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Stability of invariant foliations on almost contact manifolds

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Abstract. We prove that an invariant compact foliation of an almost cosymplectic manifold is stable. A large collection of examples is given.

1. Introduction

The following result due to H. Rummler [12] is well-known: A compact holomorphic foliation F of a Kähler manifold M is stable. This result was generalized by H. Holmann for a compact almost complex (resp. symplectic) foliation F of an almost Kähler (resp. symplectic) manifold M.

Our purpose is to extend the result for a certain kind of almost contact manifolds, the so called almost cosymplectic manifolds. We say that a foliation F on an almost contact manifold is invariant if its leaves are invariant submanifolds of M. Then we prove our main result: A compact invariant foliation of an almost cosymplectic manifold is stable.

Finally, we exhibit several examples of stable invariant foliations on almost cosymplectic manifolds. Let us remark that the examples of 4.3 are stable invariant foliations of almost contact manifolds which do not posses almost cosymplectic structures.

2. Preliminaries

First, we recall some definitions about foliations on manifolds.

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Let F be a foliation of dimension p and codimension q on a manifold M of dimension $n = p+q$. We denote by TF the vector subbundle of TM which consists of the tangent vectors to F, and by T_xF the fiber of TF over x. If X is a vector field tangent to F (i.e., $X(x) \in T_xF$ for all $x \in M$) then we put $X \in F$.

Definition 1. (1) F is said to be *compact* if each leaf of F is compact. (2) A leaf L of a compact foliation F is said to be *stable* if every neighborhood U of L contains an invariant neighborhood V of L, i.e., V is a collection of leaves. (3) F is said to be *stable* if every leaf of F is stable.

Now, we consider the product manifold $\bar{M} = M \times S^1$. A vector field on $\bar{M}$ will be denoted by (X, aT) where X is a vector field on $\bar{M}$ which takes values in TM, T is the unit tangent vector field to S^1 and a is a function on $\bar{M}$. We shall construct two foliations on $\bar{M}$ as follows:

(1) A foliation F_1 defined by

$$T_{(x,t)}F_1 = T_xF, \quad \text{for all} \quad (x,t) \in M \times S^1 .$$

Then F_1 is a foliation on $\bar{M}$ of dimension p and codimension $q+1$. The leaf $(L_1)_{(x,t)}$ of F_1 passing through (x,t) is precisely $L_x \times \{t\}$ which can be identified with L_x. Hence F is compact if and only if F_1 is compact.

(2) A foliation F_2 defined by

$$T_{(x,t)}F_2 = T_xF \oplus T_tS^1, \quad \text{for all} \quad (x,t) \in M \times S^1 .$$

Then F_2 is a foliation on $\bar{M}$ of dimension $p+1$ and codimension q. The leaf $(L_2)_{(x,t)}$ of F_2 passing through (x,t) is precisely $L_x \times S^1$. As above, F is compact if and only if F_2 is compact.

Proposition 1. *The following three assertions are equivalent: (1) F is stable. (2) F_1 is stable. (3) F_2 is stable.*

3. Invariant foliations on an almost contact manifold

Let $(M, \varphi, \eta, \xi, \langle\ ,\ \rangle)$ be an almost contact metric manifold of dimension $2n+1$. The fundamental 2–form Φ of M is defined by

$$\Phi(X,Y) = \langle X, \varphi Y\rangle, \quad \text{for all} \quad X, Y \in \chi(M) ,$$

where $\chi(M)$ denotes the Lie algebra of vector fields on M.

Let us recall some well-known definitions (see [1]). The almost contact metric manifold M is said to be:

contact iff $\Phi = d\eta$;
normal iff $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$;
almost cosymplectic iff $d\Phi = d\eta = 0$;
cosymplectic iff it is normal and almost cosymplectic.

