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Transformations on the set of all n-dimensional subspaces of a Hilbert space preserving orthogonality

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To the memory of Professor B. Brindza

Abstract. In our paper we generalize Uhlhorn's version of Wigner's famous unitary-antiunitary theorem by describing the transformations preserving the orthogonality between higher dimensional subspaces under certain conditions.

1. Introduction and Statement of the Results

Wigner's classical unitary-antiunitary theorem has several formulations. One of them describes the bijections on the set of all 1-dimensional subspaces of a Hilbert space which preserve the angles between those subspaces. This fundamental result has been extended in (at least) three directions:

- if the underlying Hilbert space is at least three-dimensional, then, keeping the condition of bijectivity, the assumption of preserving angles can be replaced by the rather mild condition of *preserving orthogonality in both directions*; this is called UHLHORN's version of Wigner's theorem (cf. [9]),

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- keeping the condition of preserving angles, the assumption of bijectivity can be omitted (in this case the transformation is induced by a linear or conjugate linear isometry instead of a unitary or antiunitary operator; see [1], [8]),
- Molnár [7] extended Wigner's result to higher dimensional subspaces, namely he obtained the following result. If n is a positive integer, H is a Hilbert space with dimension not less than n and $n = 1$ or $\dim H \neq 2n$ then any transformation ϕ on the set of all n-dimensional subspaces of H, which preserves the so-called principal angles (see the definition below) between those subspaces, is of the form $\phi(M) = V[M]$, where V is a linear or conjugate linear isometry on H. Moreover, if H is an infinite dimensional Hilbert space, then a surjective transformation ϕ on the set of all infinite dimensional subspaces of H, which preserves the principal angles between those subspaces, is of the form $\phi(M) = U[M]$, where U is a unitary operator or antiunitary operator on H.

For further generalizations see e.g. [3]–[6]. In this paper we extend Wigner's theorem simultaneously in all the three directions above.

We introduce some concepts and notation. Let H be a (real or complex) Hilbert space and for any $n \in \mathbb{N} \cup \{\infty\}$ set

$$H_n = \big\{M \subseteq H \mid \dim M = n,\ \operatorname{codim} M \geq n,\ M \text{ is a closed subspace}\big\},$$

$$H_{(n)} = \big\{M \subseteq H \mid \dim M \geq n,\ \operatorname{codim} M \geq n,\ M \text{ is a closed subspace}\big\}.$$

We say that a transformation $\phi : H_n \to H_n$ *preserves orthogonality in both directions*, if for any $M, N \in H_n$ we have

$$M \perp N \ \Leftrightarrow \ \phi(M) \perp \phi(N).$$

For any closed subspace $M \subseteq H$, let P_M denote the orthogonal projection to M. Following Molnár [7] we say that ϕ *preserves principal angles* if for any $K, L \in H_n$ the positive operators $P_K P_L P_K$ and $P_{\phi(K)} P_{\phi(L)} P_{\phi(K)}$ are unitarily equivalent. It is clear that if ϕ preserves principal angles then it also preserves orthogonality in both directions.

We now present our results. In the Theorem below we characterize the transformations ϕ on H_n which preserve orthogonality in both directions under certain natural conditions. Our basic idea is to show that ϕ is induced by a transformation acting on the 1-dimensional subspaces of H, and then we apply UHLHORN's result [9].

To formulate the Theorem, we remark that if $\dim H < 2n$ then there do not exist two orthogonal n-dimensional subspaces of H, thus in this case the condition of preserving orthogonality has no meaning. In the Proposition we show that the case in which $\dim H = 2n \in \mathbb{N}$ is singular in a certain sense. Further, if $n = \infty$ then subspaces of finite codimension are clearly not orthogonal to any infinite dimensional subspace, and so the property of preserving orthogonality cannot imply anything for them. Therefore, in the case when $n = \infty$, we consider subspaces of infinite dimension and infinite codimension. In the case $2n < \dim H \le 3n$ the proof of Step 5 would be much longer, hence we omit that case. These justify the assumption (1) below.

