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On the first Zassenhaus conjecture for integral group rings

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Dedicated to the memory of Professor Béla Brindza

Abstract. It was conjectured by H. Zassenhaus that a torsion unit of an integral group ring of a finite group is conjugate to a group element within the rational group algebra. The object of this note is the computational aspect of a method developed by I. S. Luthar and I. B. S. Passi which sometimes permits an answer to this conjecture. We illustrate the method on certain explicit examples. We prove with additional arguments that the conjecture is valid for any 3-dimensional crystallographic point group. Finally we apply the method to generic character tables and establish a p-variation of the conjecture for the simple groups $PSL(2,p)$.

1. Introduction

Let $V(\mathbb{Z}G)$ be the group of units of augmentation 1 of an integral group ring $\mathbb{Z}G$ of a finite group G. With respect to the structure of $V(\mathbb{Z}G)$ H. Zassenhaus stated around 1976 the following conjectures.

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ZC–1. Let u be a unit of finite order of $V(\mathbb{Z}G)$. Then u is conjugate within $\mathbb{Q}G$ to an element of G.[1]

ZC–2. Let H be a subgroup of $V(\mathbb{Z}G)$ with the same order as G. Then H is conjugate within $\mathbb{Q}G$ to G.

ZC–3. Let H be a finite subgroup of $V(\mathbb{Z}G)$. Then H is conjugate within $\mathbb{Q}G$ to a subgroup of G.

It was shown by K. W. Roggenkamp and L. L. Scott that the conjectures ZC–2 and ZC–3 are in general not true [16], [19], [13].

No counterexample however is known to the conjecture ZC–1. The conjectures ZC–2 and ZC–3 are valid[2] for many important classes of finite groups, e.g. nilpotent groups [17], [22]. The results indicate that the following variation of the Zassenhaus conjectures – a Sylow-like theorem for $V(\mathbb{Z}G)$ – may be true.

p-ZC–3. Let H be a finite p-subgroup of $V(\mathbb{Z}G)$. Then H is conjugate within $\mathbb{Q}G$ to a p-subgroup of G.

For a recent survey on the Zassenhaus conjectures and variations of them we refer to [12].

One of our goals is to provide algorithms and programs for the GAP – package LAGUNA which decide for a given group G whether the conjecture ZC – i is valid or not [11]. For computational aspects in integral group rings related to the Zassenhaus conjectures see [3], [2].

The main purpose of this note is to discuss the method developed by I. S. Luthar and I. B. S. Passi for the conjecture ZC–1 [14] under the view of computational aspects. We illustrate the Luthar–Passi method for $A_4 \times S_3$ and for the octahedral group $S_4 \times C_2$. In the second case the method alone does not suffice to establish the Zassenhaus conjecture whereas in the first case it suffices.

In [10] it is shown that ZC–1 is valid for all finite groups of order less than 71. We remark that the Luthar–Passi method plays an essential role to establish this result.

It seems to be unknown whether ZC–1 is true for a direct product of finite groups $G \times H$ provided it holds for each of its factors. We show

[1] Elements of G are called the trivial units of $\mathbb{Z}G$.

[2] We say that ZC–i is valid for a finite group G if $V(\mathbb{Z}G)$ has the property stated in ZC–i.

that this is the case when $H = C_2$. As a consequence, we obtain that the conjecture ZC–1 is valid for all finite 3-dimensional crystallographic point groups.

The method may also be applied to generic character tables. In [21] the simple groups $PSL(2, p^f)$ have been studied with it. In the last section it is shown that for the linear groups $PSL(2, p)$ the variation p-ZC–3 holds for the describing characteristic p.

2. The Luthar–Passi method

The ingredients for this method are the following results.

