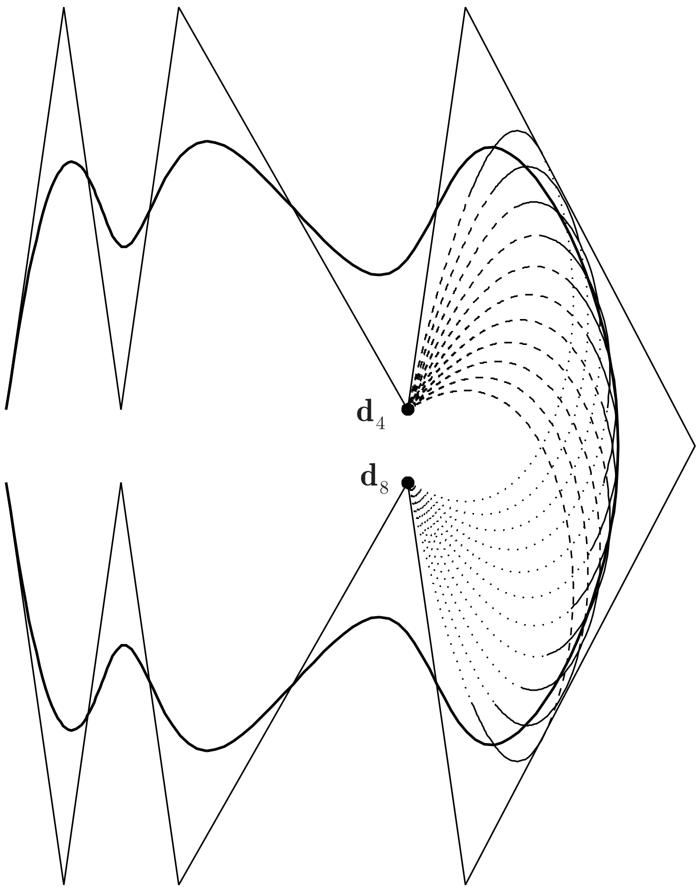

Figure 1. A cubic B-spline curve and its extended paths ($n = 12$, $k = 4$, $i = 8$)

It is easy to show that, by altering a knot of higher multiplicity, Theorem 2 will be of the following form.

Theorem 3. *Altering the knot u_i of multiplicity m, points of the extended paths of the arcs $\mathbf{s}_{i-1}(u), \ldots, \mathbf{s}_{i+m-1}(u)$ satisfy the equalities*

$$\lim_{u_i \to -\infty} \mathbf{s}_{i+j}(u, u_i) = \mathbf{d}_{i+m-1}, \quad \lim_{u_i \to \infty} \mathbf{s}_{i+j}(u, u_i) = \mathbf{d}_{i-k},$$

$$(j = -1, 0, \ldots, m-1), \quad \forall u \in [u_{i+j}, u_{i+j+1}).$$

3. Envelope of the paths

Here we will prove that the curve (4) is also an envelope of the paths of points of the arcs $\mathbf{s}_{i-1}(u)$ and $\mathbf{s}_i(u)$.

Theorem 4. *Curve* (4) *is an envelope of the family of paths*

$$\mathbf{p}(u, u_i) = \sum_{l=0}^{n} \mathbf{d}_l N_l^k(u, u_i), \quad u_i \in [u_{i-1}, u_{i+1}) \tag{8}$$

(where $u \in [u_{i-1}, u_{i+1}]$ is the family parameter) which is obtained by the modification of a single multiplicity knot u_i of curve (2).

Proof. First of all, we show that

$$\left.\frac{\partial N_l^k(u, u_i)}{\partial u_i}\right|_{u=u_i} = -\frac{1}{k-1}\left(\left.\frac{\partial N_l^k(u, u_i)}{\partial u}\right|_{u=u_i}\right) \qquad \forall k \geq 2; \tag{9}$$

$$(l = i-k, \dots, i), \qquad u, u_i \in [u_{i-1}, u_{i+1}].$$

For this purpose we increase the multiplicity of the knot u_i by one in the knot vector $U = \{u_0, u_1, \dots, u_{n+k}\}$. The elements of the new knot vector $\bar{U}$ are defined by

$$\bar{u}_j = \begin{cases} u_j, & j < i \\ u_i, & j = i \\ u_{j-1}, & j > i. \end{cases} \tag{10}$$

