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On the equivalence of variational problems and the homogeneity of Lagrangian functions

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Dedicated to Professor Makoto Matsumoto

Introduction

Let $\mathcal{L}$ be a Lagrangian function on a C^∞ manifold M. In the present paper we are mainly concerned with an Euler–Lagrange equation of the variation of $\int_{t_0}^{t_1} \mathcal{L}(x,\dot{x})dt$. $\mathcal{E}_i$ denotes a Lagrange operator such that $\mathcal{E}_i(\mathcal{L}) := d_t(\dot{\partial}_i\mathcal{L}) - \partial_i\mathcal{L}$ and $\mathcal{E}_i(\mathcal{L}) = 0$ is the Euler–Lagrange equation for $\mathcal{L}$. Since 1966 A. Moór had studied a problem of the equivalence of some types. In a case of order 1, his problem is as follows: If two given fundamental functions $F(x,\dot{x})$ and $F^*(x,\dot{x})$ are related by $\mathcal{E}_i(F^*) = \lambda(x)\mathcal{E}_i(F)$, (or $\mathcal{E}_i(F^*) = \varphi_i^k(x)\mathcal{E}_k(F)$ in succeeding papers) for every curve in the space, that is, for any $(x,\dot{x})$, he called F and F^* equivalent and discussed the problem under the assumption that F and F^* are (p)-homogeneous of degree 1 and determined a relation between F and F^* ([2], [5] and [6]). Recently M. Kirkovits was concerned with an analogous problem in regular Lagrangian functions $\mathcal{L}(x,y)$ and $\mathcal{L}^*(x,y)$, and discussed an equivalence of type $\mathcal{E}_i(\mathcal{L}^*) = \lambda(x,y)\mathcal{E}_i(\mathcal{L})$ for any (x,y) and concluded that λ must be a constant and $\mathcal{L}^* = \lambda\mathcal{L} + \partial_i\varphi + \text{const.}$, for some $\varphi(x)$ ([3]). In the present paper we study an equivalence problem of two regular Lagrangian functions $\mathcal{L}$ and $\mathcal{L}^*$ of the following type and obtain theorems below.

Definition. $\mathcal{L}$ and $\mathcal{L}^*$ are called equivalent, denoted as $\mathcal{L} \sim \mathcal{L}^*$, if the solution curves of $\mathcal{E}_i(\mathcal{L}) = 0$ and $\mathcal{E}_i(\mathcal{L}^*) = 0$ locally coincide within a parameter t at every point (x) and for every direction (y) on M.

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Theorem I. *Let $\mathcal{L}$ and $\mathcal{L}^*$ be Lagrangian functions such that there exist Finslerian fundamental functions L and L^* respectively and $\mathcal{L} = L^{p(x)}$ and $\mathcal{L}^* = {L^*}^{p^*(x)}$ are satisfied, where $p(x)$ and $p^*(x)$ are C^∞ and never take values 1 and 0. Then $\mathcal{L} \sim \mathcal{L}^*$ iff*

$$F_0{}^i{}_0 - {F^*}_0{}^i{}_0 = \frac{1}{p(p-1)}\left\{(\lambda-1)L^2 p^i - (2\lambda-1)p_0 y^i\right\} - \frac{1}{p^*(p^*-1)}\left\{(\lambda^*-1){L^*}^2 {p^*}^i - (2\lambda^*-1){p^*}_0 y^i\right\}$$

where $F_j{}^i{}_k$ are coefficients of the Cartan connection defined by L and $F_0{}^i{}_0 = F_j{}^i{}_k\, y^j y^k$ and the same for symbols with $*$. (cf. Proposition 2.1)

Theorem II. *In cases that $\mathcal{L}(x,y)$ is not assumed to be homogeneous, we have $\mathcal{L} \sim \mathcal{L}^{(p)}$ iff $y^s\partial_s\mathcal{L} - \widetilde{G}^s\dot{\partial}_s\mathcal{L} = 0$, where p is a real number not equal to* 1 *and $\mathcal{L} + (p-1)\|\dot{\partial}\mathcal{L}\|^2 \neq 0$.*