Theorem A ([14]). *Suppose that u is an element of $V(\mathbb{Z}G)$ of order k. Let z be a primitive k-th root of unity. Then for every integer l and any ordinary character χ of G, the number*

$$\mu_l(u, \chi) = \frac{1}{k} \sum_{d|k} Tr_{\mathbb{Q}(z^d)/\mathbb{Q}} \{\chi(u^d) \cdot z^{-dl}\} \tag{1}$$

is a non-negative integer. Let φ be a $\mathbb{C}$ - representation which affords χ. Then $\mu_l(u, \chi)$ is the multiplicity of z^l in the Jordan canonical form of $\varphi(u)$. In particular the degree of χ bounds $\mu_l(u, \chi)$.

Let $u = \sum \alpha_g g$ be a normalized non-central torsion unit of order k and let $\nu_i = \varepsilon_{C_i}(u)$ be the partial augmentation[3] of u with respect to the conjugacy class C_i. Then by well known theorems of G. Higman and S. D. Berman [18, Theorem 10, p. 102]

$$\nu_1 = 0 \quad \text{and more general} \quad \nu_j = 0$$

if the class C_j consists of a central element. Because u is normalized this implies

$$\nu_2 + \nu_3 + \cdots + \nu_m = 1,$$

where m denotes the class number of G.

[3]Let $u = \sum \alpha_g g \in \mathbb{Z}G$. Then the partial augmentation with respect to the conjugacy class C_i is defined as $\nu_i = \varepsilon_{C_i}(u) = \sum_{g \in C_i} \alpha_g$.

Theorem B ([5]). *Let u be a torsion unit of $V(\mathbb{Z}G)$. The order of u divides the exponent of G.*

Theorem C ([15, Theorem 2.7]). *Let u be a torsion unit of $V(\mathbb{Z}G)$. Let C be a conjugacy class of G. If p is a prime dividing the order of a representative of C but not the order of u then the partial augmentation $\varepsilon_C(u)$ is zero.*

Now the key result in order to establish the conjecture ZC–1 is the following one.

Theorem D ([14], [15, Theorem 2.5]). *Let u be a normalized unit of $\mathbb{Z}G$ of order k. Then u is conjugate in $\mathbb{Q}G$ to an element $g \in G$ if and only if for each d dividing k there is precisely one conjugacy class C_{i_d} with partial augmentation $\varepsilon_{C_{i_d}}(u^d) \neq 0$.*

Starting with the ordinary character table of a finite group G Theorem A yields restrictions on the multiplicities $\mu_l(u, \chi)$. These multiplicities are via

$$\chi(u^d) = \sum_j \varepsilon_{C_j}(u^d)\chi(g_j), \qquad g_j \in C_j$$

related with the partial augmentations of the conjugacy classes. The restrictions on the multiplicities lead to bounds on the partial augmentations. Additional information on the partial augmentation comes inductively from the quotients of G and theoretical statements like Theorem B. The starting point of the induction is given by the fact that ZC–1 is valid for the nilpotent quotients of G [22]. For some groups, e.g. $A_4 \times S_3$ this leads finally via Theorem C to ZC–1 or to statements for elements of p-power order which are relevant with respect to the conjecture p-ZC–3 (see the last section). The Luthar–Passi method is also of interest in the context of constructing a possible counterexample to ZC–1. It gives precise information about the partial augmentation of a candidate for a counterexample.

3. The group $A_4 \times S_3$

In this section we use the Luthar–Passi method to establish the following result.

Proposition 1. *Every normalized torsion unit u in $\mathbb{Z}G$, is rational conjugate to a group element, where $G = A_4 \times S_3$.*

Proof. The group G has the following character table (easily verified with [8]):

