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Modular group algebras with maximal Lie nilpotency indices

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Dedicated to the memory of Professor Jenő Erdős

Abstract. In the present paper we give the full description of the Lie nilpotent modular group algebras which have maximal Lie nilpotency indices.

1. Introduction

Let R be an associative algebra with identity. The algebra R can be regarded as a Lie algebra, called the associated Lie algebra of R, via the Lie commutator $[x, y] = xy - yx$, for every $x, y \in R$. Set $[x_1, \dots, x_{n-1}, x_n] = [[x_1, \dots, x_{n-1}], x_n]$, where $x_1, \dots, x_n \in R$. The *n-th lower Lie power* $R^{[n]}$ of R is the associative ideal generated by all Lie commutators $[x_1, \dots, x_n]$, where $R^{[1]} = R$ and $x_1, \dots, x_n \in R$. By induction, we define the *n-th upper Lie power* $R^{(n)}$ of R as the associative ideal generated by all Lie commutators $[x, y]$, where $R^{(1)} = R$ and $x \in R^{(n-1)}$, $y \in R$.

An algebra R is called *Lie nilpotent* if there exists m such that $R^{[m]} = 0$. The minimal integers m, n such that $R^{[m]} = 0$ and $R^{(n)} = 0$ are called *the Lie nilpotency index* and *the upper Lie nilpotency index* of R and they are denoted by $t_L(R)$ and $t^L(R)$, respectively.

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An algebra R is called *Lie hypercentral* if for every sequence $\{a_i\}$ of elements of R there exists some n such that $[a_1,\dots,a_n]=0$.

Let KG be the group algebra of a group G over a field K of characteristic $\mathrm{char}(K)=p>0$. According to [2], [9] for the noncommutative group algebras KG, the following statements are equivalent: (a) KG is Lie nilpotent; (b) KG is Lie hypercentral; (c) G is nilpotent and its commutator subgroup G' is a finite p-group. It is well known ([8, 13]) that if KG is Lie nilpotent then $t_L(KG)\le t^L(KG)\le |G'|+1$. Moreover, according to [1], if $\mathrm{char}(K)>3$ then $t_L(KG)=t^L(KG)$. But the question of when $t_L(KG)=t^L(KG)$ for $\mathrm{char}(K)=2,3$ is still open.

In the present paper we investigate the group algebras KG for which $t_L(KG)$ is *maximal*, i.e. $t_L(KG)=|G'|+1$. In particular, if G is a finite p-group and $\mathrm{char}(K)\ge 5$, then, as Shalev proved in [12], $\mathtt{t}_L(KG)$ is maximal if and only if G' is cyclic. We give a complete characterization by proving the following:

Theorem 1. *Let KG be a Lie nilpotent group algebra with $\mathrm{char}(K)=p>0$. Then $t_L(KG)=|G'|+1$ if and only if one of the following conditions holds:*

(1) *G' is cyclic;*

(2) *$p=2$ and G' is the noncyclic of order 4 and $\gamma_3(G)\ne 1$.*

Corollary 1. *Let KG be a Lie nilpotent group algebra with $\mathrm{char}(K)=p>0$. If $t^L(KG)=|G'|+1$, then $t_L(KG)=t^L(KG)$.*

By Du's Theorem ([4]), the previous result lists also the group algebras KG whose group of units $U(KG)$ has maximal nilpotency class under the assumption that G is a finite p-group. Note that for G a 2-group of maximal class and K a field with $\mathrm{char}(K)=2$, Konovalov in ([7]) proved that $U(KG)$ has maximal nilpotency class.

We use the standard notation for a group G: $\Phi(G)$ denotes the Frattini subgroup of G; $g^h=h^{-1}gh$ and $(g,h)=g^{-1}h^{-1}gh$, $(g,h\in G)$; $\gamma_i(G)$ means the i-th term of the lower central series of G, i.e.

$$\gamma_1(G)=G,\quad \gamma_{i+1}(G)=\big(\gamma_i(G),G\big)\qquad (i\ge 1).$$

Moreover, C_n is the cyclic group of order n and set

$$\mathrm{Q}_{2^n}=\langle a,b\mid a^{2^{n-1}}=1,\ b^2=a^{2^{n-2}},\ a^b=a^{-1}\rangle,\qquad \text{with } n\ge 3;$$

$$\mathrm{D}_{2^n} = \langle a, b \mid a^{2^{n-1}} = b^2 = 1,\ a^b = a^{-1} \rangle, \qquad \text{with } n \geq 3;$$

$$\mathrm{SD}_{2^n} = \langle a, b \mid a^{2^{n-1}} = b^2 = 1,\ a^b = a^{-1+2^{n-2}} \rangle, \qquad \text{with } n \geq 4;$$

$$\mathrm{MD}_{2^n} = \langle a, b \mid a^{2^{n-1}} = b^2 = 1,\ a^b = a^{1+2^{n-2}} \rangle, \qquad \text{with } n \geq 4.$$