Theorem. *Let H be a Hilbert space and $n \in \mathbb{N} \cup \{\infty\}$ with*

$$\begin{cases} \dim H > 3n & \text{if } n \in \mathbb{N}, \\ \dim H = \infty & \text{if } n = \infty, \end{cases} \tag{1}$$

and let $\phi : H_n \to H_n$.

If $\phi : H_n \to H_n$ is surjective, *then ϕ preserves orthogonality in both directions if and only if there exists a unique bijection $\psi : H_1 \to H_1$ which preserves orthogonality in both directions, and for any $K \in H_n$ we have*

$$\phi(K) = \operatorname{span}\{\psi(X) \mid X \in H_1,\ X \subseteq K\}, \tag{2}$$

where span *denotes the generated linear subspace.*

Thus, by Uhlhorn's theorem, if $n \in \mathbb{N} \cup \{\infty\}$ is such that (1) holds and $\phi : H_n \to H_n$ is surjective, then ϕ preserves orthogonality in both directions if and only if there exists a unitary or antiunitary operator U on H such that for any $K \in H_n$ we have

$$\phi(K) = U[K]. \tag{3}$$

If H is finite dimensional, *then ϕ preserves orthogonality in both directions (surjectivity is not assumed) if and only if there exists a unique transformation $\psi : H_1 \to H_1$ which preserves orthogonality in both directions and for any $K \in H_n$ (2) holds. Moreover, if ϕ preserves principal* *angles then ψ also preserves angles, thus in this case ϕ is of the form* (3) *with a unitary or antiunitary operator U on H.*

Remark. We make some short remarks.

- We learn from [7] that if ϕ preserves principal angles then ϕ is of the form (2) even in the case $n \le \dim H < 3n$.
- As for the case $\dim H = 2n \in \mathbb{N}$, observe that the bijection ϕ defined by $\phi(K) = K^{\perp}$ ($K \in H_n$) preserves principal angles, but is not of the form (3) (cf. [7]).
- We mention that the Theorem implies Molnár's result [7] in the case when H is finite dimensional.
- Our Theorem implies that surjective transformations on H_n which preserve orthogonality in both directions are the same as surjective transformations which preserve principle angles. In the case $n = 1$ this is a trivial consequence of Uhlhorn's theorem.
- We note that in every step except Step 5 the condition $\dim H > 2n$ is enough, and we use the condition $\dim H > 3n$ in Step 5 only. It is possible to prove our theorem for $\dim H > 2n$, but in that case the proof of Step 5 is much longer, hence that case will be omitted.

The following Proposition shows that the Theorem is not valid if $\dim H = 2n \in \mathbb{N}$.

Proposition. *If $2 \le n \in \mathbb{N}$ and $\dim H = 2n$, then there exists a bijection $\phi : H_n \to H_n$ which preserves orthogonality in both directions but is not of the form* (2).

2. Proofs

Proof of the Theorem. For any $M \in H_{(n)}$ let

$$\psi(V) = \overline{\operatorname{span}\{\phi(K) \mid K \in H_n,\ K \subseteq V\}}. \tag{4}$$

Since $V \in H_{(n)}$, there exists $M \in H_n$ with $M \perp V$. Now for any $K \in H_n$ with $K \subseteq V$, we have $M \perp K$ which implies $\phi(M) \perp \phi(K)$. Thus $\phi(M) \in H_n$ and $\phi(M) \perp \psi(V)$, hence $\psi(V) \in H_{(n)}$. Therefore $\psi : H_{(n)} \to H_{(n)}$.

Let $n \in \mathbb{N} \cup \{\infty\}$ such that (1) holds, and let $\phi : H_n \to H_n$ be an operator which preserves orthogonality in both directions. As in the statement of the Theorem, *in the case* $\dim H = \infty$ *we also assume that* ϕ *is surjective.* Clearly, if $n = \infty$ then $\dim H = \infty$.