	$1a$	$3a$	$3b$	$2a$	$2b$	$6a$	$6b$	$2c$	$3c$	$3d$	$3e$	$6c$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1	-1	1	1	1	1
χ_3	1	ω	$\bar{\omega}$	1	1	ω	$\bar{\omega}$	1	1	ω	$\bar{\omega}$	1
χ_4	1	ω	$\bar{\omega}$	1	-1	$-\omega$	$-\bar{\omega}$	-1	1	ω	$\bar{\omega}$	1
χ_5	1	$\bar{\omega}$	ω	1	1	$\bar{\omega}$	ω	1	1	$\bar{\omega}$	ω	1
χ_6	1	$\bar{\omega}$	ω	1	-1	$-\bar{\omega}$	$-\omega$	-1	1	$\bar{\omega}$	ω	1
χ_7	2	2	2	2	0	0	0	0	-1	-1	-1	-1
χ_8	2	2ω	$2\bar{\omega}$	2	0	0	0	0	-1	$-\omega$	$-\bar{\omega}$	-1
χ_9	2	$2\bar{\omega}$	2ω	2	0	0	0	0	-1	$-\bar{\omega}$	$-\omega$	-1
χ_{10}	3	0	0	-1	-3	0	0	1	3	0	0	-1
χ_{11}	3	0	0	-1	3	0	0	-1	3	0	0	-1
χ_{12}	6	0	0	-2	0	0	0	0	-3	0	0	1

where ω denotes a primitive third root of unity.

Note that by Theorem B the possible orders of non-trivial torsion units of $V(\mathbb{Z}[A_4 \times S_3])$ are 2, 3 or 6, and according to Theorem C we get:

$$\begin{aligned}
&\nu_{3a} = \nu_{3b} = \nu_{3c} = \nu_{3d} = \nu_{3e} = \nu_{6x} = 0 \quad \text{when } k = 2;\\
&\nu_{2a} = \nu_{2b} = \nu_{2c} = \nu_{6x} = 0 \quad \text{when } k = 3; \qquad (2)\\
&\nu_{3a} = \nu_{3b} = \nu_{2a} = \nu_{2b} = \nu_{2c} = \nu_{3c} = \nu_{3d} = \nu_{3e} = \nu_{6x} = 0 \quad \text{when } k = 6,
\end{aligned}$$

where $6x$ denotes one of the following conjugacy classes: $6a$, $6b$, $6c$.

Let $u \in V(\mathbb{Z}G)$ be a non-trivial involution. According to (2) we get $\nu_{2a} + \nu_{2b} + \nu_{2c} = 1$ and by (1)

$$\begin{aligned}
\mu_0(u, \chi_2) &= \frac{1}{2}(\nu_{2a} - \nu_{2b} - \nu_{2c} + 1); & \mu_1(u, \chi_{11}) &= \frac{1}{2}(\nu_{2a} - 3\nu_{2b} + \nu_{2c} + 3);\\
\mu_1(u, \chi_2) &= \frac{1}{2}(1 - \nu_{2a} + \nu_{2b} + \nu_{2c}); & \mu_0(u, \chi_{11}) &= \frac{1}{2}(3 - \nu_{2a} + 3\nu_{2b} - \nu_{2c});\\
\mu_0(u, \chi_7) &= \nu_{2a} + 1; & \mu_1(u, \chi_{10}) &= \frac{1}{2}(\nu_{2a} + 3\nu_{2b} - \nu_{2c} + 3);\\
\mu_1(u, \chi_7) &= 1 - \nu_{2a}; & \mu_0(u, \chi_{10}) &= \frac{1}{2}(-\nu_{2a} - 3\nu_{2b} + \nu_{2c} + 3).
\end{aligned}$$

Since $\mu_i(u,\chi_j) \geq 0$, it follows that

$$(\nu_{2a}, \nu_{2b}, \nu_{2c}) \in \{(0,0,1),\ (0,1,0),\ (1,0,0)\}.$$

Let $u \in V(\mathbb{Z}G)$ be a non-trivial unit of order 3. Put

$$\nu_1 = \nu_{3a}, \quad \nu_2 = \nu_{3b}, \quad \nu_3 = \nu_{3c}, \quad \nu_4 = \nu_{3d}, \quad \nu_5 = \nu_{3e}.$$

