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Hermite–Hadamard-type inequalities for generalized 3-convex functions

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Abstract. The aim of this paper is to present Hermite–Hadamard type inequalities for generalized 3-convex functions. A particular result for generalized 4-convex functions is also obtained.

1. Introduction

In 1937, E. F. Beckenbach ([Bec37]; see also [BB45]) introduced a notion which subsumes most of the convexity notions of higher-order:

Definition 1. Let $I \subset \mathbb{R}$ be a real interval and denote by $\mathcal{C}(I)$ the set of all real valued functions defined on I. We say that $\Omega_n(I) \subset \mathcal{C}(I)$ is an *n-parameter Beckenbach family* on I if, for any system of points $(x_1, y_1), \dots, (x_n, y_n) \in I \times \mathbb{R}$ with pairwise distinct first coordinates, there exists a unique element $\omega \in \Omega_n$ such that $\omega(x_k) = y_k$ for all $k = 1, \dots, n$.

A function $\omega_0 : I \to \mathbb{R}$ is called generalized n-convex with respect to $\Omega_n(I)$ if it intersects the elements of $\Omega_n(I)$ alternately in a certain sense. More precisely, we have the following

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Definition 2. Let $\Omega_n(I)$ be an n-parameter Beckenbach family. A function $\omega_0 : I \to \mathbb{R}$ is said to be *generalized n-convex (nth-order convex) with respect to $\Omega_n(I)$ (in the sense of Beckenbach)* if, for all $x_1 < \dots < x_n$ in I, the function $\omega \in \Omega_n(I)$ determined uniquely by the condition

$$\omega_0(x_k) = \omega(x_k) \qquad (k = 1, \dots, n)$$

satisfies the inequalities

$$\begin{aligned} (-1)^n(\omega_0(y) - \omega(y)) &\geq 0 \qquad (y \leq x_1) \\ (-1)^{n+k}(\omega_0(y) - \omega(y)) &\geq 0 \qquad (x_k \leq y \leq x_{k+1},\ k = 1, \dots, n-1) \\ (\omega_0(y) - \omega(y)) &\geq 0 \qquad (x_n \leq y). \end{aligned}$$

Throughout in this paper, we restrict our investigations to Beckenbach families generated by the linear span of certain continuous functions $\omega_1, \dots, \omega_n : I \to \mathbb{R}$. As it can be easily checked, the linear span $\mathfrak{L}(\omega_1, \dots, \omega_n)$ is an n-parameter Beckenbach family if and only if $(\omega_1, \dots, \omega_n)$ is a *Tchebychev system*, i.e., $\omega_1, \dots, \omega_n : I \to \mathbb{R}$ are continuous functions and

$$\begin{vmatrix} \omega_1(x_1) & \cdots & \omega_1(x_n) \\ \vdots & \ddots & \vdots \\ \omega_n(x_1) & \cdots & \omega_n(x_n) \end{vmatrix} \neq 0$$

is valid whenever $x_1, \dots, x_n$ are pairwise distinct elements of I. We say that $\omega_0 : I \to \mathbb{R}$ is generalized n-convex with respect to the Tchebychev system $(\omega_1, \dots, \omega_n)$ if ω_0 is generalized n-convex with respect to the Beckenbach family $\Omega_n(I) = \mathfrak{L}(\omega_1, \dots, \omega_n)$. For the properties of Tchebychev systems and their applications see [KS66] and [Kar68].

In the particular case when the Tchebychev system is generated by the affine functions, the generalized convexity so obtained coincides with the ordinary convexity. For a convex function (in the traditional sense) $\omega_0 : I \to \mathbb{R}$ we have the Hermite–Hadamard inequality ([Had93], [ML85], [NP04])

$$\omega_0\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \omega_0(x)dx \leq \frac{\omega_0(a)+\omega_0(b)}{2} \qquad (a, b \in I).$$

This inequality provides a lower and an upper estimate for the integral average of ω_0 using certain base points of the domain. For generalized convex functions with respect to arbitrary dimensional Tchebychev systems with polynomial basis, or with respect to 2 dimensional Tchebychev systems with arbitrary basis, analogous inequalities hold (cf. [BP02], [BP03]). The aim of this paper is to present Hermite–Hadamard type inequalities for generalized convex functions with respect to 3 dimensional Tchebychev systems with arbitrary basis. A particular result for 4 dimensional Tchebychev systems is also obtained.