2. Preliminaries

Let K be a field of characteristic $p > 0$ and G a group. We consider a sequence of subgroups of G setting

$$\mathfrak{D}_{(m)}(G) = G \cap (1 + KG^{(m)}), \qquad (m \geq 1).$$

The subgroup $\mathfrak{D}_{(m)}(G)$ is called the m-th *Lie dimension subgroup* of KG. It is possible to describe the $\mathfrak{D}_{(m)}(G)$'s in terms of the lower central series of G in the following manner ([8], p. 44)

$$\mathfrak{D}_{(m+1)}(G) = \begin{cases} G & \text{if } m = 0; \\ G' & \text{if } m = 1; \\ \big(\mathfrak{D}_{(m)}(G), G\big)(\mathfrak{D}_{(\lceil \frac{m}{p} \rceil + 1)}(G))^p & \text{if } m \geq 2. \end{cases} \tag{1}$$

where $\lceil \frac{m}{p} \rceil$ is the smallest integer greater than $\frac{m}{p}$.

Put $p^{d_{(m)}} := [\mathfrak{D}_{(m)}(G) : \mathfrak{D}_{(m+1)}(G)]$. If KG is Lie nilpotent, according to Jennings' theory ([10]) for the Lie dimension subgroups, we get that

$$t^L(KG) = 2 + (p-1) \sum_{m \geq 1} m d_{(m+1)}.$$

Lemma 1 ([11], [12]). *Let K be a field with $\mathrm{char}(K) = p > 0$ and G a nilpotent group such that G' is a finite p-group with $\exp(G') = p^l$.*

(1) *If $d_{(m+1)} = 0$ and m is a power of p, then $\mathfrak{D}_{(m+1)}(G) = 1$.*

(2) *If $d_{(m+1)} = 0$ and p^{l-1} divides m, then $\mathfrak{D}_{(m+1)}(G) = 1$.*

Lemma 2. *Let $p, s, n \in \mathbb{N}$ and $m_0, \ldots, m_{s-1}$ the non-negative integers such that $s < n$ and $\sum_{i=0}^{s-1} m_i = n$. Then $\sum_{i=0}^{s-1} m_i p^i < \sum_{i=0}^{n-1} p^i$.*

Proof. By the assumptions, there exist integers $0 \leq j_1 < \cdots < j_k \leq s-1$ such that $m_{j_l} > 1$ for every $1 \leq l \leq k$. Since $p^s > p^{j_k}$, we obtain that

$$\begin{aligned}\sum_{i=0}^{s-1} m_i p^i &= \sum_{i=0}^{s-1} p^i + \sum_{i=1}^{k}(m_{j_i}-1)p^{j_i}\\ &\leq \sum_{i=0}^{s-1} p^i + p^{j_k}(n-s) < \sum_{i=0}^{s-1} p^i + \sum_{i=s}^{n-1} p^i. \qquad \square\end{aligned}$$

Lemma 3. *Let K be a field with $\mathrm{char}(K) = p > 0$ and G a nilpotent group such that $|G'| = p^n$. Then $t^L(KG) = |G'|+1$ if and only if $d_{(p^i+1)} = 1$ and $d_{(j)} = 0$, where $0 \leq i \leq n-1$, $j \neq p^i + 1$ and $j > 1$.*

Proof. If $d_{(p^i+1)} = 1$ for $0 \leq i \leq n-1$ and $d_{(j)} = 0$ for $j > 1$, then

$$t^L(KG) = 2 + (p-1)\sum_{i=0}^{n-1} p^i = 1 + p^n = |G'| + 1.$$

In order to prove the other implication, we preliminarily remark that

$$\sum_{m \geq 2} d_{(m)} = n \tag{2}$$

that is an immediate consequence of the definition of $d_{(j)}$'s. Now we suppose that there exists $0 \leq j \leq n-1$ such that $d_{(p^j+1)} = 0$. Let s be the minimal integer for which $d_{(p^s+1)} = 0$. From (1) it follows at once that $s \neq 0$ and by (1) of Lemma 1 we have that $\mathfrak{D}_{(p^s+1)}(G) = 1$ and so $d_{(r)} = 0$ for every $r \geq p^s + 1$. It is immediate by (2) that $\alpha = \sum_{i=0}^{s-1} d_{(p^i+1)} \leq n$. Let us consider the following two cases: $\alpha = n$ and $\alpha < n$. If $\alpha = n$, then, according to Lemma 2, we have that

$$t^L(KG) = 2 + (p-1)\sum_{i=0}^{s-1} p^i d_{(p^i+1)} < 2 + (p-1)\sum_{i=0}^{n-1} p^i = |G'| + 1.$$

