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On a composite functional equation arising in utility theory

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Abstract. The aim of this paper is to solve the composite functional equation

$$F\big(x+G(tH(y-x))\big)=R\big((1-t)K(x),tK(y)\big) \qquad (x,y\in I,\ x\le y,\ t\in[0,1])$$

that arises in utility theory. Here all functions involved are considered to be unknown.

1. Introduction

In this paper we consider the following functional equation

$$F\big(x+G(tH(y-x))\big)=R\big((1-t)K(x),tK(y)\big) \qquad (x,y\in I,\ x\le y,\ t\in[0,1]), \tag{1}$$

where

(i) I is a real interval of the form $[0,c[$, $[0,c]$ $(c>0)$, or $[0,\infty[$;

(ii) $H: I\to\mathbb{R}$ is a strictly increasing continuous function such that $H(0)=0$;

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(iii) $K : I \to \mathbb{R}$ is a strictly increasing continuous function such that $K(0) = 0$;

(iv) $G : H(I) \to \mathbb{R}$ is an arbitrary function;

(v) $F : J \to \mathbb{R}$ is an injective function whose domain J is chosen so that the left hand side of (1) is defined;

(vi) $R : D \to \mathbb{R}$, where D is defined by

$$D := \big\{(a,b) \mid a,b \geq 0,\ a+b \in K(I)\big\}.$$

Observe that

$$D = \big\{((1-t)u, tv) \mid t \in [0,1],\ u,v \in K(I),\ u \leq v\big\},$$

therefore, the right hand side of (1) has sense.

This equation is a generalization of an equation derived from the axiomatic treatment of various choice models in utility theory. For the mathematical psychology background see the monograph Luce [2]. It should be noted that (1) can be handled by using only elementary tools of the theory of functional equations while the solution of some more general equations involves hard results of the theory of real functions and convexity (cf. [3]).

This paper is organized as follows. In the next section, we analyze the consequences of two simple substitutions, $x = y$ and $x = 0$ in (1). We obtain a new functional equation (see (8) below) whose structure is similar to that of (1), but contains only one unknown function H. The third section contains the main results of this paper. We solve (8) without further regularity assumptions and present the solution of (1) as well.

2. Basic reductions

In this section, we make the two simple substitutions $x = y$ and $x = 0$ in (1) and derive their consequences. In our first result we eliminate the unknown functions F and R.

Theorem 1. *Assume that* (i)–(vi) *hold. Then the unknown functions* F, G, H, K *and* R *satisfy* (1) *if and only if*

$$R(a,b) = F\left(K^{-1}(a+b) + G(0)\right) \quad \text{if } (a,b) \in D, \tag{2}$$

and G, H and K fulfill the following functional equation

$$K\big(x+G(tH(y-x))-G(0)\big)=(1-t)K(x)+tK(y) \qquad (x,y\in I,\ x<y,\ t\in\,]0,1]). \tag{3}$$

Proof. Assume that (1) holds. We prove (2) first. Substituting $x=y$ into (1), we get

$$F\big(x+G(0)\big)=R\big((1-t)K(x),tK(x)\big) \qquad (x\in I,\ t\in[0,1]) \tag{4}$$

If $(a,b)\in D$ and $a+b>0$, then

$$t=\frac{b}{a+b}, \qquad x=K^{-1}(a+b)$$

solves the following system of equations

$$a=(1-t)K(x), \qquad b=tK(x).$$

Hence, by (4), we get (2) for $a+b>0$. If $a=b=0$, then, with $x=0$ in (4), the equation (2) can also be verified.

Equation (2) now can be applied to express the right hand side of (1). Using the injectivity of F, we obtain

$$x+G(tH(y-x))=K^{-1}\big((1-t)K(x)+tK(y)\big)+G(0) \qquad (x,y\in I,\ x\le y,\ t\in[0,1]), \tag{5}$$

whence (3) follows at once.

Conversely, assume that R is expressed via (2) and (3) is satisfied. Then, (3) can be rearranged to get (5). Applying F to both sides of (5) and using formula (2), we obtain that (1) is valid for $x,y\in I$, $x\le y$, $t\in[0,1]$. □

In the next result we analyze the consequences of the substitution $x=0$ in (3).

Theorem 2. *Assume that* (i)–(iv) *hold. Then the unknown functions G,H, and K satisfy* (3) *if and only if there exists a positive constant c such that*

$$K(x)=cH(x) \qquad (x\in I), \tag{6}$$

$$G(u)=G(0)+H^{-1}(u), \qquad (u\in H(I)), \tag{7}$$

and the function H fulfills the following functional equation

$$H\big(x+H^{-1}(tH(y-x))\big)=(1-t)H(x)+tH(y) \qquad (x,y\in I,\ x\le y,\ t\in[0,1]). \tag{8}$$

Proof. Assume that (3) holds. Substituting $x=0$, we get

$$K\big(G(tH(y))-G(0)\big)=tK(y) \qquad (y\in I,\ t\in[0,1]). \tag{9}$$

The substitution $t=1$ and the injectivity of K yield

$$G(H(y))-G(0)=y \qquad (y\in I). \tag{10}$$

Hence (7) follows. Substituting $t=H(z)/H(y)$ into (9) and using (10), we get

$$K(z)=\frac{H(z)}{H(y)}K(y) \qquad (z,y\in I,\ z\le y).$$

This equality immediately implies the existence of a constant $c>0$ such that (6) holds. Using (6) and (7), the functional equation (3) obviously reduces to (8).