We now consider the product manifold $\bar{M} = M \times S^1$. An almost complex structure J on $\bar{M}$ is defined by

$$J(X, aT) = (\varphi X - a\xi,\ \eta(X)T)\,.$$

Then the product Riemannian metric on $\bar{M}$ (also denoted by $\langle\ ,\ \rangle$) is Hermitian with respect to J and we have the following

Proposition 2. *$(M, \varphi, \eta, \xi, \langle\ ,\ \rangle)$ is almost cosymplectic (resp., normal, cosymplectic) if and only if $(\bar{M}, J, \langle\ ,\ \rangle)$ is almost Kähler (resp., Hermitian, Kähler).*

Remark. In fact, this proposition is just the aim of the definition of normality (see [1], p. 48).

Definition 2. Let $(M, \varphi, \eta, \xi, \langle\ ,\ \rangle)$ be an almost contact metric manifold. A foliation F on M is said to be *invariant* if $\varphi X \in F$, for any vector field $X \in F$.

In other words, F is invariant if and only if its leaves are invariant submanifolds of M (see [13]).

We easily see that there occur only two cases for any invariant foliation F on an almost contact metric manifold M.

(1) If the vector field ξ is never tangent to F, then F has even dimension. In fact, for any vector field $X \in F$ we have

$$\varphi^2 X = -X + \eta(X)\xi\,,$$

which implies $\varphi^2 X = -X$ and $\eta(X) = 0$. Then φ induces an almost complex structure on TF (or, equivalently, on each leaf of F). Thus F has even dimension. Furthermore, on $\bar{M}$ we have

$$J(X, aT) = (\varphi X - a\xi, 0), \quad \text{for all} \quad X \in F\,.$$

Thus $J(X, 0) = (\varphi X, 0)$ and hence F_1 is invariant by J, i.e., F_1 is an almost complex foliation on the almost Hermitian manifold $\bar{M}$.

(2) If the vector field ξ is tangent to F, then F has odd dimension. In fact, for any vector field $X \in F$ we have

$$\varphi^2 X = -X + \eta(X)\xi\,.$$

Hence the leaves of F are almost contact manifolds endowed with the restriction of φ, η and ξ. Furthermore, on $\bar{M}$ we have

$$J(X, aT) = (\varphi X - a\xi,\ \eta(X)T) \in F_2, \quad \text{for all} \quad X \in F\,.$$

Thus F_2 is a foliation on $\bar{M}$ invariant by J, i.e., an almost complex foliation on the almost Hermitian manifold $\bar{M}$.

Remark. From the above results we easily deduce that the vector field ξ is either everywhere tangent or nowhere tangent to F. In fact, if $\xi_x \notin T_xF$ (resp. $\xi_x \in T_xF$) at a point $x \in M$, then F has even (resp. odd) dimension.

The following result has been proved by HOLMANN [8].

Proposition 3. *A compact almost complex foliation on an almost Kähler manifold is stable.*

Then from Propositions 1, 2 and 3, we easily deduce our main result:

Theorem 1. *A compact invariant foliation F on an almost cosymplectic manifold M is stable.*

4. Examples

4.1. *The manifolds $M(1,r)$*

Let $H(1,r)$ be the generalized Heisenberg group which consists of the real matrices of the form

$$\begin{bmatrix} I_r & X & Z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

where $X^t = (x_1, \ldots, x_r),\ Z^t = (z_1, \ldots, z_r) \in R^r,\ y \in R$. Then $H(1,r)$ is a connected simply-connected nilpotent Lie group of dimension $2r+1$.

Denote by (x_i, y, z_i) the global coordinate system of $H(1,r)$. Then we have a set of linearly independent left invariant 1–forms on $H(1,r)$:

$$\bar{\alpha}_i = dx_i,\ \bar{\beta} = dy,\ \bar{\gamma}_i = dz_i - x_i dy\,;$$

its dual basis of left invariant vector fields is given by

$$\bar{X}_i = \partial/\partial x_i,\ \bar{Y} = \partial/\partial y + \sum_{i=1}^{r} x_i(\partial/\partial z_i),\ \bar{Z}_i = \partial/\partial z_i\,.$$