If $\alpha < n$ by (2) there exists at least one $j > 1$ such that $d_{(j)} \neq 0$ and $j \neq p^i + 1$. Suppose that $d_{(j_1)}, \ldots, d_{(j_k)}$ are all of such $d_{(j)}$'s, where $j_1 < \cdots < j_k$. Clearly, $j_k \leq p^s$. According to Lemma 2 for the case $\alpha > s$, we obtain that

$$t^L(KG) = 2 + (p-1)\sum_{i=0}^{s-1} p^i d_{(p^i+1)} + (p-1)\sum_{i=1}^{k}(j_i - 1)d_{(j_i)}$$

$$\leq 2 + (p-1)\sum_{i=0}^{\alpha-1} p^i + (p-1)(j_k - 1)(n-\alpha)$$
$$< 2 + (p-1)\sum_{i=0}^{\alpha-1} p^i + (p-1)p^s(n-\alpha)$$
$$< 2 + (p-1)\sum_{i=0}^{\alpha-1} p^i + (p-1)\sum_{i=\alpha}^{n-1} p^i = |G'| + 1.$$

So, if $t^L(KG)$ is maximal, then $d_{(p^j+1)} > 0$ for each $0 \leq j \leq n-1$ and, by (2), the lemma is proved. □

Corollary 2. *Let K be a field with $\mathrm{char}(K) = p > 0$ and G a nilpotent group with $|G'| = p^n$. If $t^L(KG) = |G'| + 1$, then $|\mathfrak{D}_{(p^i+1)}(G)| = p^{n-i}$, for $0 \leq i \leq n$.*

Proof. By Lemma 3, it is easy to check that

$$\mathfrak{D}_{(p^0+1)}(G) \supset \mathfrak{D}_{(p^1+1)}(G) \supset \mathfrak{D}_{(p^1+2)}(G) = \cdots$$
$$\cdots = \mathfrak{D}_{(p^i+1)}(G) \supset \mathfrak{D}_{(p^i+2)}(G) = \cdots$$
$$\cdots = \mathfrak{D}_{(p^s+1)}(G) \supset \mathfrak{D}_{(p^s+2)}(G) = 1$$

for some $s \in \mathbb{N}$. Clearly, $|\mathfrak{D}_{(p^s+1)}(G)| = p$, $|\mathfrak{D}_{(p^{s-1}+1)}(G)| = p^2$ and $|\mathfrak{D}_{(p^i+1)}(G)| = p^{s-i+1}$, so $|\mathfrak{D}_{(p^0+1)}(G)| = p^{s+1} = p^n$ and $s = n-1$. □

Lemma 4. *Let K be a field with $\mathrm{char}(K) = p > 0$ and G a nilpotent group with G' a finite p-group such that $t^L(KG) = |G'| + 1$.*

(1) *If $p > 2$ then G' is cyclic.*

(2) *If $p = 2$ then G' has at most two generators.*

Proof. Assume that $|G'| = p^n$. Let us prove that

$$|\Phi(G')| \geq \begin{cases} p^{n-1} & \text{if } p \neq 2; \\ 2^{n-2} & \text{if } p = 2. \end{cases} \tag{3}$$

First, set $p \neq 2$, $1 < a < p$ and suppose that $|\Phi(G')| \leq p^{n-2}$, where $n \geq 2$. Since $\exp(G'/\Phi(G')) = p$, we have that $\exp(G') = p^k \leq p^{n-1}$ for some k. By Lemma 3, we get $d_{(ap^{n-2}+1)} = 0$ and p^{k-1} divides p^{n-2}. Then, by (2) of Lemma 1, we obtain that $\mathfrak{D}_{(ap^{n-2}+1)}(G) = 1$. But $ap^{n-2} < p^{n-1}$ and $\mathfrak{D}_{(p^{n-1}+1)}(G) \neq 1$, which is a contradiction.