Conversely, if H fulfills (8) and K,G are given via (6) and (7), then it is easy to see that (3) is also satisfied. □

3. Main results

In the next theorem we present the general solution of (8).

Theorem 3. *Assume that* (i) *and* (ii) *hold. Then H satisfies* (8) *if and only if*

- *either there exists a positive constant α such that $H(x)=\alpha x$ for $x\in I$,*
- *or there exist constants α,β such that $\alpha\beta<0$ and $H(x)=\alpha(1-e^{\beta x})$ for $x\in I$.*

Proof. Let $0<x<z<y$ $(x,y,z\in I)$ and $t:=\frac{H(z-x)}{H(y-x)}$. Then, clearly $t\in\,]0,1[$. Making this substitution, (8) reduces to

$$H(z)=H(x)+\frac{H(z-x)}{H(y-x)}(H(y)-H(x)) \qquad (0<x<z<y).$$

Thus
$$\frac{H(z)-H(x)}{H(z-x)}=\frac{H(y)-H(x)}{H(y-x)} \qquad (0<x<y<z).$$
Keeping $x \in I$ fixed, we can see that
$$y \mapsto \frac{H(y)-H(x)}{H(y-x)} \qquad (y \in I,\ y>x)$$
does not depend on y, that is, there exists a function $\Phi : I^\circ \to]0,\infty[$ such that
$$H(y)-H(x)=H(y-x)\Phi(x) \qquad (x,y \in I^\circ,\ x \le y). \tag{11}$$
(Here I° denotes the interior of I.) The function H being continuous, Φ must be continuous as well. Using this equation, for $x<y<z$, we get
$$\begin{aligned} H(z-x)\Phi(x) &= H(z)-H(x)=(H(z)-H(y))+(H(y)-H(x)) \\ &= H(z-y)\Phi(y)+H(y-x)\Phi(x). \end{aligned}$$
Hence
$$\frac{H(z-x)-H(y-x)}{H(z-y)}=\frac{\Phi(y)}{\Phi(x)} \qquad (x,y,z \in I^\circ,\ x<y<z).$$
On the other hand, by (11),
$$\frac{H(z-x)-H(y-x)}{H(z-y)}=\Phi(y-x) \qquad (x,y,z \in I^\circ,\ x<y<z).$$
Therefore, Φ satisfies the following exponential Cauchy equation (on a restricted domain)
$$\Phi(y)=\Phi(x)\Phi(y-x) \qquad (x,y \in I^\circ,\ x<y).$$
By the continuity of Φ, there exists a constant β such that
$$\Phi(x)=e^{\beta x}, \qquad (x \in I).$$
(See [1].) Thus (11) can be rewritten as
$$H(y)-H(x)=H(y-x)e^{\beta x} \qquad (x,y \in I,\ x \le y). \tag{12}$$

We distinguish two cases. If $\beta=0$, then H satisfies an additive Cauchy equation (on a restricted domain), and hence H must be of the form $H(x) = \alpha x$, where α must be a positive constant (because H is strictly increasing, see [1]).

If $\beta \neq 0$, then applying (12) twice, we get, for $x, y \in I$ $(x < y)$,

$$H(x) + H(y-x)e^{\beta x} = H(y) = H(y-x) + H(x)e^{\beta(y-x)},$$

that is,

$$\frac{H(x)}{1-e^{\beta x}} = \frac{H(y-x)}{1-e^{\beta(y-x)}}. \tag{13}$$

This equality yields that, for all $x, z \in I^\circ$

$$\frac{H(x)}{1-e^{\beta x}} = \frac{H(z)}{1-e^{\beta z}}.$$

Indeed, if $x, z \in I^\circ$, then there exists a positive number p such that $x+p, z+p \in I$. Hence, applying (13) for $y = x+p$ and $y = z+p$, we get

$$\frac{H(x)}{1-e^{\beta x}} = \frac{H(p)}{1-e^{\beta p}} = \frac{H(z)}{1-e^{\beta z}}.$$

Therefore, there exists a constant α such that

$$H(x) = \alpha(1-e^{\beta x}) \qquad (x \in I^\circ).$$

Using the continuity, this equality extends to I as well. The function H is increasing, hence $\alpha\beta < 0$.

A straightforward computation shows that both forms of H are solutions of (8). $\square$

Finally, summarizing our results we list the solutions of equation (1).

Theorem 4. *Assume that* (i)–(vi) *hold. Then the unknown functions* F, G, H, K *and* R *satisfies* (1) *if and only if*

- *either there exists a positive constant* α *such that* $H(x) = \alpha x$ *for* $x \in I$,
- *or there exist constants* α, β *such that* $\alpha\beta < 0$ *and* $H(x) = \alpha(1-e^{\beta x})$ *for* $x \in I$,

there exist a positive constant c and a constant d such that

$$K(x) = cH(x) \qquad (x \in I),$$

$$G(u) = d + H^{-1}(u), \qquad (u \in H(I)),$$

$$R(a,b) = F\left(K^{-1}(a+b) + d\right) \qquad \text{if} \quad (a,b) \in D,$$

and F is arbitrary.

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