Now, we define $M(1,r) = \Gamma(1,r)\backslash H(1,r)$, where $\Gamma(1,r)$ is the discrete subgroup of matrices with integer entries. Thus $M(1,r)$ is a compact nilmanifold of dimension $2r+1$ (see [2]). Since $\{\bar{\alpha}_i, \bar{\beta}, \bar{\gamma}_i\}$ are invariant under the action of $\Gamma(1,r)$ there exists a linearly independent set of 1–forms $\{\alpha_i, \beta, \gamma_i\}$ on $M(1,r)$ such that

$$\pi^*\alpha_i = \bar{\alpha}_i,\ \pi^*\beta = \bar{\beta},\ \pi^*\gamma_i = \bar{\gamma}_i\,,$$

where $\pi : H(1,r) \to M(1,r)$ is the canonical projection. Thus we obtain

$$d\alpha_i = 0,\ d\beta = 0,\ d\gamma_i = -\alpha_i \wedge \beta, \quad 1 \le i \le r\,.$$

If we denote by $\{X_i, Y, Z_i\}$ the dual basis of vector fields, then we have

$$[X_i, Y] = Z_i, \quad 1 \leq i \leq r\,,$$

and all the other brackets are zero.

Alternatively, $M(1,r)$ may be seen as the total space of a T^{r+1}–bundle over T^r. In fact, consider the representation

$$\varrho : Z^r \to \mathrm{Diff}(T^{r+1})$$

defined as follows: $\varrho(n_1, \ldots, n_r)$ is the transformation of T^{r+1} covered by the linear transformation of R^{r+1} given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ n_1 & 1 & 0 & \cdots & 0 \\ n_2 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ n_r & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Then ϱ induces an action A of Z^r on $R^r \times T^{r+1}$ defined by

$$A((n_1, \ldots, n_r), ((x_1, \ldots, x_r), [y, z_1, \ldots, z_r])) =$$
$$((x_1 + n_1, \ldots, x_r + n_r), \varrho(n_1, \ldots, n_r)[y, z_1, \ldots, z_r])\,.$$

Hence we obtain a T^{r+1}–bundle $p : R^r \times_{Z^r} T^{r+1} \to T^r$ with projection p given by

$$p[(x_1, \ldots, x_r), [y, z_1, \ldots, z_r]] = [x_1, \ldots, x_r]\,.$$

In fact, this bundle is the suspension of the representation ϱ (see [7]).

Now, it is easy to identify $M(1,r)$ with $R^r \times_{Z^r} T^{r+1}$ in such a way that $p : M(1,r) \to T^r,\ p[x_1, y, z_1] = [x_1]$, is a T^{r+1}–bundle.

We know that $M(1,r)$ can have no cosymplectic structures (see [4]). However, $M(1,r)$ possesses some interesting non-normal almost cosymplectic structures.

Next, we shall describe two examples of stable invariant foliations of almost cosymplectic structures on $M(1,r)$.

Example 4.1.1. Suppose $r \geq 2$ and define a Riemannian metric on $M(1,r)$ by

$$\langle\ ,\ \rangle = \sum_{i=1}^{r} (\alpha_i^2 + \gamma_i^2) + \beta^2\,.$$

Now, let $(\varphi, \eta, \xi, \langle\ ,\ \rangle)$ be the almost contact metric structure on $M(1,r)$, $r \geq 2$, given by

$$\varphi = \sum_{i=2}^{r} (\gamma_i \otimes X_i - \alpha_i \otimes Z_i) - \beta \otimes Z_1 + \gamma_1 \otimes Y, \quad \eta = \alpha_1,\ \xi = X_1\,.$$

Then

$$\Phi = \beta \wedge \gamma_1 + \sum_{i=2}^{r} \alpha_i \wedge \gamma_i \, .$$

Thus, we have

$$d\Phi = 0, \ d\eta = 0$$

and hence $(\varphi, \xi, \eta, \langle \ , \ \rangle)$ is almost cosymplectic.

Now, consider on $M(1,r)$ the foliation F globally spanned by $\{X_2, \ldots, X_r, Y, Z_1, \ldots, Z_r\}$. Then F is a foliation of dimension $2r$ whose leaves are precisely the fibres of the fibration $q : M(1,r) \to S^1, \ q[x_i, y, z_i] = [x_1]$. Thus, they are $2r$–dimensional tori T^{2r}. Hence F is a compact invariant foliation and, from Theorem 1, it is stable. We notice that the result follows directly since the leaves of F are the fibres of q, which is a fibration with compact fibres [11].