According to (2) we get $\sum_{j=1}^{5} \nu_j = 1$. By (1)

$$\begin{aligned}
\mu_0(u,\chi_3) &= \frac{1}{3}(-\nu_1 - \nu_2 + 2\nu_3 - \nu_4 - \nu_5 + 1) \geq 0;\\
\mu_1(u,\chi_3) &= \frac{1}{3}(-\nu_1 + 2\nu_2 - \nu_3 - \nu_4 + 2\nu_5 + 1) \geq 0;\\
\mu_2(u,\chi_3) &= \frac{1}{3}(2\nu_1 - \nu_2 - \nu_3 + 2\nu_4 - \nu_5 + 1) \geq 0;\\
\mu_0(u,\chi_7) &= \frac{1}{3}(4\nu_1 + 4\nu_2 - 2\nu_3 - 2\nu_4 - 2\nu_5 + 2) \geq 0;\\
\mu_1(u,\chi_7) &= \frac{1}{3}(-2\nu_1 - 2\nu_2 + \nu_3 + \nu_4 + \nu_5 + 2) \geq 0;\\
\mu_0(u,\chi_8) &= \frac{1}{3}(-2\nu_1 - 2\nu_2 - 2\nu_3 + \nu_4 + \nu_5 + 2) \geq 0;\\
\mu_1(u,\chi_8) &= \frac{1}{3}(-2\nu_1 + 4\nu_2 + \nu_3 + \nu_4 - 2\nu_5 + 2) \geq 0;\\
\mu_2(u,\chi_8) &= \frac{1}{3}(4\nu_1 - 2\nu_2 + \nu_3 - 2\nu_4 + \nu_5 + 2) \geq 0;\\
\mu_0(u,\chi_{10}) &= 2\nu_3 + 1 \geq 0; \qquad \mu_1(u,\chi_{10}) = -\nu_3 + 1 \geq 0,
\end{aligned}$$

so we obtain only the following trivial solution:

$$\begin{aligned}(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) \in \{&(0,0,0,0,1),\ (0,0,0,1,0),\ (0,0,1,0,0),\\ &(0,1,0,0,0),\ (1,0,0,0,0)\}.\end{aligned}$$

Let $u \in V(\mathbb{Z}G)$ be a non-trivial unit of order 6. Clearly, $\chi(u^3)$ coincides either with $\chi(2a)$ or $\chi(2b)$ or $\chi(2c)$ and $\chi(u^2)$ coincides either with $\chi(3a)$ or $\chi(3b)$ or $\chi(3c)$ or $\chi(3d)$ or $\chi(3e)$. By (1) we obtain 15 systems of inequalities. These have no integral solutions, except for the case $\chi(u^3) = \chi(2b)$ and $\chi(u^2) = \chi(3a)$. In this exceptional case, it is easy to see that

$$\begin{aligned}
&\mu_1(u,\chi_2) = \mu_2(u,\chi_2), \quad \mu_0(u,\chi_3) = \mu_3(u,\chi_3), \quad \mu_1(u,\chi_3) = \mu_4(u,\chi_3),\\
&\mu_0(u,\chi_4) = \mu_3(u,\chi_4), \quad \mu_1(u,\chi_4) = \mu_4(u,\chi_4), \quad \mu_1(u,\chi_7) = \mu_2(u,\chi_7),\\
&\mu_0(u,\chi_8) = \mu_3(u,\chi_8), \quad \mu_1(u,\chi_8) = \mu_4(u,\chi_8).
\end{aligned}$$

Using these additional relations, we obtain that

$$(\nu_{2a}, \nu_{3a}, \nu_{2c}, \nu_{2d}, \nu_{4a}, \nu_{6a}, \nu_{2e}, \nu_{4b}) = (0,0,0,0,0,1,0,0,0,0,0).$$

Thus Theorem D completes the proof. $\square$

4. The octahedral group $S_4 \times C_2$

In this section we want to give an example of a group for which the Luthar–Passi method is not sufficient to proof ZC–1.