2. Preliminary results

First we list up some known results for generalized higher-order convex functions. One of the most important states continuity and integrability properties for them (cf. [BP04]).

Theorem A. *Let $(\omega_1, \dots, \omega_n)$ be a Tchebychev system on the interval I. If $\omega_0 : I \to \mathbb{R}$ is a generalized n-convex function with respect to this system and $n \geq 2$, then ω_0 is continuous on the interior of I. Furthermore, if $[a, b] \subset I$, then ω_0 is bounded on $[a, b]$.*

The subsequent theorems present Hermite–Hadamard type inequalities for generalized higher-order convex functions. Two cases are considered: the odd- and the even-order one. The left and the right hand side inequalities are investigated independently in both cases.

Theorem B. *Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_{2n+1})$ be a Tchebychev system on the interval $[a, b]$ and $\rho : [a, b] \to \mathbb{R}$ be a positive integrable function. If a system of points $\xi_1 < \dots < \xi_n$ in $]a, b[$ satisfies the inclusion*

$$\int_a^b \boldsymbol{\omega}\rho \in \mathcal{L}\big(\boldsymbol{\omega}(a), \boldsymbol{\omega}(\xi_1), \dots, \boldsymbol{\omega}(\xi_n)\big), \tag{1}$$

then there exists uniquely determined positive constants $\alpha_0, \dots, \alpha_n$ such that

$$\int_a^b \boldsymbol{\omega}\rho = \alpha_0 \boldsymbol{\omega}(a) + \sum_{i=1}^{n} \alpha_i \boldsymbol{\omega}(\xi_i).$$

Furthermore, if $\omega_0 : [a,b] \to \mathbb{R}$ is a generalized $(2n+1)$-convex function with respect to $\boldsymbol{\omega}$, then the following inequality holds

$$\int_a^b \omega_0\rho \geq \alpha_0\omega_0(a) + \sum_{i=1}^{n} \alpha_i\omega_0(\xi_i).$$

Theorem C. *Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_{2n+1})$ be a Tchebychev system on the interval $[a,b]$ and $\rho : [a,b] \to \mathbb{R}$ be a positive integrable function. If a system of points $\eta_1 < \cdots < \eta_n$ in $]a,b[$ satisfies the inclusion*

$$\int_a^b \boldsymbol{\omega}\rho \in \mathcal{L}\big(\boldsymbol{\omega}(\eta_1), \ldots, \boldsymbol{\omega}(\eta_n), \boldsymbol{\omega}(b)\big), \tag{2}$$

then there exist uniquely determined positive constants $\beta_1, \ldots, \beta_{n+1}$ such that

$$\int_a^b \boldsymbol{\omega}\rho = \sum_{i=1}^{n} \beta_i\boldsymbol{\omega}(\eta_i) + \beta_{n+1}\boldsymbol{\omega}(b).$$

Furthermore, if $\omega_0 : [a,b] \to \mathbb{R}$ is a generalized $(2n+1)$-convex function with respect to $\boldsymbol{\omega}$, then the following inequality holds

$$\int_a^b \omega_0\rho \leq \sum_{i=1}^{n} \beta_i\omega_0(\eta_i) + \beta_{n+1}\omega_0(b).$$

In the even-order case analogous results can be verified: the correspondence of the inclusion (1) contains only interior base points, while the correspondence of the inclusion (2) contains some other interior base points plus both endpoints.

In fact, Theorem B, Theorem C and their even-order analogue give sufficient conditions for the existence of Hermite–Hadamard type inequalities. The existence of ξ_k and η_k in the inclusions of these theorems are guaranteed for polynomially higher-order convexity and generalized 2-convexity.

3. The main result

First we formulate the left hand side Hermite–Hadamard inequality for generalized 3-convex functions.