Remark. If $r = 1$, then we put

$$\varphi = -\beta \otimes Z + \gamma \otimes Y, \ \ \eta = \alpha, \ \xi = X \, .$$

Therefore we deduce

$$\Phi = \beta \wedge \gamma$$

and hence $(\varphi, \xi, \eta, \langle \ , \ \rangle)$ is almost cosymplectic. Proceeding as above, we obtain an invariant foliation F globally spanned by $\{Y, Z\}$. The leaves of F are the fibres of the fibration $q : M(1,1) \to S^1, \ q[x,y,z] = [x]$, which are 2–dimensional tori T^2. Therefore, F is stable.

Example 4.1.2. Consider the product manifold $N = M(1,r) \times T^2$. Then N may be seen as the total space of a T^{r+2}–bundle over T^{r+1} with projection $p \times pr_1 : N \to T^{r+1}$, where $pr_1 : T^2 \to S^1$ is the projection defined by $pr_1[u,v] = [u]$. We notice that N can have no cosymplectic structures since $N \times S^1$ can have no Kähler structures (see [4]).

We denote by $\{U, V\}$ the global basis of unit vector fields on T^2 and by $\{du, dv\}$ its dual basis of 1–forms.

Now, let $(\varphi', \eta', \xi', \langle \ , \ \rangle')$ be the almost contact metric structure on N given by

$$\varphi' = \varphi - \alpha_1 \otimes U + du \otimes X_1, \ \ \eta' = dv, \ \xi' = V \, .$$

(Here $\langle \ , \ \rangle'$ denotes the product metric on N).

Then we obtain $\Phi' = \Phi + \alpha_1 \wedge du$, and hence $(\varphi', \eta', \xi', \langle \ , \ \rangle')$ is almost cosymplectic.

Consider the $(2r+1)$–dimensional foliation F' on N globally spanned by $\{X_2, \ldots, X_r, Y, Z_1, \ldots, Z_r, V\}$. Then F' is invariant and its leaves are precisely the fibres of the fibration $q' : N \to T^2, \ q'([x_i, y, z_i], [u,v]) = [x_1, u]$. Thus, they are $(2r+1)$–dimensional tori T^{2r+1}. Consequently, from

Theorem 1 we deduce that F' is stable. Alternatively, the result follows directly since the leaves of F' are the fibres of q', which is a fibration with compact fibres.

4.2. *The manifolds $M(k_1, \ldots, k_r)$*

Let $G(k_1, \ldots, k_r)$ be the connected simply-connected $(2r+1)$–dimensional Lie group of real matrices of the form

$$\begin{bmatrix} e^{k_1 z} & 0 & \cdots & 0 & 0 & 0 & x_1 \\ 0 & e^{-k_1 z} & \cdots & 0 & 0 & 0 & y_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e^{k_r z} & 0 & 0 & x_r \\ 0 & 0 & \cdots & 0 & e^{-k_r z} & 0 & y_r \\ 0 & 0 & \cdots & 0 & 0 & 1 & z \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $x_i, y_i, z \in R$ and $k_1, \ldots, k_r$ are fixed non-zero real numbers, with $k_i = kz_i,\ z_i \in Z,\ k \in R$. An easy computation shows that

$$\{\bar{\alpha}_i = e^{-k_i z} dx_i,\ \bar{\beta}_i = e^{k_i z} dy_i,\ \bar{\gamma} = dz\}$$

is a family of linearly independent left invariant 1–forms on $G(k_1, \ldots, k_r)$. The corresponding dual basis of left invariant vector fields on $G(k_1, \ldots, k_r)$ is

$$\left\{ \bar{X}_i = e^{k_i z} \frac{\partial}{\partial x_i},\ \bar{Y}_i = e^{-k_i z} \frac{\partial}{\partial y_i},\ \bar{Z} = \frac{\partial}{\partial z} \right\} .$$

We have

$$[\bar{X}_i, \bar{Z}] = -k_i \bar{X}_i, \quad [\bar{Y}_i, \bar{Z}] = k_i \bar{Y}_i, \quad 1 \le i \le r\,,$$

and all the other brakects are zero. Then we easily show that $G(k_1, \ldots, k_r)$ is a solvable non-nilpotent Lie group.