Our theorem will be proved in several steps.

Step 1. *For any* $V_1, V_2 \in H_{(n)}$ *we have*

$$V_1 \subseteq V_2 \Leftrightarrow \psi(V_1) \subseteq \psi(V_2) \quad \textit{and} \quad V_1 \perp V_2 \Leftrightarrow \psi(V_1) \perp \psi(V_2).$$

Now ψ *is clearly injective.*

Moreover, if $n < \infty$, *then for any* $K \in H_n$ *obviously* $\psi(K) = \phi(K)$ *holds.*

Let $V_1, V_2 \in H_{(n)}$. If $V_1 \subseteq V_2$ then, by (4), it is trivial that $\psi(V_1) \subseteq \psi(V_2)$.

If $V_1 \nsubseteq V_2$ then, by $\operatorname{codim} V_2 \geq n$, there exists $M \in H_n$ for which $M \perp V_2$ and $M \not\perp V_1$. Now there exists $N \in H_n$ with $N \subseteq V_1$, $M \not\perp N$. So $\phi(M) \not\perp \phi(N) \subseteq \psi(V_1)$, thus $\phi(M) \not\perp \psi(V_1)$. For any $K \in H_n$, $K \subseteq V_2$ we have $M \perp K$, which implies $\phi(M) \perp \phi(K)$. Hence, by (4), $\phi(M) \perp \psi(V_2)$. Thus $\psi(V_1) \not\perp \phi(M) \perp \psi(V_2)$, which yields $\psi(V_1) \nsubseteq \psi(V_2)$.

If $V_1 \perp V_2$ then for any $M, N \in H_n$, $M \subseteq V_1$, $N \subseteq V_2$ we have $\phi(M) \perp \phi(N)$ which gives $\psi(V_1) \perp \psi(V_2)$.

If $V_1 \not\perp V_2$ then there exist $M, N \in H_n$ with $M \subseteq V_1$, $N \subseteq V_2$ such that $M \not\perp N$. Now $\psi(V_1) \supseteq \phi(M) \not\perp \phi(N) \subseteq \psi(V_2)$, thus $\psi(V_1) \not\perp \psi(V_2)$.

Step 2. *For any* $V \in H_{(n)}$ *we have*

$$\dim \psi(V) \geq \dim V, \tag{5}$$

$$\operatorname{codim} \psi(V) \geq \operatorname{codim} V. \tag{6}$$

Hence for any $m \in \mathbb{N} \cup \{\infty\}$ *for which* (1) *holds, we have* $\psi : H_{(m)} \to H_{(m)}$. *Moreover, for any* $K \in H_n$ *we have* $\phi(K) = \psi(K)$, *which implies that* ϕ *is also injective.*

If $\dim V = \infty$ then (5) is trivial. Assume that $n \leq \dim V < \infty$. Let $V_k \in H_k$ $(n \leq k \leq \dim V)$ with $V_n \subsetneqq V_{n+1} \subsetneqq \cdots \subsetneqq V_{\dim V} = V$. By Step 1, we have $\psi(V_n) \subsetneqq \psi(V_{n+1}) \subsetneqq \cdots \subsetneqq \psi(V_{\dim V})$, whence $n = \dim \psi(V_n) < \dim \psi(V_{n+1}) < \cdots < \dim \psi(V_{\dim V}) = \dim V$, which implies (5).

Now, applying (5) to $V^\perp$, we obtain that $\dim\psi(V^\perp) \ge \dim V^\perp$. By Step 1, we have $\psi(V) \perp \psi(V^\perp)$, whence $\operatorname{codim}\psi(V) = \dim\psi(V)^\perp \ge \dim\psi(V^\perp) \ge \dim V^\perp = \operatorname{codim} V$ (6).

If $\dim H < \infty$ then $\phi(K) = \psi(K)$ is now trivial.