Theorem 1. *Let* $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ *be a Tchebychev system on* $[a,b]$, *and* $\rho : [a,b] \to \mathbb{R}$ *be a positive integrable function. Then,*

(1) *there exists a unique element* ξ *of* $]a,b[$ *which fulfills the inclusion*

$$\int_a^b \boldsymbol{\omega}\rho \in \mathfrak{L}\left(\boldsymbol{\omega}(a), \boldsymbol{\omega}(\xi)\right);$$

(2) *there exist uniquely determined positive constants* c_1, c_2 *such that*

$$\int_a^b \boldsymbol{\omega}\rho = c_1\boldsymbol{\omega}(a) + c_2\boldsymbol{\omega}(\xi);$$

(3) *if* $\omega_0 : [a,b] \to \mathbb{R}$ *is generalized* 3*-convex with respect to* $\boldsymbol{\omega}$, *then the following inequality holds*

$$\int_a^b \omega_0\rho \geq c_1\omega_0(a) + c_2\omega_0(\xi).$$

Proof. Due to Theorem B, we suffice to prove only the first assertion of the theorem. Define the function $F : [a,b] \to \mathbb{R}$ by the formula

$$F(x) := \begin{vmatrix} \boldsymbol{\omega}(a) & \int_a^x \boldsymbol{\omega}\rho & \int_a^b \boldsymbol{\omega}\rho \end{vmatrix} := \begin{vmatrix} \omega_1(a) & \int_a^x \omega_1\rho & \int_a^b \omega_1\rho \\ \omega_2(a) & \int_a^x \omega_2\rho & \int_a^b \omega_2\rho \\ \omega_3(a) & \int_a^x \omega_3\rho & \int_a^b \omega_3\rho \end{vmatrix}.$$

Then, F is continuous on $[a,b]$ and $F(a) = F(b) = 0$. Further on, due to the Tchebychev property of $\boldsymbol{\omega}$ and the positivity of ρ, $F(x) \neq 0$ if $x \in]a,b[$. For simplicity, we may assume that F is positive on $]a,b[$. According to Weierstrass' theorem, there exists a ξ such that

$$F(\xi) = \max_{[a,b]} F.$$

Clearly, $\xi \in]a,b[$. Then, for $x \in]\xi, b]$, we have the inequality

$$0 \geq \frac{F(x) - F(\xi)}{\int_\xi^x \rho} = \begin{vmatrix} \boldsymbol{\omega}(a) & \frac{\int_\xi^x \boldsymbol{\omega}\rho}{\int_\xi^x \rho} & \int_a^b \boldsymbol{\omega}\rho \end{vmatrix}. \tag{3}$$

On the other hand, for $k = 1, 2, 3$ the following estimates hold

$$\min_{[\xi,x]} \omega_k \le \frac{\min_{[\xi,x]} \omega_k \int_\xi^x \rho}{\int_\xi^x \rho} \le \frac{\int_\xi^x \omega_k \rho}{\int_\xi^x \rho} \le \frac{\max_{[\xi,x]} \omega_k \int_\xi^x \rho}{\int_\xi^x \rho} \le \max_{[\xi,x]} \omega_k.$$

Therefore, tending to ξ from the right in formula (3), the central column of the determinant tends to $\boldsymbol{\omega}(\xi)$, and

$$\begin{vmatrix} \boldsymbol{\omega}(a) & \boldsymbol{\omega}(\xi) & \int_a^b \boldsymbol{\omega}\rho \end{vmatrix} \le 0$$

follows. Choosing $x \in [a, \xi[$ and applying the same argument, we get the opposite inequality and arrive at

$$\begin{vmatrix} \boldsymbol{\omega}(a) & \boldsymbol{\omega}(\xi) & \int_a^b \boldsymbol{\omega}\rho \end{vmatrix} = 0.$$

Thus, the linear independence of $\boldsymbol{\omega}(a)$ and $\boldsymbol{\omega}(\xi)$ yields that $\int_a^b \boldsymbol{\omega}\rho \in \mathcal{L}(\boldsymbol{\omega}(a), \boldsymbol{\omega}(\xi))$.