Now, let $B \in SL(2, Z)$ be an unimodular matrix, with positive real eigenvalues and different λ and λ^{-1} and let (a, b), $(c, d) \in R^2$ be the corresponding eigenvectors. We consider the discrete subgroup $\Gamma(k_1, \ldots, k_r)$ of $G(k_1, \ldots, k_r)$ which consits of the matrices of the form

$$\begin{bmatrix} \lambda^{z_1 p} & 0 & \cdots & 0 & 0 & 0 & n_1 a + m_1 b \\ 0 & \lambda^{-z_1 p} & \cdots & 0 & 0 & 0 & n_1 c + m_1 d \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda^{z_r p} & 0 & 0 & n_r a + m_r b \\ 0 & 0 & \cdots & 0 & \lambda^{-z_r p} & 0 & n_r c + m_r d \\ 0 & 0 & \cdots & 0 & 0 & 1 & (p/k) \ln \lambda \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}$$

with $n_i, m_i, p \in Z$. We denote by $M(k_1, \dots, k_r) = \Gamma(k_1, \dots, k_r) \setminus G(k_1, \dots, k_r)$ the space of right cosets. Thus $M(k_1, , \dots, k_r)$ is a compact solvmanifold of dimension $2r + 1$.

If $\pi : G(k_1, \dots, k_r) \to M(k_1, \dots, k_r)$ is the canonical projection, then we have a basis $\{\alpha_i, \beta_i, \gamma\}$ of 1–forms on $M(k_1, \dots, k_r)$ such that

$$\pi^*\alpha_i = \bar{\alpha}_i, \quad \pi^*\beta_i = \bar{\beta}_i, \quad \pi^*\gamma = \bar{\gamma}\,,$$
$$d\alpha_i = k_i\ \alpha_i \wedge \gamma,\ d\beta_i = -k_i\ \beta_i \wedge \gamma,\ d\gamma = 0\,.$$

The corresponding dual basis of vector fields is denoted by $\{X_i, Y_i, Z\}$, and we have

$$[X_i, Z] = -k_i X_i, \quad [Y_i, Z] = k_i Y_i, \quad 1 \le i \le r\,,$$

all the other brackets being zero.

Alternatively, the manifold $M(k_1, \dots, k_r)$ may be seen as the total space of a T^{2r}–bundle over S^1. In fact, let $T^{2r} = R^{2r}/H^{2r}$ the $2r$–dimensional tori, where $H^{2r} \cong Z^{2r}$ is the discrete subgroup of the integral linear combinations of the basis of R^{2r} given by $\{(a, c, 0, \dots, 0)$, $(b, d, 0, \dots, 0), (0, 0, a, c, 0, \dots, 0), (0, 0, b, d, 0, \dots, 0), \dots, (0, \dots, 0, a, c)$, $(0, \dots, 0, b, d)\}$ and let $\varrho : Z \to \mathrm{Diff}(T^{2r})$ be the representation defined as follows: $\varrho(n)$ represents the transformation of T^{2r} covered by the linear transformation of R^{2r} given by the matrix

$$\begin{bmatrix} \lambda^{z_1 n} & 0 & \cdots & 0 & 0 \\ 0 & \lambda^{-z_1 n} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda^{z_r n} & o \\ 0 & 0 & \cdots & 0 & \lambda^{-z_r n} \end{bmatrix}$$

This representation determinates an action A of Z on $R \times T^{2r}$which is defined as follows:

$$A(n, (z, [x_1, y_1, \dots, x_r y_r])) = (z + n, \varrho(n)[x_1, y_1, \dots, x_r, y_r])\,.$$

Then $p : R \times_Z T^{2r} \to S^1$ is a T^{2r}–bundle where the projection p is given by

$$p[z, [x_1, y_1, \dots, x_r, y_r]] = [z]\,.$$

Now, it is easy to see that $\psi : R \times_Z T^{2r} \to M(k_1, \dots, k_r)$ given by $\psi([z, [x_1, y_1, \dots, x_r, y_r]]) = [x_1, y_1, \dots, x_r, y_r, ((\ln \lambda)/k)z]$ is a diffeomorphism, in such a way that $p : M(k_1, \dots, k_r) \to S^1$, $p[x_1, y_1, z] = [(k/\ln \lambda)z]$ is a T^{2r}–bundle over S^1.