The group $G = S_4 \times C_2$ has the following character table (easily verified with [8]):

	$1a$	$2a$	$2b$	$3a$	$2c$	$2d$	$4a$	$6a$	$2e$	$4b$
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	−1	−1	1	1	1	−1	−1	−1	1
χ_3	1	−1	1	1	1	−1	−1	1	1	−1
χ_4	1	1	−1	1	1	−1	1	−1	−1	−1
χ_5	2	0	2	−1	2	0	0	−1	2	0
χ_6	2	0	−2	−1	2	0	0	1	−2	0
χ_7	3	−1	−3	0	−1	1	1	0	1	−1
χ_8	3	−1	3	0	−1	−1	1	0	−1	1
χ_9	3	1	−3	0	−1	−1	−1	0	1	1
χ_{10}	3	1	3	0	−1	1	−1	0	−1	−1

According to Theorem B the values 2, 3, 4, 6 and 12 are possible orders of a non-trivial normalized torsion unit u of $\mathbb{Z}G$. By Theorem D units of order 3 are conjugate to group elements. The Luthar–Passi method gives a positive answer of ZC–1 for units of order 4, 6 and 12, provided that the conjecture holds for elements of order 2. For an involution u of $\mathbb{Z}G$ the Luthar–Passi method gives 22 possible sets of partial augmentations of u, that would be in contradiction to ZC–1. In 20 of these cases the sum of partial augmentations of the classes of elements of order four is different from 0. Looking at the reduction $G \to G/C_2$ we see that the induced surjective ring homomorphism $\mathbb{Z}G \to \mathbb{Z}[G/C_2] = \mathbb{Z}S_4$ maps the conjugacy classes of order 4 of $S_4 \times C_2$ onto the unique conjugacy class C

of elements of order 4 of S_4. Thus the image of u is a unit which has at the conjugacy class C a partial augmentation different from zero. But for S_4 the conjecture ZC–1 holds [6].

In the remaining two cases the involution u has one of the following sets of partial augmentations different from 0:

$$(\nu_{2a}, \nu_{4a}, \nu_{4b}) = (1, -1, 1);$$
$$(\nu_{2d}, \nu_{4a}, \nu_{4b}) = (1, 1, -1).$$

If one considers now only the reduction maps $G \to G/N$, where N is an arbitrary normal subgroup of G, then no contradiction occurs. In fact an involution of this type would map modulo C_2 to a unit which has the same type as a transposition of $\mathbb{Z}S_4$. With respect to the other reductions the image has also always the same type as a trivial unit.

Using the fact that ZC–1 is valid for S_4 [6], the conjecture ZC–1 follows however for $S_4 \times C_2$ from the following result [10].

Proposition 2. *The conjecture ZC–1 holds for $G \times C_2$ provided it holds for G.*

Proof. Let $C_2 = \langle t \rangle$. Let κ be the surjective ring homomorphism $\mathbb{Z}[G \times C_2] \longrightarrow \mathbb{Z}G$ induced by the projection $\pi : G \times C_2 \longrightarrow G$ and denote by α the ring homomorphism which is induced by the identity on G and $t \mapsto -1 \in \mathbb{Z}G$.

Two conjugacy classes of $G \times C_2$ have the same image $\overline{C}$ in G if and only if they are of the form $C_1 = C \times \{1\}$ and $C_2 = C \times \{t\}$, where C is a conjugacy class of G. Let $u \in V(\mathbb{Z}G)$ be of order k. Denote the partial augmentations of u with respect to C_1 and C_2 by ν_1 and ν_2. The image of u under κ has with respect to $\overline{C}$ the partial augmentation $\nu_1 + \nu_2$ and under α the partial augmentation $\nu_1 - \nu_2$.

Because ZC–1 is valid for $\mathbb{Z}G$ we get that $\nu_1 + \nu_2 \in \{0, 1\}$ and that $\nu_1 - \nu_2 \in \{0, 1, -1\}$. Note that the unit $\alpha(u)$ may be not normalized. It follows that $\nu_1 + \nu_2 = 0$ if and only if $\nu_1 = \nu_2 = 0$ and $\nu_1 + \nu_2 \neq 0$ if and only if $\nu_1 = 1$ and $\nu_2 = 0$ or $\nu_1 = 0$ and $\nu_2 = 1$. Thus there is only one partial augmentation of u which is different from zero. By Theorem C we conclude that u is conjugate to an element of $G \times C_2$. □

We remark that it seems to be unknown whether the conjecture ZC–1 is valid for a direct product $G \times H$ if it holds for G and H.