Finally, let $\dim H = \infty$ and let $K \in H_n$ be arbitrary. Now ϕ is surjective by assumption, and, by (4), we have $\phi(K) \subseteq \psi(K)$. Suppose on the contrary that $\phi(K) \subsetneqq \psi(K)$. Then there exists $L \in H_n$ with $\phi(L) \not\perp \psi(K)$ and $\phi(L) \perp \phi(K)$. Hence by (4) there exists $M \in H_n$, $M \subseteq K$ with $\phi(M) \not\perp \phi(L)$. Now Step 1 yields $K \supseteq M \not\perp L \perp K$, which is a contradiction. Therefore $\phi(K) = \psi(K)$.

Step 3. *If ϕ is a bijection then define ψ^{-1} for ϕ^{-1} as ψ was defined above for ϕ. Now for any $V \in H_{(n)}$ we have $\psi^{-1}(\psi(V)) = \psi(\psi^{-1}(V)) = V$. Thus, if ϕ is a bijection, then so is ψ.*

Let $V \in H_{(n)}$ and $K \in H_n$ be arbitrary with $K \subseteq \psi(V)$. Suppose on the contrary that $\phi^{-1}(K) \not\subseteq V$. Then there exists $L \in H_n$ with $L \not\perp \phi^{-1}(K)$, $L \perp V$. Now $\phi(L) \not\perp K \subseteq \psi(V) \perp \phi(L)$, which is a contradiction. Thus for any $K \in H_n$ with $K \subseteq \psi(V)$ we have $\phi^{-1}(K) \subseteq V$. This yields $\psi^{-1}\big(\psi(V)\big) \subseteq V$. For any $K \in H_n$ with $K \subseteq V$ we have $\phi(K) \subseteq \psi(V)$, which implies $K = \phi^{-1}\big(\phi(K)\big) \subseteq \psi^{-1}\big(\psi(V)\big)$. Hence $V \subseteq \psi^{-1}\big(\psi(V)\big)$, and thus $\psi^{-1}\big(\psi(V)\big) = V$.

Step 4. *For any $V \in H_{(n)}$ we have $\dim\psi(V) = \dim V$ and $\psi(V^\perp) = \psi(V)^\perp$.*

If $\dim H < \infty$ then by Steps 1 and 2 we are done.

If ϕ is bijective then, applying (5) to ψ and ψ^{-1}, by Step 3 we obtain that $\dim V = \dim\psi^{-1}\big(\psi(V)\big) \ge \dim\psi(V) \ge \dim V$, which gives $\dim\psi(V) = \dim V$. Moreover,

$$\psi^{-1}\big(\psi(V^\perp)^\perp\big) \perp \psi^{-1}\big(\psi(V^\perp)\big) = V^\perp,$$

whence $\psi^{-1}\big(\psi(V^\perp)^\perp\big) \subseteq V$. Steps 1 and 3 now yield that $\psi(V^\perp)^\perp \subseteq \psi(V)$. This implies $\psi(V^\perp) \supseteq \psi(V)^\perp \supseteq \psi(V^\perp)$.

Step 5. *For any $X \in H_1$ and $V \in H_n$ with $V \perp X$, let*

$$\psi_V(X) = \psi(V)^\perp \cap \psi(V + X).$$

Then $\dim \psi_V(X) = 1$, *and* $\psi_V(X)$ *does not depend on* V. *Now let* $\psi(X) = \psi_V(X)$.

Let $V \in H_n$ be arbitrary such that $V \perp X$. Step 1 implies $\psi(V) \subsetneqq \psi(V+X)$, whence $\dim \psi_V(X) \geq 1$. If $\dim H < \infty$ then, by Step 4, we infer that $\dim \psi(V+X) = \dim (V+X) = \dim (V) + 1 = \dim \psi(V) + 1$, whence $\dim \psi_V(X) = 1$. Suppose temporarily that $\dim H = \infty$. Then ϕ and ψ are bijections. If $\dim \psi_V(X) > 1$ then there exists $U \in H_{(n)}$ such that $\psi(V) \subsetneqq U \subsetneqq \psi(V+X)$. By Step 1 now we get $V \subsetneqq \psi^{-1}(U) \subsetneqq V + X$, which is a contradiction. Thus

$$\dim \psi_V(X) = 1 \tag{7}$$

for any $V \in H_n$ with $V \perp X$.