Next, we shall describe an example of stable invariant foliation on $M(k_1, \dots, k_r)$.

Define a Riemannian metric on $M(k_1,\dots,k_r)$ by

$$\langle\ ,\ \rangle = \sum_{i=1}^{r}(\alpha_i^2+\beta_i^2)+\gamma^2\,.$$

Now, let $(\varphi,\eta,\xi,\langle\ ,\ \rangle)$ be the almost contact metric structure on $M(k_1,\dots,k_r)$ given by

$$\varphi = \sum_{i=1}^{r}(\beta_i\otimes X_i-\alpha_i\otimes Y_i),\quad \eta=\gamma,\quad \xi = Z\,.$$

Then we obtain

$$\Phi = \sum_{i=1}^{r}\alpha_i\wedge\beta_i\,,$$

and hence $d\Phi=0$. Therefore $(\varphi,\xi,\eta,\langle\ ,\ \rangle)$ is almost cosymplectic.

Denote by F the $2r$–dimensional foliation globally spanned by $\{X_1,\dots, X_r,Y_1,\dots,Y_r\}$. Then the leaves of F are precisely the fibres of the fibration $p: M(k_1,\dots,k_r)\to S^1$, which are $2r$–dimensional tori T^{2r}. Since F is invariant and its leaves are compact, from Theorem 1 we deduce that F is stable. As above, the result follows directly since the leaves of F are the fibres of p, which is a fibration with compact fibres.

Remark. If $r=1$ and $k_1=k$, it is known that $M(k)$ has no cosymplectic structures (see [10]). In fact, $M(k)$ can have no normal structures since $M(k)\times S^1$ can have no complex structures (see [9], [6], [5]).

4.3. *The manifolds $M(r,1)$*

Let $H(r,1)$ be the generalized Heisenberg group which consists of the real matrices of the form

$$\begin{bmatrix} 1 & X & z\\ 0 & I_r & Y\\ 0 & 0 & 1\end{bmatrix}$$

where $X=(x_1,\dots,x_r)$, $Y^t=(y_1,\dots y_r)\in R^r$, $z\in R$. Then $H(r,1)$ is a connected simply-connected nilpotent Lie group of dimension $2r+1$.

Denote by (x_i,y_i,z) the global coordinates of $H(r,1)$. Then we have a set of linearly independent left invariant 1–forms

$$\bar\alpha_i = dx_i,\ \bar\beta_i=dy_i,\ \bar\gamma = dz-\sum_{i=1}^{r}x_i dy_i\,,$$

and its dual basis of left invariant vector fields is given by

$$\bar X_i=\partial/\partial x_i,\ \bar Y_i=\partial/\partial y_i + x_i(\partial/\partial z),\ \bar Z=\partial/\partial z\,.$$

Now, we set $M(r,1) = \Gamma(r,1) \setminus H(r,1)$, where $\Gamma(r,1)$ is the discrete subgroup of matrices with integer entries. Thus $M(r,1)$ is a compact nilmanifold of dimension $2r+1$ (see [3]). As in the Example 4.1, we obtain a global basis of 1–forms $\{\alpha_i, \beta_i, \gamma\}$ on $M(r,1)$ such that

$$\pi^*\alpha_i = \bar{\alpha}_i,\ \pi^*\beta_i = \bar{\beta}_i,\ \pi^*\gamma = \bar{\gamma}\,,$$

where $\pi : H(r,1) \to M(r,1)$ is the canonical projection. Thus we obtain

$$d\alpha_i = 0, \quad d\beta_i = 0, \quad d\gamma = -\sum_{i=1}^{r} \alpha_i \wedge \beta_i\,.$$

If we denote by $\{X_i, Y_i, Z\}$ the dual basis of vector fields, then we have

$$[X_i, Y_i] = Z\,,$$

and all the other brackets are zero.