For the uniqueness, assume indirectly that the inclusions

$$\int_a^b \boldsymbol{\omega}\rho \in \mathcal{L}\left(\boldsymbol{\omega}(a), \boldsymbol{\omega}(\xi)\right) \qquad \int_a^b \boldsymbol{\omega}\rho \in \mathcal{L}\left(\boldsymbol{\omega}(a), \boldsymbol{\omega}(\bar{\xi})\right)$$

hold with some $\xi \ne \bar{\xi}$ from $]a, b[$. Taking into consideration (2) and after rearranging, the identity

$$0 = (c_1 - \bar{c}_1)\boldsymbol{\omega}(a) + c_2\boldsymbol{\omega}(\xi) - \bar{c}_2\boldsymbol{\omega}(\bar{\xi})$$

follows. Obviously, the coefficients $c_1 - \bar{c}_1, c_2, \bar{c}_2$ cannot be equal to zero simultaneously, therefore

$$\begin{vmatrix} \boldsymbol{\omega}(a) & \boldsymbol{\omega}(\xi) & \boldsymbol{\omega}(\bar{\xi}) \end{vmatrix} = 0$$

which contradicts the Tchebychev property of $\boldsymbol{\omega}$. □

For the right hand side inequality analogous result remains true which can be verified with a similar argument, therefore the proof is omitted.

Theorem 2. *Let $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ be a Tchebychev system on $[a, b]$, and $\rho : [a, b] \to \mathbb{R}$ be a positive integrable function. Then,*

(1) *there exists a unique element η of $]a, b[$ which fulfills the inclusion*

$$\int_a^b \boldsymbol{\omega}\rho \in \mathfrak{L}\left(\boldsymbol{\omega}(\eta), \boldsymbol{\omega}(b)\right);$$

(2) *there exist uniquely determined positive constants d_1, d_2 such that*

$$\int_a^b \boldsymbol{\omega}\rho = d_1\boldsymbol{\omega}(\eta) + d_2\boldsymbol{\omega}(b);$$

(3) *if $\omega_0 : [a, b] \to \mathbb{R}$ is generalized 3-convex with respect to $\boldsymbol{\omega}$, then the following inequality holds*

$$\int_a^b \omega_0\rho \le d_1\omega_0(\eta) + d_2\omega_0(b).$$

Keeping the notations of Theorem 1 and Theorem 2, the coefficients c_1, c_2 and d_1, d_2 depend linearly on $\omega_k(a), \omega_k(\xi)$ and $\omega_k(\eta), \omega_k(b)$, respectively. Therefore, in concrete cases, the main difficulty is to determine the points ξ and η. Not claiming completeness, we list some examples when they can be determined explicitly.

Example 1. If the Tchebychev system $(\omega_1, \omega_2, \omega_3)$ is defined on $[a, b]$ by $\omega_1(x) = 1$, $\omega_2(x) = \sinh x$, $\omega_3(x) = \cosh x$ and $\rho \equiv 1$, then

$$\xi = 2\,\mathrm{artanh}\left(\frac{\sinh b - \sinh a - (b-a)\cosh a}{\cosh b - \cosh a - (b-a)\sinh a}\right) - a$$

$$\eta = 2\,\mathrm{artanh}\left(\frac{\sinh b - \sinh a - (b-a)\cosh b}{\cosh b - \cosh a - (b-a)\sinh b}\right) - b.$$

Example 2. If the Tchebychev system $(\omega_1, \omega_2, \omega_3)$ is defined on $[a, b] \subset]-\pi, \pi[$ by $\omega_1(x) = 1$, $\omega_2(x) = \sin x$, $\omega_3(x) = \cos x$ and $\rho \equiv 1$, then

$$\xi = 2\arctan\left(\frac{\sin a - \sin b + (b-a)\cos a}{\cos a - \cos b - (b-a)\sin a}\right) - a$$

$$\eta = 2\arctan\left(\frac{\sin a - \sin b + (b-a)\cos b}{\cos a - \cos b - (b-a)\sin b}\right) - b.$$

Example 3. If the Tchebychev system $(\omega_1, \omega_2, \omega_3)$ is defined on $[a,b]$ by $\omega_1(x) = 1$, $\omega_2(x) = \exp x$, $\omega_3(x) = \exp 2x$ and $\rho \equiv 1$, then

$$\xi = \log\left(\frac{\exp 2b - \exp 2a - 2(b-a)\exp 2a}{2(\exp b - \exp a - (b-a)\exp a)} - \exp a\right)$$

$$\eta = \log\left(\frac{\exp 2b - \exp 2a - 2(b-a)\exp 2b}{2(\exp b - \exp a - (b-a)\exp b)} - \exp b\right).$$