Suppose now that $n \in \mathbb{N}$. We show that for any $V_1, V_2 \in H_n$ with $V_1, V_2 \perp X$ we have $\psi_{V_1}(X) = \psi_{V_2}(X)$. By $\dim H > 3n$ there is $V \in H_n$ with $V \perp X, V_1, V_2$. We have $X + V_1 + V \in H_{(n)}$, whence $\dim \psi(X + V_1 + V) = \dim(X+V_1+V) = 2n+1$. Now $\dim\bigl(\psi(X+V_1)\bigr) = \dim\bigl(\psi(X+V)\bigr) = n+1$ and $\psi(X+V_1), \psi(X+V) \subseteq \psi(X+V_1+V)$ yield $\psi(X+V_1) \cap \psi(X+V) \neq \emptyset$. Let $Y \subseteq \psi(X+V_1) \cap \psi(X+V)$ with $\dim Y = 1$. Then $Y \subseteq \psi(X+V_1) \perp \psi(V)$ and $Y \subseteq \psi(X+V)$, hence $Y = \psi_V(X)$. Similarly, $Y = \psi_{V_1}(X)$, thus $\psi_{V_1}(X) = \psi_V(X)$. Similarly again, we obtain $\psi_{V_2}(X) = \psi_V(X)$, hence $\psi_{V_1}(X) = \psi_{V_2}(X)$.

Suppose now that $n = \infty$. Then ψ is a bijection. Let $V_1, V_2 \in H_n$ be arbitrary with $V_1, V_2 \perp X$. Then there exist $k \in \mathbb{N}$ and $U_1, \ldots, U_k \in H_n$ with $V_1 = U_1$, $V_2 = U_k$, $U_1, \ldots, U_k \perp X$, $U_i \perp U_{i+1}$ and $U_i + U_{i+1} + X \in H_{(n)}$ for any $1 \leq i < k$. Let $1 \leq i < k$ be arbitrary. Now $\psi(U_i) \perp \psi(U_{i+1}) + \psi_{U_i+U_{i+1}}(X) \subseteq \psi(U_i + U_{i+1} + X)$ implies $U_i \perp \psi^{-1}\bigl(\psi(U_{i+1}) + \psi_{U_i+U_{i+1}}(X)\bigr) \subseteq U_i + U_{i+1} + X$. Thus $\psi^{-1}\bigl(\psi(U_{i+1}) + \psi_{U_i+U_{i+1}}(X)\bigr) \subseteq U_{i+1} + X$, whence $\psi(U_{i+1}) + \psi_{U_i+U_{i+1}}(X) \subseteq \psi(U_{i+1} + X)$. By (7) and $\psi_{U_i+U_{i+1}}(X) \perp \psi(U_{i+1})$ we obtain $\psi_{U_i+U_{i+1}}(X) = \psi_{U_{i+1}}(X)$. Similarly $\psi_{U_i+U_{i+1}}(X) = \psi_{U_i}(X)$, which yields $\psi_{U_i}(X) = \psi_{U_{i+1}}(X)$, and so we are ready.

Step 6. *For any* $X \in H_1$ *and* $V \in H_{(n)}$, $X \subseteq V$ *implies that* $\psi(X) \subseteq \psi(V)$.

