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Quasi-arithmetic elements of a given class of means

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Dedicated to the memory of Béla Brindza and Jenő Erdős

Abstract. We determine the quasi-arithmetic elements of a given class of means, which is defined by two unknown generating functions and a parameter.

1. Introduction

Let $I \subset \mathbb{R}$ be a non-empty open interval. A function $M : I^2 \to I$ is called a symmetric strict mean on I, if it has the following properties:

(i) If $x, y \in I$ and $x \neq y$, then $\min\{x, y\} < M(x, y) < \max\{x, y\}$,

(ii) $M(x, y) = M(y, x)$ if $x, y \in I$,

(iii) M is continuous in I^2.

Let $\mathcal{CM}(I)$ denote the set of continuous and strictly monotone real functions defined on the interval I.

A function $N : I^2 \to I$ is called a quasi-arithmetic mean on I, if there exists a $\chi \in \mathcal{CM}(I)$, so that

$$N(x, y) = \chi^{-1}\left(\frac{\chi(x) + \chi(y)}{2}\right) =: A_\chi(x, y)$$

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for all $x, y \in I$. Then this function χ is called the generating function of the quasi-arithmetic mean N. If $\chi = id$ then $A_\chi(x,y) = \frac{x+y}{2} =: A(x,y)$ is the well-known elementary arithmetic mean ([1], [12]).

It is obvious that any quasi-arithmetic mean $N : I^2 \to I$ is a symmetric strict mean on I.

Let a set $\mathcal{K}(I)$ of symmetric strict means be given on the interval I. Our general question is which elements are quasi-arithmetic in $\mathcal{K}(I)$. In other words all those elements $M \in \mathcal{K}(I)$ should be determined for which the equation

$$M = A_\chi \quad \text{on } I^2$$

has a solution $\chi \in \mathcal{CM}(I)$.

In this paper we discuss the problem in the case of the class $\mathcal{K}(I)$ of mean values defined below.

Let $L : I^2 \to I$ be a quasi-arithmetic mean whose generating function is $\psi \in \mathcal{CM}(I)$. Moreover, let $q \in \,]0,1]$ and $\varphi \in \mathcal{CM}(I)$. Then the function

$$L_\varphi^{(q)}(x,y) := \varphi^{-1}\left(q\varphi(x) + q\varphi(y) + (1-2q)\varphi(L(x,y))\right) \tag{1}$$

defined for $x, y \in I$, is a symmetric strict mean on I ([5]). As in the case $q = \frac{1}{2}$ $L_\varphi^{(q)}$ in (1) is a quasi-arithmetic mean therefore we look for the quasi-arithmetic means in the class

$$\mathcal{K}(I) := \left\{ (A_\psi)_\varphi^{(q)} : I^2 \to I \;\middle|\; \varphi, \psi \in \mathcal{CM}(I),\ q \in \,]0,1] \setminus \left\{\tfrac{1}{2}\right\} \right\}.$$

Earlier in some special cases the problem has already been studied. Thus, in the case $\psi = \mathrm{id}$ and $q = 1$ see the papers [2], [7], [8], [6]. In the case $\psi = \mathrm{id}$ and $q = \frac{1}{3}$ it was solved by Kahlig–Matkowski [13] under a strong smoothness condition.

2. Equivalent solutions

Define $\varphi, \psi \in \mathcal{CM}(I)$. We say that φ and ψ are equivalent on I if there exist constants $a \neq 0$ and b so that

$$\psi(x) = a\varphi(x) + b \quad \text{for all } x \in I.$$

In notation $\psi(x) \sim \varphi(x)$ if $x \in I$ or $\psi \sim \varphi$ on I. It is well-known that $A_\varphi = A_\psi$ on I^2 if and only if $\varphi \sim \psi$ on I. Moreover, when L is fix then

$L_\varphi^{(q)} = L_\psi^{(q)}$ on I^2 if and only if $\varphi \sim \psi$ on I (see Daróczy–Páles [5], Daróczy [2], [12]). It means that the elements of $\mathcal{K}(I)$ are described by $\varphi, \psi \in \mathcal{CM}(I)$ apart from equivalence where $q \in \,]0,1]\setminus\{\frac{1}{2}\}$. That is, if $\varphi \sim f$ and $\psi \sim g$ on I then

$$(A_\psi)_\varphi^{(q)} = (A_g)_f^{(q)} \quad \text{on } I^2 \quad \text{for all } q \in \,]0,1] \setminus \left\{\frac{1}{2}\right\}.$$

Introduce the following notations:

$$\varepsilon_p(x) := \begin{cases} x & \text{if } p = 0 \\ e^{px} & \text{if } p \neq 0 \end{cases} \qquad (x \in \mathbb{R})$$

and

$$P_+(I) = \{\, p \mid I + p \subset \mathbb{R}_+\}$$

$$P_-(I) = \{\, p \mid -I + p \subset \mathbb{R}_+\}$$

where $\mathbb{R}_+$ denotes the set of positive real numbers.

3. A solution of the problem

The following theorem yields a solution of the problem in question under a weak smoothness condition.