5. 3-dimensional crystallographic point groups

Theorem. *ZC–1 is valid for all finite 3-dimensional crystallographic point groups.*

Proof. By the classification of 3-dimensional crystallographic point groups the maximal ones are $D_6 \times C_2$, $D_4 \times C_2$, $S_4 \times C_2$, where D_n denotes the dihedral group of order $2n$.

ZC–1 is known for $D_6 \times C_2$ and all its subgroups by [20], for the 2-group $D_4 \times C_2$ and all its subgroups by [22]. The subgroups of $S_4 \times C_2$ are of type G resp. $G \times C_2$, where G is either metacyclic, or isomorphic to A_4 or S_4. ZC–1 for A_4 has been proved in [1], and for S_4 in [6]. So it remains the case of the octahedral group $S_4 \times C_2$. Thus the previous section completes the proof. $\square$

Remark. Let H be a subgroup of the finite group G. Suppose that ZC–1 holds for G. Then we may consider $\mathbb{Z}H$ as subring of $\mathbb{Z}G$ via the inclusion of H in G. Because ZC–1 holds, a torsion unit u of $V(\mathbb{Z}H)$ is conjugate within $\mathbb{Q}G$ to $g \in G$. If C is a conjugacy class of G with $C \cap H = \varnothing$ then the partial augmentation $\varepsilon_C(u) = 0$. Therefore u is conjugate to $h \in H$ within $\mathbb{Q}G$. This leads naturally to the question whether this conjugation may be realized in $\mathbb{Q}H$. If this is the case ZC–1 would follow for H and the proof of ZC–1 for all finite groups would be reduced to that one for finite symmetric groups.

In the context of the proof of the theorem above it would then suffice to establish ZC–1 for the maximal finite crystallographic point groups.

Thus it might be worthwhile to consider the following version of the Zassenhaus conjecture ZC–1:

ZC–1$_R$ Suppose that u is a torsion unit of $V(\mathbb{Z}G)$ then there exists a ring R containing $\mathbb{Z}G$ as subring such that u is conjugate in R to an element of G.

6. Generic character tables

Let p be an odd prime ≥ 5. The group $PSL(2,p)$ has the following generic character tables ([7] §10, [4]). In the second row we list the orders of the elements in the respective conjugacy classes.

Define $\varepsilon \in \{\pm 1\}$ by $p \equiv \varepsilon \pmod 4$.

	C_1	C_2	C_3	$C_4(j)$	$C_5(j)$
	1	p	p	$\frac{p-1}{2}$	$\frac{p+1}{2}$
χ_1	1	1	1	1	1
χ_2	p	0	0	1	-1
χ_3	$\frac{p+\varepsilon}{2}$	$\frac{1}{2}(\varepsilon - \sqrt{\varepsilon p})$	$\frac{1}{2}(\varepsilon + \sqrt{\varepsilon p})$	$(-1)^j \frac{1+\varepsilon}{2}$	$(-1)^j \frac{1-\varepsilon}{2}$
χ_4	$\frac{p+\varepsilon}{2}$	$\frac{1}{2}(\varepsilon + \sqrt{\varepsilon p})$	$\frac{1}{2}(\varepsilon - \sqrt{\varepsilon p})$	$(-1)^j \frac{1+\varepsilon}{2}$	$(-1)^j \frac{1-\varepsilon}{2}$
$\chi_5(k)$	$p+1$	1	1	$a^{2jk} + a^{-2jk}$	0
$\chi_6(k)$	$p-1$	-1	-1	0	$-b^{2jk} - b^{-2jk}$

with $a = e^{\frac{2\pi i}{p-1}}$ and $b = e^{\frac{2\pi i}{p+1}}$.