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1. Preliminaries

Terms and notations are mostly taken from [1] and [4]. Let M be an n-dim. C^∞ manifold. M_x denotes a tangent space to M at x in M. We are concerned with a tangent bundle $T^0(M)$ whose fibre at x consists of only non-zero vectors at every point x in M. (x^i, y^i) denotes canonical coordinates on $T^0(M)$. A Lagrangian function $\mathcal{L}(x,y)$ which we study, is a positive C^∞ function on $T^0(M)$ and assumed to be regular, i.e., rank $({g^{(1)}}_{ij}) = n$ throughout the paper, where ${g^{(1)}}_{ij}$ denotes $\dot{\partial}_i\dot{\partial}_j\mathcal{L}$. We adopt abbreviations d_t, ∂_i and $\dot{\partial}_i$ in place of d/dt, $\partial/\partial x^i$ and $\partial/\partial y^i$. $g^{(1)ij}$ denotes the inverse of ${g^{(1)}}_{ij}$. The Euler–Lagrange equation of the variational problem of the integral

$$J = \int_{t_0}^{t_1} \mathcal{L}(x,\dot{x})dt \tag{1.1}$$

with respect to curves $x(t)$ is $\mathcal{E}_i(\mathcal{L}) = 0$, and from the regularity of ${g^{(1)}}_{ij}$ it is equivalent to ${d_t}^2x^i + G^{(1)i}(x, d_tx) = 0$, where we put

$$G^{(1)i} := g^{(1)ij}(y^s(\partial_s\dot{\partial}_j\mathcal{L}) - \partial_j\mathcal{L}). \tag{1.2}$$

In this paper we identify parameter curves with their parameters, i.e. we distinguish a curve from its parameter changed one. If $x(t)$ satisfies

$\mathcal{E}_i(\mathcal{L}) = 0$ we say that it is an extremal of a variational problem of (1.1) of $\mathcal{L}$, and if all extremals coincide with those of $\mathcal{L}^*$ we have defined that $\mathcal{L} \sim \mathcal{L}^*$. Thus we have

Proposition 1.1. *$\mathcal{L} \sim \mathcal{L}^*$ iff $G^{(1)i} - {G^*}^{(1)i} = 0$ for any (x, y).*

When (M, L) is a Finsler manifold if we put $\mathcal{L} = \frac{1}{2}L^2$ according to the usual manner, we have ${g^{(1)}}_{ij} = g_{ij}$ and $G^{(1)i} = 2G^i = {F_0}^i{}_0$, where ${F_j}^i{}_k$ comes from the Cartan connection $C\Gamma({F_j}^i{}_k, {N^i}_k, {C_j}^i{}_k)$ of (M, L), defined uniquely from conditions

$$
\begin{aligned}
&g_{ij|k} = \partial_k g_{ij} - {N^r}_k \dot{\partial}_r g_{ij} - g_{rj}{F_i}^r{}_k - g_{ir}{F_j}^r{}_k = 0\\
(1.3)\qquad &g_{ij}|_k = \dot{\partial}_k g_{ij} - g_{rj}{C_i}^r{}_k - g_{ir}{C_j}^r{}_k = 0\\
&{F_j}^i{}_k = {F_k}^i{}_j, \quad {C_j}^i{}_k = {C_k}^i{}_j \quad \text{and} \quad y^s {F_s}^i{}_j = {N^i}_j
\end{aligned}
$$

where $_{|}$ and $|$ denote h- and v-covariant differentiations respectively [4].