Now, set $p = 2$ and we suppose that $|\Phi(G')| \leq 2^{n-3}$, where $n \geq 3$. By Lemma 3 we have that $d_{(3\cdot 2^{n-3}+1)} = 0$. Since $3 \cdot 2^{n-3} < 2^{n-1}$, by (2) of Lemma 1 we get that $\mathfrak{D}_{(3\cdot 2^{n-3}+1)}(G) = 1$ and $\mathfrak{D}_{(2^{n-1}+1)}(G) \neq 1$, which is a contradiction either. □

Lemma 5. *Let K be a field with* $\mathrm{char}(K) = 2$*, G a nilpotent group such that G' is a 2-generated finite 2-group and let $t^L(KG) = |G'| + 1$. If either $\gamma_2(G)^2 \subset \gamma_3(G)$ or $\gamma_3(G) \cap \gamma_2(G)^2 = 1$ then $|\gamma_3(G)| = 2$ and $\gamma_2(G) \cong C_2 \times C_2$.*

Proof. Assume that $|G'| = 2^n$. Let G be nilpotent of class $\mathrm{cl}(G) = t \leq n+1$ and $\gamma_2(G)^2 \subset \gamma_3(G)$. Then, by Theorem III.2.13 ([6], p. 266), we have that $\gamma_k(G)^2 \subseteq \gamma_{k+1}(G)$ for every $k \geq 2$. Let us prove by induction on i that $\mathfrak{D}_{(2^i+1)}(G) = \gamma_{i+2}(G)$. It follows at once that $\mathfrak{D}_{(2)}(G) = \gamma_2(G)$ and $\mathfrak{D}_{(3)}(G) = \gamma_3(G)$. According to Lemma 3 we have that

$$\begin{aligned}
\mathfrak{D}_{(2^{i+1}+1)}(G) &= \mathfrak{D}_{(2^i+2)}(G) \\
&= \big(\mathfrak{D}_{(2^i+1)}(G), G\big) \cdot \mathfrak{D}_{(\lceil 2^{i-1}+\frac{1}{2}\rceil+1)}(G)^2 \\
&= \big(\gamma_{i+2}(G), G\big) \cdot \mathfrak{D}_{(2^{i-1}+2)}(G)^2 \\
&= \gamma_{i+3}(G) \cdot \mathfrak{D}_{(2^{i-1}+2)}(G)^2 \\
&= \gamma_{i+3}(G) \cdot \mathfrak{D}_{(2^i+1)}(G)^2 \\
&= \gamma_{i+3}(G) \cdot \gamma_{i+2}(G)^2 = \gamma_{i+3}(G).
\end{aligned}$$

It follows that $\mathfrak{D}_{(2^{t-1}+1)}(G) = \gamma_{t+1}(G) = 1$, but by Lemma 3 we have that $\mathfrak{D}_{(2^{n-1}+1)}(G) \neq 1$, so $t > n$ and $t = n+1$.

Obviously, for $i \geq 1$

$$\begin{aligned}
\mathfrak{D}_{(2^i+2)}(G) &= \big(\mathfrak{D}_{(2^i+1)}(G), G\big) \cdot \mathfrak{D}_{(2^{i-1}+2)}(G)^2 \\
&= \gamma_{i+3}(G) \cdot \mathfrak{D}_{(2^i+1)}(G)^2 \\
&= \gamma_{i+3}(G) \cdot \gamma_{i+2}(G)^2 = \gamma_{i+3}(G);
\end{aligned}$$

$$\begin{aligned}
\mathfrak{D}_{(2^i+3)}(G) &= \big(\mathfrak{D}_{(2^i+2)}(G), G\big) \cdot \mathfrak{D}_{(2^{i-1}+2)}(G)^2 \\
&= \gamma_{i+4}(G) \cdot \gamma_{i+2}(G)^2.
\end{aligned}$$

Since $\mathfrak{D}_{(2^i+2)}(G)=\mathfrak{D}_{(2^i+3)}(G)$ for $i\geq 1$ we get that

$$\gamma_{i+3}(G)=\gamma_{i+4}(G)\cdot\gamma_{i+2}(G)^2.$$

According to $\gamma_3(G)^2\supset\gamma_4(G)^2\supset\cdots$ it follows that

$$\gamma_4(G)=\gamma_3(G)^2\cdot\gamma_4(G)^2\cdot\gamma_5(G)^2\cdots\gamma_t(G)^2=\gamma_3(G)^2.$$

Since $\Phi(\gamma_3(G))=\gamma_3(G)^2$ we have that

$$[\gamma_3(G):\Phi(\gamma_3(G))]=[\gamma_3(G):\gamma_4(G)]=2,$$

so $\gamma_3(G)$ is cyclic. According to Theorem 12.5.1 in [5], the 2-generated group $\gamma_2(G)$ with cyclic subgroup of index 2 is one of the following groups: Q_{2^n}, D_{2^n}, SD_{2^n}, MD_{2^n}, or $C_2\times C_{2^{n-1}}$, and therefore $\gamma_2(G)^2=\gamma_4(G)$.