Normalized B-spline basis functions on these knots are denoted by $\bar{N}_j^k(u)$. The derivative of $N_l^k(u)$ with respect to u_i at $u = u_i$ is

$$\left.\frac{\partial N_l^k(u, u_i)}{\partial u_i}\right|_{u=u_i} = \begin{cases} 0, & l < i-k \\ \dfrac{\bar{N}_{i-k+1}^k(u_i)}{u_i - u_{i-k+1}}, & l = i-k \\ \dfrac{\bar{N}_{l+1}^k(u_i)}{u_{l+k} - u_{l+1}} - \dfrac{\bar{N}_l^k(u_i)}{u_{l+k-1} - u_l}, & l = i-k+1, \dots, i-1 \\ -\dfrac{\bar{N}_i^k(u_i)}{u_{i+k-1} - u_i}, & l = i \\ 0, & l > i \end{cases}$$

(c.f. [11]). On the basis of Boehm's knot insertion algorithm (see [2]) the relation between $N_l^k(u)$ and $\bar{N}_l^k(u)$ is

$$N_l^k(u) = \begin{cases} \bar{N}_l^k(u), & l < i-k \\ \dfrac{u_i - \bar{u}_l}{\bar{u}_{l+k} - \bar{u}_l}\bar{N}_l^k(u) & \\ \quad + \dfrac{\bar{u}_{l+k+1} - u_i}{\bar{u}_{l+k+1} - \bar{u}_{l+1}}\bar{N}_{l+1}^k(u), & l = i-k, \dots, i-1 \\ \bar{N}_{l+1}^k(u), & l > i-1. \end{cases} \tag{11}$$

For the cases $i-k$ and i property (9) directly follows. For the rest, we consider the derivative of (11) with respect to u at $u = u_i$ which is

$$(k-1)\left(\frac{\bar{N}_l^{k-1}(u_i)}{\bar{u}_{l+k} - \bar{u}_l}\frac{u_i - \bar{u}_l}{\bar{u}_{l+k-1} - \bar{u}_l} - \frac{\bar{N}_{l+2}^{k-1}(u_i)}{\bar{u}_{l+k+1} - \bar{u}_{l+1}}\frac{\bar{u}_{l+k+1} - u_i}{\bar{u}_{l+k+1} - \bar{u}_{l+2}}\right.$$

$$\left. + \bar{N}_{l+1}^{k-1}(u_i)\left(\frac{\bar{u}_{l+k+1} - u_i}{(\bar{u}_{l+k+1} - \bar{u}_{l+1})(\bar{u}_{l+k} - \bar{u}_{l+1})} - \frac{u_i - \bar{u}_l}{(\bar{u}_{l+k} - \bar{u}_l)(\bar{u}_{l+k} - \bar{u}_{l+1})}\right)\right).$$

After some rearrangements and using assignments (10) we obtain (9).

Now, consider an arbitrary member of the family of curves (8) at the family parameter $u = \widetilde{u}$. It is obvious that the point of the path $\mathbf{p}(\widetilde{u}, u_i)$ at $u_i = \widetilde{u}$ coincides with the point of the member $u_i = \widetilde{u}$ of the family (5) at $u = \widetilde{u}$. This means $\mathbf{p}(\widetilde{u}, \widetilde{u}) = \mathbf{s}(\widetilde{u}, \widetilde{u})$. Applying property (9) one can write

$$\frac{\partial}{\partial u_i}\mathbf{p}(\widetilde{u}, u_i)\bigg|_{u_i=\widetilde{u}} = \frac{-1}{k-1}\frac{\partial}{\partial u}\mathbf{s}(u, u_i)\bigg|_{u=\widetilde{u}, u_i=\widetilde{u}},$$

i.e., at the common point the tangent line is common as well. From Theorem 3 in [6] we know that

$$\mathbf{b}(u_i) = \mathbf{s}(u_i, u_i) \quad \text{and} \quad \frac{\partial}{\partial v}\mathbf{b}(v)\bigg|_{v=u_i} = \frac{k-2}{k-1}\frac{\partial}{\partial u}\mathbf{s}(u, u_i)\bigg|_{u=u_i},$$

that yields (by substituting $u = u_i = \widetilde{u}$)

$$\mathbf{b}(\widetilde{u}) = \mathbf{p}(\widetilde{u}, \widetilde{u}) \quad \text{and} \quad \frac{\partial}{\partial v}\mathbf{b}(v)\bigg|_{v=\widetilde{u}} = (2-k)\frac{\partial}{\partial u_i}\mathbf{p}(\widetilde{u}, u_i)\bigg|_{u_i=\widetilde{u}}$$

which completes the proof. □

The envelope of the family of paths of a cubic curve can be seen in Figure 2.