2. Pointwise homogeneous Lagrangian manifolds and an equivalence problem

A (p)-homogeneous function $f(x, y)$ of degree p with respect to (y) is defined as a function which satisfies $f(x, ky) = k^p f(x, y)$ for any positive constant k, and there is a well known relation $y^s(\dot{\partial}_s f) = pf$, and a Finsler fundamental function $L(x, y)$ is in fact a (p)-homogeneous function with respect to (y) of degree 1. We generalize the notion of (p)-homogeneity:

Definition 2.1. Let $\{M, \mathcal{L}\}$ be a Lagrangian manifold. $\mathcal{L}(x, y)$ on $T^0(M)$ is called pointwise (p)-homogeneous with a degree function $p(x)$ if it satisfies $\mathcal{L}(x, ky) = k^{p(x)}\mathcal{L}(x, y)$ for any positive constant k.

We are concerned with positive pointwise (p)-homogeneous Lagrangian functions such that there exists a (p)-homogeneous $L(x, y)$ of degree 1, called an associated fundamental function for $\mathcal{L}$ such that $\mathcal{L}$ is expressed as $p(x)$-th power of L ([1]); $\mathcal{L}(x, y) = (L(x, y))^{p(x)}$. (M, L) is an associated Finsler manifold. To preserve regularity we assume that $p(x)$ is C^∞ and never takes 0 and 1 as its values. We denote simply $\bar{\mathcal{G}} = L^{p(x)}$ and, if there is no confusion, omit $p(x)$ to avoid complications. We put successively, $\bar{\mathcal{G}}_i := \dot{\partial}_i \bar{\mathcal{G}} = pL^{p-1} l_i$ and $\bar{\mathcal{G}}_{ij} := \dot{\partial}_i \dot{\partial}_j \bar{\mathcal{G}} = pL^{p-2}[g_{ij} + (p-2) l_i l_j]$, where $l_i := \dot{\partial}_i L$ and g_{ij} is the metric tensor of the associated (M, L). Moreover we define $\bar{G}^i$ as $\bar{G}^i := \bar{\mathcal{G}}^{ij}(y^s(\partial_s \bar{\mathcal{G}}_i) - \partial_i \bar{\mathcal{G}})$. We have the regularity from $\det(\bar{\mathcal{G}}_{ij})$ and get the inverse.

Lemma 2.1.

(1) $\det(\bar{\mathcal{G}}_{ij}) = p^n L^{n(p-2)}(p-1)\det\,(g_{ij})$

(2) $\bar{\mathcal{G}}^{ij} = \dfrac{1}{pL^{p-2}}\left[g^{ij} - \dfrac{p-2}{p-1}l^i l^j\right],\ l^i := g^{ij}l_j$

Proposition 2.1. *$G^{(p)i}$ of $\{M, \mathcal{L}\},\ \mathcal{L} = L^{p(x)}$ is given as*

$$G^{(p)i} = F_0{}^i{}_0 + \frac{1}{p(p-1)}\left\{(2\lambda-1)p_0 y^i - (\lambda-1)L^2 p^i\right\} \tag{2.1}$$

where $\lambda = 1 + (p-1)\log L,\ p_i = \partial_i p,\ p^i = g^{ij}p_j$ and $p_0 = p_s y^s$

Proof. From (1.3) $y^i{}_{|j}$ and $L_{|j}$ vanish with respect to $C\Gamma$ of (M, L), therefore we have

$$\begin{aligned} &\partial_h \bar{\mathcal{G}} = \bar{\mathcal{G}}_{|h} + F_0{}^r{}_h \bar{\mathcal{G}}_r = p_h L^p \log L + F_0{}^r{}_h \bar{\mathcal{G}}_r \\ &y^s \partial_s \bar{\mathcal{G}}_h = (1 + p\log L)L^{p-1}p_0 l_h + F_0{}^r{}_0 \bar{\mathcal{G}}_{rh} + \bar{\mathcal{G}}_r F_h{}^r{}_0\ . \end{aligned} \tag{2.2}$$

Paying attention to $l^i = y^i/L$ and substituting from (2) of lemma 2.1 and (2.2) into the definition of $\bar{G}^i$ we get (2.1)

Referring to Proposition 2.1 we can consider that corresponding to a Finsler manifold (M, L) there arises a family of pointwise homogeneous Lagrangian manifolds $\{M, \mathcal{L}\}$ which have (M, L) as an associated Finsler manifold. Though there may occur negative values of the Lagrangian function, we formally put $\bar{\mathcal{G}} = \log L$ as in the case of $p(x) = 0$ [1] and conclude just like above that $\bar{\mathcal{G}}_{ij} = L^{-2}(g_{ij} - 2l_i l_j)$.