Moreover, $\gamma_3(G)^2\subseteq\gamma_5(G)$. Indeed, the elements of the form (x,y), where $x\in\gamma_2(G)$ and $y\in G$ are generators of $\gamma_3(G)$, so we have to prove that $(x,y)^2\in\gamma_5(G)$. Evidently,

$$(x^2,y)=(x,y)(x,y,x)(x,y)=(x,y)^2(x,y,x)^{(x,y)}$$

and $(x^2,y),(x,y,x)^{(x,y)}\in\gamma_5(G)$, so $(x,y)^2\in\gamma_5(G)$ and $\gamma_3(G)^2\subseteq\gamma_5(G)$. Using the fact that $\exp(\gamma_3(G)/\gamma_5(G))$=2, since $\gamma_3(G)$ is cyclic, we obtain that $|\gamma_3(G)|$=2 and $\gamma_2(G)\cong C_2\times C_2$.

Now, let $\gamma_3(G)\cap\gamma_2(G)^2=1$. By (1) we have that $\mathfrak{D}_{(2)}(G)=\gamma_2(G)$, $\mathfrak{D}_{(3)}(G)=\gamma_2(G)^2\cdot\gamma_3(G)$ and

$$\mathfrak{D}_{(2)}(G)/\mathfrak{D}_{(3)}(G)=\gamma_2(G)/\bigl[\gamma_2(G)^2\cdot\gamma_3(G)\bigr]\cong\bigl[\gamma_2(G)/\gamma_2(G)^2\bigr]/\gamma_3(G).$$

Since $|\mathfrak{D}_{(2)}(G)/\mathfrak{D}_{(3)}(G)|=2$ and $|\gamma_2(G)/\gamma_2(G)^2|=4$, from the last equality it follows that $|\gamma_3(G)|=2$ and $\gamma_4(G)=1$.

Obviously, $(\gamma_2(G),\gamma_2(G))\subseteq\gamma_4(G)=1$, so $\gamma_2(G)$ is abelian and

$$\begin{aligned}\mathfrak{D}_{(4)}(G)=\mathfrak{D}_{(5)}(G)&=\bigl(\gamma_2(G)^2\cdot\gamma_3(G),G\bigr)(\gamma_2(G)^2\cdot\gamma_3(G))^2\\&=\gamma_2(G)^4\cdot\gamma_3(G)^2=\gamma_2(G)^4.\end{aligned}$$

It is easy to check that

$$\Phi(\gamma_2(G)^2\cdot\gamma_3(G))=(\gamma_2(G)^2\cdot\gamma_3(G))^2=\gamma_2(G)^4=\mathfrak{D}_{(5)}(G).$$

Therefore $\gamma_2(G)^2 \cdot \gamma_3(G)$ is a cyclic subgroup of index 2 in $\gamma_2(G)$ and

$$\gamma_2(G) = \langle a \rangle \times \langle b \rangle \cong C_{2^{n-1}} \times C_2 \quad (|a| = 2^{n-1},\ |b| = 2).$$

Clearly, $\gamma_2(G)^2 = \langle a^2 \rangle$ and either $\gamma_3(G) = \langle b \rangle$ or $\gamma_3(G) = \langle a^{2^{n-2}} b \rangle$. Now, let us compute the weak complement of $\gamma_3(G)$ in G' (see [3], p. 34). It is easy to see that $\nu(b) = \nu(a^{2^{n-2}} b) = 2$ and the weak complement will be $A = \langle a \rangle$. Since G is of class 3, by (ii) of Theorem 3.3 of [3] we have that

$$t_L(KG) = t^L(KG) = 2^n + 1 = t(\gamma_2(G)) + t(\gamma_2(G)/A) = 2^{n-1} + 3,$$

so $n = 2$ and $\gamma_2(G) \cong C_2 \times C_2$. □

3. Proof of Theorem 1

Let KG be a Lie nilpotent group algebra with $\mathrm{char}(K) = p > 0$ and let $t^L(KG) = |G'| + 1$. By Lemma 4 is either $p > 2$ and $\gamma_2(G)$ is cyclic or $p = 2$ and $\gamma_2(G)$ has at most 2 generators.