For the sequel let $z = e^{\frac{2\pi i}{p}}$. By Theorem C we know that the partial augmentations ν_i of a unit $u \in \mathbb{Z}[PSL(2,7)]$ of order p are zero for the classes C_1, $C_4(j)$ and $C_5(j)$. Thus for an irreducible character χ, using $\nu_2 + \nu_3 = 1$, we get $\chi(u) = \nu_2 \chi(C_2) + (1-\nu_2)\chi(C_3)$. Hence

$$\chi_3(u) = \frac{\varepsilon}{2} + (-2\nu_2 + 1)\frac{\sqrt{\varepsilon p}}{2}.$$

For the calculation of the multiplicities we use the following.
For any integer l,

$$Tr_{\mathbb{Q}(z)/\mathbb{Q}}(\sqrt{\varepsilon p} \cdot z^{-l}) = \sum_{\nu=1}^{p-1} \sqrt{\varepsilon p} \left(\frac{\nu}{p}\right) z^{-l\nu} = \sqrt{\varepsilon p} \sum_{\nu=1}^{p-1} \left(\frac{\nu}{p}\right) (z^{\nu})^{-l}$$

where $\left(\frac{\nu}{p}\right)$ denotes the Legendre symbol. Thus we get

$$Tr_{\mathbb{Q}(z)/\mathbb{Q}}(\sqrt{\varepsilon p}) = 0 \quad \text{for } l \equiv 0 \pmod p.$$

C. F. Gauss[4] proved in [9] that

$$\sum_{\nu=1}^{p-1} \left(\frac{\nu}{p}\right) e^{\frac{2\pi i \nu}{p}} = \sum_{\nu=1}^{p-1} \left(\frac{\nu}{p}\right) z^{\nu} = \sqrt{\varepsilon p}.$$

[4]We thank the referee for this reference.

Hence

$$Tr_{\mathbb{Q}(z)/\mathbb{Q}}\big(\sqrt{\varepsilon p}\cdot z^{-l}\big) = \left(\frac{-l}{p}\right)\varepsilon p = \left(\frac{-l}{p}\right) p \quad \text{for } l \not\equiv 0 \pmod{p}.$$

For the multiplicities we get by Theorem A

(i) $\mu_0(u,\chi_3) = \frac{1+\varepsilon}{2}$.

(ii) $\mu_l(u,\chi_3) = 1-\nu_2$ if $Tr_{\mathbb{Q}(z)/\mathbb{Q}}\big(\sqrt{\varepsilon p}\cdot z^{-l}\big) = +p$.

(iii) $\mu_l(u,\chi_3) = \nu_2$ if $Tr_{\mathbb{Q}(z)/\mathbb{Q}}\big(\sqrt{\varepsilon p}\cdot z^{-l}\big) = -p$.

Because the multiplicities have to be non-negative it follows that $\nu_2 = 0$ or that $\nu_2 = 1$. This shows that elements of order p are conjugate to an element of a group basis in $\mathbb{Z}[PSL(2,p)]$.

The order of a finite subgroup of $V(\mathbb{Z}G)$ has to divide the order of G. Thus finite p - subgroups of the normalized unit subgroup of $\mathbb{Z}[PSL(2,p)]$ have order p. Summarizing we get

Proposition 3. *The variation p-ZC–3 is valid for $PSL(2,p)$.*

Finally we consider p-ZC–3 in the spirit of the variation ZC-1_R for permutation groups of prime degree.

Proposition 4. *Let p be a prime and let G be a primitive permutation group of degree p. Consider G as subgroup of the symmetric group S_p. Let u be a torsion unit of order p of $V(\mathbb{Z}G)$. Then u is conjugate within $\mathbb{Q}S_p$ to an element of G. In particular p-ZC–3 is valid for S_p.*

Proof. By the remark after the Theorem it suffices to establish the result for $G = S_p$. The symmetric group S_p has only one conjugacy class K of elements of order p. By Theorem C all partial augmentations $\varepsilon_C = 0$ for all classes $C \neq K$. By Theorem D we get that u is conjugate to an element of S_p within $\mathbb{Q}G$. Because p^2 does not divide $p!$, it follows that p-ZC–3 is valid for S_p. □

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