Proposition 2.2. *We have $G^{(0)i} = F_0{}^i{}_0$, for $\{M, \mathcal{L}\},\ \mathcal{L} = \log L$.*

The proof of Theorem I. is now easy from Proposition 1.1 and Proposition 2.1.

Corollary 2.1. *Let $\mathcal{L}$ and $\mathcal{L}^*$ have the same associated L, and $p(x)$ and $q(x)$ be degree functions of $\mathcal{L}$ and $\mathcal{L}^*$ respectively. Then $\mathcal{L} \sim \mathcal{L}^*$ iff $p_i/(p(p-1)) = q_i/(q(q-1))$.*

Corollary 2.2. *Let (M, L) be a Finsler manifold and $\mathcal{L} = L^{p(x)}$. Then we have that $\mathcal{L} \sim \frac{1}{2}L^2 \sim \log L$ iff $p(x)$ is a constant $\neq 0, 1$.*

Corollary 2.3. *If $\mathcal{L}$ and $\mathcal{L}^*$ are associated with L and L^*, and the degree functions p and p^* are constants not equal to 1, then we have that $\mathcal{L} \sim \mathcal{L}^*$ iff $F_0{}^i{}_0 - F^*{}_0{}^i{}_0 = 0$ where in the case that p or p^* vanishes, $\mathcal{L}$ or $\mathcal{L}^*$ is regarded as $\log L$ or $\log L^*$ respectively.*

3. Equivalence problems for non-homogeneous $\{M, \mathcal{L}\}$

In this section we study the equivalence of general non-homogeneous $\mathcal{L}$. $\mathcal{L}$ is assumed to be C^∞, positive and regular, that is, $g^{(1)}{}_{ij}$ is regular. For simplicity we replace (1) on the right shoulder with a tilde on the top hereafter. Let p be a constant nonzero real number. We put $\mathcal{L}_i := \dot{\partial}_i \mathcal{L}$, $\mathcal{L}^i := \widetilde{g}^{ij} \mathcal{L}_j$ and $\|\dot{\partial}\mathcal{L}\|^2 := g^{ij}\mathcal{L}_i\mathcal{L}_j$, and define $g^{(p)}{}_{ij} := \dot{\partial}_i\dot{\partial}_j\mathcal{L}^p = p\mathcal{L}^{p-1}[\widetilde{g}_{ij} + (p-1)\mathcal{L}^{-1}\mathcal{L}_i\mathcal{L}_j]$.

Lemma 3.1. *The determinant and the inverse of $g^{(p)}{}_{ij}$ are given as*

(1) $\det(g^{(p)}{}_{ij}) = p^n \mathcal{L}^{n(p-1)} \det(\widetilde{g}_{ij}) \left(1 + \dfrac{p-1}{\mathcal{L}}\|\dot{\partial}\mathcal{L}\|^2\right)$

(2) *If $\mathcal{L}+(p-1)\|\dot{\partial}\mathcal{L}\|^2 \neq 0$, then $g^{(p)}{}_{ij}$ is regular and the inverse is*

$$g^{(p)ij} = \frac{1}{p\mathcal{L}^{p-1}}\left[\widetilde{g}^{ij} - \frac{p-1}{\mathcal{L} + (p-1)\|\dot{\partial}\mathcal{L}\|^2}\mathcal{L}^i\mathcal{L}^j\right]. \tag{3.1}$$

To include the case of $p = 0$, we put $\mathcal{L}^{(p)} := \mathcal{L}^p$ for $p \neq 0$ and $\mathcal{L}^{(0)} := \log L$.