Now, let $p = 2$ and $\gamma_2(G)$ a 2-generated group. Let us prove that either $\gamma_2(G)^2 \subset \gamma_3(G)$ or $\gamma_3(G) \cap \gamma_2(G)^2 = 1$.

First, suppose that $\gamma_3(G) \subseteq \gamma_2(G)^2$. It is easy to see that

$$\mathfrak{D}_{(2)}(G) = \gamma_2(G), \quad \mathfrak{D}_{(3)}(G) = \gamma_2(G)^2$$

and

$$\mathfrak{D}_{(2)}(G)/\mathfrak{D}_{(3)}(G) = \gamma_2(G)/\gamma_2(G)^2 \cong \gamma_2(G)/\Phi(\gamma_2(G)) \cong C_2$$

which contradicts to the fact that $\gamma_2(G)$ is a 2-generated group.

Finally, suppose $\gamma_3(G) \cap \gamma_2(G)^2 \neq 1$ and $\gamma_2(G)^2 \not\subset \gamma_3(G)$. Clearly,

$$\mathfrak{D}_{(2)}(G) = \gamma_2(G); \quad \mathfrak{D}_{(3)}(G) = \gamma_2(G)^2 \cdot \gamma_3(G);$$

$$\mathfrak{D}_{(2^i+1)}(G) \equiv \gamma_3(G)^{2^{i-1}} \cdot \gamma_2(G)^{2^i} \pmod{\gamma_4(G)} \qquad (i \geq 2).$$

Since $\mathfrak{D}_{(4)}(G) \equiv \mathfrak{D}_{(5)}(G) \pmod{\gamma_4(G)}$, it follows that

$$\begin{aligned} 2 &= [\mathfrak{D}_{(3)}(G) : \mathfrak{D}_{(4)}(G)] \\ &\equiv [\gamma_2(G)^2 \cdot \gamma_3(G) : (\gamma_2(G)^2 \cdot \gamma_3(G))^2] \end{aligned}$$

$$\equiv [(\gamma_2(G)^2 \cdot \gamma_3(G)) : \Phi(\gamma_2(G)^2 \cdot \gamma_3(G))] \pmod{\gamma_4(G)}$$

and $L \equiv \gamma_2(G)^2 \cdot \gamma_3(G) \pmod{\gamma_4(G)}$ is a cyclic group. Set $L = \langle a \rangle$. Since $[\gamma_2(G) : \gamma_2(G)^2] = 4$, we get that L is a subgroup of index 2 in $\gamma_2(G)/\gamma_4(G)$. Thus, by Theorem 12.5.1 of [5], we have that

$$\gamma_2(G)/\gamma_4(G) = \big\langle a, b \mid a^{2^{n-1}} = 1,\ b^2 \in \langle a \rangle \big\rangle$$

is one of the following groups: C_{2^n}, $C_{2^{n-1}} \times C_2$, Q_{2^n}, D_{2^n}, SD_{2^n} or MD_{2^n}. But in these cases $\gamma_2(G)^2/\gamma_4(G) = \langle a^2 \rangle$ and from

$$L \equiv \gamma_2(G)^2 \cdot \gamma_3(G) \pmod{\gamma_4(G)}$$

it follows that $\langle a \rangle \equiv \langle a^2 \rangle \cdot \gamma_3(G) \pmod{\gamma_4(G)}$, so $a \in \gamma_3(G)/\gamma_4(G)$ and $\gamma_2(G)^2 \subset \gamma_3(G) \pmod{\gamma_4(G)}$, a contradiction. Therefore, by Lemma 5, we have that $|\gamma_3(G)| = 2$ and $\gamma_2(G) \cong C_2 \times C_2$.

Conversely, let $p \geq 2$ and $\gamma_2(G)$ a cyclic group. By (ii) of Theorem 3.1 of [3] we get that $t_L(KG) = t^L(KG) = |G'| + 1$. Now, let $p = 2$ and G' the noncyclic group of order 4 and $\gamma_3(G) \neq 1$. Again, by (ii) of Theorem 3.3 of [3], we obtain that

$$t_L(KG) = t^L(KG) = t(G') + t(G'/A) = |G'| + 1,$$

where A is the weak complement of $\gamma_3(G)$ in G' and the proof is complete.

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