Alternatively, $M(r,1)$ may be seen as the total space of a T^{r+1}–bundle over T^r. In fact, consider the representation

$$\varrho : Z^r \to \mathrm{Diff}(T^{r+1})$$

defined as follows: $\varrho(n_1, \ldots, n_r)$ is the transformation of T^{r+1} covered by the linear transformation of R^{r+1} given by the matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \\ n_1 & n_2 & \cdots & n_r & 1 \end{bmatrix}$$

Then ϱ induces an action A of Z^r on $R^r \times T^{r+1}$ defined by

$$\begin{aligned} &A((n_1, \ldots, n_r), ((x_1, \ldots, x_r), [y_1, \ldots y_r, z])) \\ &= ((x_1 + n_1, \ldots, x_r + n_r), \varrho(n_1, \ldots, n_r)[y_1, \ldots, y_r, z])\,. \end{aligned}$$

Thus we obtain a T^{r+1}–bundle $p : R^r \times_{Z^r} T^{r+1} \to T^r$ with projection p given by

$$p[(x_1, \ldots, x_r), [y_1, \ldots, y_r, z]] = [x_1, \ldots, x_r]$$

Now, it is clear that $M(r,1)$ may be canonically identified with $R^r \times_{Z^r} T^{r+1}$ in such a way that $p : M(r,1) \to T^r$, $p[x_i, y, z_i] = [x_i]$, is a T^{r+1}–bundle.

Next, we shall describe three examples of stable invariant foliations of almost contact metric structures on $M(r,1)$ which are stable. However, $M(r,1)$ can have no almost cosymplectic structures, since $M(r,1) \times S^1$ can have no symplectic structures for $r \geq 2$ (see [3]). In the three cases, the stability of the foliation follows because the leaves are the compact fibres of a fibration.

Define a Riemannian metric on $M(r,1)$ by

$$\langle \ , \ \rangle = \sum_{i=1}^{r}(\alpha_i^2 + \beta_i^2) + \gamma^2 ,$$

and suppose $r \geq 2$.

4.3.1. We put

$$\varphi = \sum_{i=2}^{r}(\beta_i \otimes X_i - \alpha_i \otimes Y_i) - \beta_1 \otimes Z + \gamma \otimes Y_1, \quad \eta = \alpha_1, \quad \xi = X_1 ,$$

and F to be the foliation globally spanned by $\{X_2, \dots, X_r, Y_1, \dots, Y_r, Z\}$. Then F is invariant and its leaves are the fibres of the projection $q : M(r,1) \to S^1$, $q[x_i, y_i, z] = [x_1]$. Thus, they are $2r$–dimensional tori T^{2r}.

4.3.2. Suppose r even, say $r = 2s$. Then we define

$$\varphi = \sum_{i=1}^{s}(\alpha_i \otimes X_{2i} - \alpha_{2i} \otimes X_i + \beta_i \otimes Y_{2i} - \beta_{2i} \otimes Y_i), \quad \eta = \gamma, \quad \xi = Z ,$$

and F to be the foliation globally spanned by $\{Y_1, \dots, Y_{2s}, Z\}$. Then F is invariant and its leaves are the fibres of the projection $p : M(2s,1) \to T^{2s}$, i.e., $(2s+1)$–dimensional tori T^{2s+1}.

4.3.3. Suppose r odd, say $r = 2s+1$. Then we define

$$\begin{aligned}\varphi = \sum_{i=1}^{s}(\alpha_i \otimes X_{2i} - \alpha_{2i} \otimes X_i + \beta_i \otimes Y_{2i} - \beta_{2i} \otimes Y_i) \\ +\beta_{2s+1} \otimes Z - \gamma \otimes Y_{2s+1} , \\ \eta = \alpha_{2s+1}, \quad \xi = X_{2s+1} ,\end{aligned}$$

and F to be the foliation globally spanned by $\{Y_1, \dots, Y_{2s+1}, Z\}$. Then F is invariant and its leaves are the fibres of the projection $p : M(2s+1,1) \to T^{2s+1}$, i.e., $(2s+2)$–dimensional tori T^{2s+2}.

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