Example 4. If, for $p > 0$, the Tchebychev system $(\omega_1, \omega_2, \omega_3)$ is defined on $[a,b] \subset [0, +\infty[$ by $\omega_1(x) = 1$, $\omega_2(x) = x^p$, $\omega_3(x) = x^{2p}$ and $\rho \equiv 1$, then

$$\xi = \left(\frac{p+1}{2p+1} \cdot \frac{b^{2p+1} - a^{2p+1} - (2p+1)(b-a)a^{2p}}{b^{p+1} - a^{p+1} - (p+1)(b-a)a^p} - a^p\right)^{1/p}$$

$$\eta = \left(\frac{p+1}{2p+1} \cdot \frac{b^{2p+1} - a^{2p+1} - (2p+1)(b-a)b^{2p}}{b^{p+1} - a^{p+1} - (p+1)(b-a)b^p} - b^p\right)^{1/p}.$$

The particular case $p = 1$ of the last example gives a corollary of [BP02] for so called 3-monotone functions. For 3 dimensional Tchebychev systems generated by arbitrary power functions, the interior base points in general, cannot be expressed explicitly.

The proof of Theorem 1 is applicable for generalized 2-convexity, and gives a different approach that was followed in [BP03]. Observe, that the "uniqueness" part of the proof can be generalized for arbitrary Tchebychev systems: if inclusion (1) (analogously, (2), or their equivalence in the even-order case) is satisfied, then it is satisfied uniquely. Using the key idea of the "existence" part, we can state right hand side Hermite–Hadamard type inequality for generalized 4-convex functions.

Theorem 3. *Let $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3, \omega_4)$ be a Tchebychev system on $[a,b]$, and $\rho : [a,b] \to \mathbb{R}$ be a positive integrable function.*

(1) *There exists a unique element ξ of $]a,b[$ which fulfills the inclusion*

$$\int_a^b \boldsymbol{\omega}\rho \in \mathcal{L}\left(\boldsymbol{\omega}(a), \boldsymbol{\omega}(\xi), \boldsymbol{\omega}(b)\right);$$

(2) *there exist uniquely determined positive constants c_1, c_2, c_3 such that*

$$\int_a^b \boldsymbol{\omega}\rho = c_1\boldsymbol{\omega}(a) + c_2\boldsymbol{\omega}(\xi) + c_3\boldsymbol{\omega}(b);$$

(3) *if* $\omega_0 : [a,b] \to \mathbb{R}$ *is generalized* 4*-convex with respect to* $\boldsymbol{\omega}$*, then the following inequality holds*

$$\int_a^b \omega_0\rho \leq c_1\omega_0(a) + c_2\omega_0(\xi) + c_3\omega_0(b).$$

Hint. Apply the same argument as in the proof of Theorem 1 for the function $F : [a,b] \to \mathbb{R}$ defined by the formula

$$F(x) := \begin{vmatrix} \boldsymbol{\omega}(a) & \int_a^x \boldsymbol{\omega}\rho & \boldsymbol{\omega}(b) & \int_a^b \boldsymbol{\omega}\rho \end{vmatrix}$$

$$:= \begin{vmatrix} \omega_1(a) & \int_a^x \omega_1\rho & \omega_1(b) & \int_a^b \omega_1\rho \\ \omega_2(a) & \int_a^x \omega_2\rho & \omega_2(b) & \int_a^b \omega_2\rho \\ \omega_3(a) & \int_a^x \omega_3\rho & \omega_3(b) & \int_a^b \omega_3\rho \\ \omega_4(a) & \int_a^x \omega_4\rho & \omega_4(b) & \int_a^b \omega_4\rho \end{vmatrix}. \qquad \square$$

Unfortunately, the method fails if someone tries to use it for stating left hand side Hermite–Hadamard type inequality for generalized 4-convex function, because, according to a result of [BP04], the existence of two interior base points should be guaranteed. Due to similar reasons, the "existence" part in the proof of Theorem 1 cannot be applied for generalized n-convex functions if $n > 4$.

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