Theorem 3.1. *Suppose that $\varphi, \psi \in \mathcal{CM}(I)$ and $q \in]0,1]\setminus\{\frac{1}{2}\}$, furthermore there exists a non-empty open interval $J \subset I$ on which φ and ψ are continuously differentiable. Then*

$$(A_\psi)_\varphi^{(q)} : I^2 \to I$$

is a quasi-arithmetic mean on I if and only if one of the following cases holds

(i) *If $q \notin \{1, \frac{1}{4}\}$ then $\psi \sim \varphi$ on I,*

(ii) *If $q = 1$ then there exists $p \in \mathbb{R}$ so that $\psi \sim \varepsilon_p \circ \varphi$ on I,*

(iii) *If $q = \frac{1}{4}$ then either $\psi \sim \varphi$ on I or there exists $p \in P_+(\varphi(I))$ so that $\psi \sim \log\sqrt{\varphi + p}$ on I or there exists $p \in P_-(\varphi(I))$ so that $\psi \sim \log\sqrt{-\varphi + p}$ on I.*

Proof of Theorem 3.1. Without a loss of generality we can assume that there exists a non-empty open interval $J \subset I$ on which φ and ψ are continuously differentiable and $\varphi'(x) \neq 0$ and $\psi'(x) \neq 0$ if $x \in J$. As $(A_\psi)_\varphi^{(q)}$ is quasi-arithmetic on I therefore there exists a $\chi \in \mathcal{CM}(I)$ so that the functional equation

$$\begin{aligned}\varphi^{-1}\left(q\varphi(x) + q\varphi(y) + (1-2q)\varphi\left(\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right)\right)\right)\\ = \chi^{-1}\left(\frac{\chi(x)+\chi(y)}{2}\right)\end{aligned} \tag{2}$$

holds for all $x, y \in I$. Introduce new variables

$$u = \varphi(x), \quad v = \varphi(y) \qquad (u, v \in \varphi(I) := K)$$

and functions

$$f := \chi \circ \varphi^{-1}, \quad g := \psi \circ \varphi^{-1} \qquad (f, g \in CM(K)). \tag{3}$$

Then from (2) we obtain the functional equation

$$\frac{1}{2q} f^{-1}\left(\frac{f(u)+f(v)}{2}\right) + \left(1 - \frac{1}{2q}\right) g^{-1}\left(\frac{g(u)+g(v)}{2}\right) = \frac{u+v}{2} \tag{4}$$

for all $u, v \in K$. Clearly, with the notation $\mu := \frac{1}{2q} > 0$ $\mu \neq 1$ and from (4) we have

$$\mu A_f(u, v) + (1-\mu) A_g(u, v) = \frac{u+v}{2} \tag{5}$$

for all $u, v \in K$, where $f, g \in \mathcal{CM}(K)$. Because of the condition of the theorem $g = \psi \circ \varphi^{-1}$ is continuously differentiable on some real open interval $K_0 \subset K$. Hence by (4) f is also continuously differentiable with non-zero derivative on the same interval (see [3], [9]). Thus, according to [10] (Theorem 6) solutions for (4) can be determined. By virtue of extension theorem by [4], [11] these solutions can be uniquely extended to K, hence we obtain the following cases:

(i) If $\mu \in \{\frac{1}{2}, 1\}$ then $g \sim \varepsilon_0$ on K;

(ii) If $\mu = \frac{1}{2}$ then there exists $p \in \mathbb{R}$ so that $g \sim \varepsilon_p$ on K;

(iii) If $\mu = 2$ then either $g \sim \varepsilon_0$ on K or there exists $p \in P_+(K)$ so that $g \sim \log\sqrt{x+p}$ on K or there exists $p \in P_-(K)$ so that $g \sim \log\sqrt{-x+p}$ on K.

From these cases by (3) we have the proposition of the theorem and in each case we can check that the obtained element belonging to $\mathcal{K}(I)$ is really quasi-arithmetic. $\square$

Note: (a) In the case (ii) the functional equation (5) is the original Matkowski–Sutô problem that was generally solved by Daróczy–Páles in [8].

(b) Therefore it is expectable that the statement of the theorem holds even without any smoothness condition.

(c) According to the theorem if we give an arbitrary function $\varphi \in \mathcal{CM}(I)$ and a number $q \in \,]0,1]\setminus\{\frac{1}{2}\}$ then we can decide on $\psi \in \mathcal{CM}(I)$ for which $(A_\psi)_\varphi^{(q)}$ will be quasi-arithmetic mean. For example define $I := \,]0,1[$, $\varphi(x) = -\log x$ and $q = \frac{1}{4}$. Then the case (iii) holds and $\varphi(I) = \mathbb{R}_+$. Therefore it is either $\psi \sim -\log$ on $]0,1[$ or there exists $p \in P_+(\mathbb{R}_+) = \{\, p \mid \mathbb{R}_+ + p \subset \mathbb{R}_+\} = \{\, p \mid p \geq 0\}$ so that $\psi \sim \log\sqrt{-\log + p}$ on $]0,1[$ and the third case can not hold because $P_-(\mathbb{R}+) = \emptyset$.

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