Let $X \in H_1$ and $V \in H_{(n)}$ with $X \subseteq V$. If $\dim V > n$ or $n = \infty$ then there exists $V_0 \in H_n$ for which $V_0 \subseteq V \cap X^{\perp}$, whence, by Step 5, we obtain that

$$\psi(X) = \psi_{V_0}(X) \subseteq \psi(V_0 + X) \subseteq \psi(V).$$

Now assume that $n < \infty$ and $\dim V = n$. For any $U \in H_n$ with $U \subseteq V^\perp$, we have $\psi(X) = \psi_U(X) = \phi(U)^\perp \cap \psi(U + X) \perp \phi(U)$, and thus $\psi(X) \perp \psi(V^\perp)$. Step 4 now yields $\psi(X) \subseteq \psi(V^\perp)^\perp = \psi(V)$.

Step 7. *For any $X, Y \in H_1$, we have*

$$X \perp Y \Leftrightarrow \psi(X) \perp \psi(Y).$$

Suppose that $X \perp Y$. Then there are $M, N \in H_n$ such that $M \perp X$, $M \perp Y$, $N \perp X$, $N \perp M$ and $Y \subseteq N$. By Steps 1 and 6, we obtain that $\psi(X) \subseteq \psi(X + M) \perp \phi(N) \supseteq \psi(Y)$, whence $\psi(X) \perp \psi(Y)$.

Now suppose that $X \not\perp Y$ and suppose on the contrary that $\psi(X) \perp \psi(Y)$. Then there exists an $M \in H_n$ with $M \perp X$, $M \perp Y$. By Step 6, it is easy to see that $\phi(M) \perp \psi(X)$ and $\phi(M) \perp \psi(Y)$. Thus $\psi(M + X) = \psi(X) \oplus \phi(M) \perp \psi(Y)$. Let $K_X, N \in H_n$ such that $X \subseteq K_X \subseteq M + X$ and $Y, K_X \perp N$. Then $\phi(K_X) \perp \phi(N)$ and $\phi(K_X) \subseteq \psi(M + X) \perp \psi(Y)$. Hence $\psi(Y + N) = \psi(Y) \oplus \phi(N) \perp \phi(K_X)$. Now let $K_Y \in H_n$ with $Y \subseteq K_Y \subseteq Y + N$. Then $\phi(K_Y) \subseteq \psi(Y + N) \perp \phi(K_X)$, which implies $\phi(K_X) \perp \phi(K_Y)$. This leads to $X \subseteq K_X \perp K_Y \supseteq Y$, thus $X \perp Y$, which is a contradiction. Therefore $\psi(X) \not\perp \psi(Y)$ indeed.

Step 8. *If ϕ is a bijection then $\psi : H_1 \to H_1$ is also a bijection.*

Let ϕ be a bijection. Now there exists $\psi^{-1} : H_1 \to H_1$ corresponding to ϕ^{-1} as ψ corresponds to ϕ. We shall apply the above results also to ψ^{-1}. Let $V \in H_1$. Then there exist $V_1, V_2 \in H_n$ for which $V = V_1 \cap V_2$. Hence $\psi^{-1}(V) \subseteq \phi^{-1}(V_1) \cap \phi^{-1}(V_2)$, thus $\psi\big(\psi^{-1}(V)\big) \subseteq \phi\big(\phi^{-1}(V_1)\big) = V_1$ and $\psi\big(\psi^{-1}(V)\big) \subseteq \phi\big(\phi^{-1}(V_2)\big) = V_2$. This implies that $\{0\} \neq \psi\big(\psi^{-1}(V)\big) \subseteq V_1 \cap V_2 = V$. By $\dim V = 1$, we obtain that $\psi\big(\psi^{-1}(V)\big) = V$. Similarly, $\psi^{-1}\big(\psi(V)\big) = V$. Now it is already clear that ψ is a bijection.

Step 9. *For any $V \in H_{(n)}$, we have $\psi(V) = \operatorname{span}\big\{\psi(X) \mid X \in H_1,\ X \subseteq V\big\}$.*

By Step 6, it is clear that $\psi(V) \supseteq \operatorname{span}\{\psi(X) \mid X \in H_1,\ X \subseteq V\}$.

If $\dim H < \infty$ then, by Steps 4 and 7, we are ready.