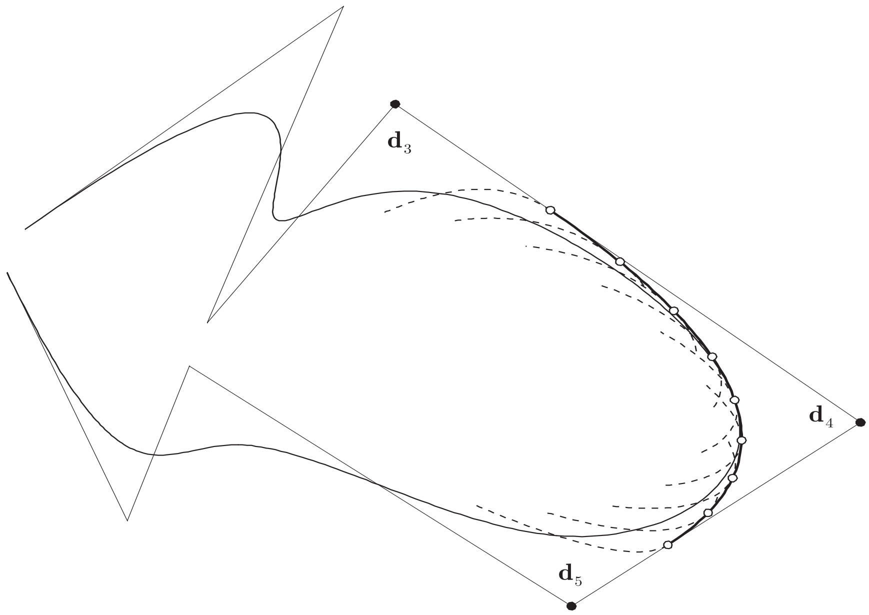

Figure 2. Some paths and their envelope in the case of a cubic B-spline curve ($n = 8$, $k = 4$) obtained by altering the knot value u_6

4. Further research

Equation (6) is a parametrization of a rational surface Φ. Both sets of isoparametric curves of surface patch (6) are of degree $k-1$, therefore an upper bound for the degree of Φ is $2\,(k-1)^2$. Since this bound is for the general case, it is possible that there is a lower upper bound for this specific surface.

In Section 2 we have shown that control points $\mathbf{d}_i$ and $\mathbf{d}_{i-k}$ are on the extension of the patch (6), and are of singular parametrization. It could be examined, whether these control points are singular points of the algebraic surface Φ itself.

References

[1] C. K. Au and M. M. F. Yuen, Unified approach to NURBS curve shape modification, *Computer-Aided Design* **27** (1995), 85–93.

[2] W. Boehm, Inserting new knots into B-spline curves, *Computer-Aided Design* **12** (1980), 199–201.

[3] C. de Boor, A Practical Guide to Splines, *Springer-Verlag*, *Berlin*, 2001.

[4] J. Hoschek and D. Lasser, Fundamentals of Computer Aided Geometric Design, *A. K. Peters*, *Wellesley, Massachusetts*, 1993.

[5] I. Juhász, Weight-based shape modification of NURBS curves, *Computer Aided Geometric Design* **16** (1999), 377–383.

[6] I. Juhász and M. Hoffmann, The effect of knot modifications on the shape of B-spline curves, *Journal for Geometry and Graphics* **5** (2001), 111–119.

[7] I. Juhász and M. Hoffmann, Modifying a knot of B-spline curves, *Computer Aided Geometric Design* **20** (2003), 243–245.

[8] I. Juhász and M. Hoffmann, Constrained shape modification of cubic B-spline curves by means of knots, *Computer-Aided Design* **36** (2004), 437–445.

[9] L. Piegl, Modifying the shape of rational B-splines, Part 1: curves, *Computer-Aided Design* **21** (1989), 509–518.

[10] L. Piegl and W. Tiller, The NURBS book, *Springer-Verlag*, *Berlin*, 1995.

[11] L. Piegl and W. Tiller, W., Computing the derivative of NURBS with respect to a knot, *Computer Aided Geometric Design* **15** (1998), 925–934.

[12] J. Sánchez-Reyes, A simple technique for NURBS shape modification, *IEEE Comp. Graph. and Appl.* **17** (1997), 52–59.

[13] L. L. Schumaker, Spline Functions: Basic Theory, *Wiley*, *New-York*, 1981.

MIKLÓS HOFFMANN
INSTITUTE OF MATHEMATICS AND COMPUTER SCIENCE
KÁROLY ESZTERHÁZY COLLEGE
H-3300 EGER, LEÁNYKA 4
HUNGARY

E-mail: hofi@ektf.hu

IMRE JUHÁSZ
DEPARTMENT OF DESCRIPTIVE GEOMETRY
UNIVERSITY OF MISKOLC
H-3515 MISKOLC, EGYETEMVÁROS
HUNGARY

E-mail: agtji@gold.uni-miskolc.hu

(Received June 23, 2003; revised February 5, 2004)