Now let ϕ be surjective and let $Y \in H_1$ be arbitrary with $Y \subseteq \psi(V)$. Then, by Step 8, ψ is also a bijection. Applying Steps 3 and 6 to ψ^{-1}, we deduce that $\psi^{-1}(Y) \subseteq \psi^{-1}\big(\psi(V)\big) = V$. Hence $Y = \psi(\psi^{-1}(Y)) \subseteq \operatorname{span}\{\psi(X) \mid X \in H_1,\, X \subseteq V\}$. Thus $\psi(V) \subseteq \operatorname{span}\{\psi(X) \mid X \in H_1,\, X \subseteq V\}$, which completes the proof.

Step 10. *If ϕ preserves principal angles then ψ also preserves angles.*

Let $X, Y \in H_1$ be arbitrary, and let $N_X, N_Y \in H_{(n-1)}$ be orthogonal subspaces which are also orthogonal both to X and to Y, and for which $K_X = X \oplus N_X \in H_n$ and $K_Y = Y \oplus N_Y \in H_n$. Let e_α $(\alpha \in A)$ be pairwise orthogonal 1-dimensional subspaces of N_X with $N_X = \operatorname{span}\{e_\alpha \mid \alpha \in A\}$, and similarly, let f_β $(\beta \in B)$ be pairwise orthogonal 1-dimensional subspaces of N_Y with $N_Y = \operatorname{span}\{f_\beta \mid \beta \in B\}$.

By Steps 7 and 9, we have

$$P_{K_X} = P_X + \sum_{\alpha \in A} P_{e_\alpha}, \quad P_{\phi(K_X)} = P_{\psi(X)} + \sum_{\alpha \in A} P_{\psi(e_\alpha)},$$

$$P_{K_Y} = P_Y + \sum_{\beta \in B} P_{f_\alpha}, \quad P_{\phi(K_Y)} = P_{\psi(Y)} + \sum_{\beta \in B} P_{\psi(f_\beta)},$$

and so

$$\begin{aligned}
&P_{\psi(X)} P_{\psi(Y)} P_{\psi(X)} \\
&\quad = \Big(P_{\psi(X)} + \sum_{\alpha \in A} P_{\psi(e_\alpha)}\Big)\Big(P_{\psi(Y)} + \sum_{\beta \in B} P_{\psi(f_\beta)}\Big)\Big(P_{\psi(X)} + \sum_{\alpha \in A} P_{\psi(e_\alpha)}\Big) \\
&\quad = P_{\phi(K_X)} P_{\phi(K_Y)} P_{\phi(K_X)} = U_{K_X,K_Y} P_{K_X} P_{K_Y} P_{K_X} U^*_{K_X,K_Y} \\
&\quad = U_{K_X,K_Y} \Big(P_X + \sum_{\alpha \in A} P_{e_\alpha}\Big)\Big(P_Y + \sum_{\beta \in B} P_{f_\alpha}\Big)\Big(P_X + \sum_{\alpha \in A} P_{e_\alpha}\Big) U^*_{K_X,K_Y} \\
&\quad = U_{K_X,K_Y} P_X P_Y P_X U^*_{K_X,K_Y}.
\end{aligned}$$

Now, by Steps 7, 8, 9 and 10, the proof of the Theorem is complete. □

Proof of the Proposition. Let

$$\mathcal{A} = \big\{\{K, K^\perp\} \mid K \in H_n\big\}.$$

By the axiom of choice, there exists a set A which contains exactly one element of each set in $\mathcal{A}$. Let $\xi : A \to A$ be an arbitrary bijection, and define $\phi : H_n \to H_n$ by

$$\phi(K) = \begin{cases} \xi(K) & \text{if } K \in A, \\ \xi(K^\perp)^\perp & \text{otherwise.} \end{cases}$$

It is clear that $\phi : H_n \to H_n$ is a bijection which preserves orthogonality in both directions. If $n \geq 2$ then it is easy to see that we may choose a bijection ξ such that ϕ is not of the form (2). $\square$

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