Lemma 3.2.

(1) $g^{(0)}{}_{ij} = \mathcal{L}^{-1}(\widetilde{g}_{ij} - \mathcal{L}^{-1}\mathcal{L}_i\mathcal{L}_j)$

(2) $\det(g^{(0)}{}_{ij}) = \mathcal{L}^{-n} \det(\widetilde{g}_{ij})(1 - \mathcal{L}^{-1}\|\dot{\partial}\mathcal{L}\|^2)$

(3) *If $\mathcal{L} - \|\dot{\partial}\mathcal{L}\|^2 \neq 0$, $g^{(0)}{}_{ij}$ is regular and its inverse is given as*

$$g^{(0)ij} = \mathcal{L}\left(\widetilde{g}^{ij} + \frac{1}{\mathcal{L} - \|\dot{\partial}\mathcal{L}\|^2}\mathcal{L}^i\mathcal{L}^j\right). \tag{3.2}$$

Proposition 3.1. *If $\mathcal{L} + (p-1)\|\dot{\partial}\mathcal{L}\|^2 \neq 0$, for real p we have*

$$G^{(p)i} = \widetilde{G}^i + \frac{p-1}{\mathcal{L} + (p-1)\|\dot{\partial}\mathcal{L}\|^2}(y^s\partial_s\mathcal{L} - \widetilde{G}^s\mathcal{L}_s)\mathcal{L}^i\ . \tag{3.3}$$

Proof. $G^{(p)i} := g^{(p)ij}(y^s\partial_s\dot{\partial}_j\mathcal{L}^{(p)} - \partial_j\mathcal{L}^{(p)})$ and Lemmas 3.1 $\sim$ 2 lead us to (3.3).

The proof of Theorem II. now follows from Proposition 3.1. We shall give other expressions of the above condition.

Connection of Berwald type. Though it is not homogeneous, $\widetilde{G}^i$ induces a Berwald type connection on $\{M, \mathcal{L}\}$. We put

$$\widetilde{G}^i_j := \frac{1}{2}\dot{\partial}_j G^i \quad \text{and} \quad \widetilde{G}_j{}^i{}_k := \dot{\partial}_k G^i_j \tag{3.4}$$

As it is easily verified, $\widetilde{G}^i_j$ and $\widetilde{G}_j{}^i{}_k$ obey the law of transformation which must be satisfied by coefficients of connections under coordinate transformations. Because of non-homogeneity of $\mathcal{L}$, the Berwald type connection $\widetilde{B\Gamma}(\widetilde{G}_j{}^i{}_k, \widetilde{G}^i_k)$ has non-homogeneous coefficients in (3.4). h- and v-covariant differentiations with respect to $\widetilde{B\Gamma}$ are denoted with double perpendicular bars. We put M and T as below and operate differentiations on them successively.

$$M := y^s\partial_s\mathcal{L} - \widetilde{G}^s\mathcal{L}_s \text{ and } T := \mathcal{L}_r(y^s\widetilde{G}^r{}_s - \widetilde{G}^r) \tag{3.5}$$

Proposition 3.2. *The following relations hold good*

(1) $M = y^s\mathcal{L}_{\|s} + T$,

(2) $\dot{\partial}_i(\mathcal{L}_{\|j}) = (\dot{\partial}_i\mathcal{L})_{\|j}$

(3) $\dot{\partial}_i M = 2\mathcal{L}_{\|i}, \dot{\partial}_i T = \mathcal{L}_{\|i} - y^s(\dot{\partial}_i\mathcal{L})_{\|s}$

(4) $\dot{\partial}_i\dot{\partial}_j M = 2\dot{\partial}_j(\mathcal{L}_{\|i})$,

(5) $(\dot{\partial}_j\mathcal{L})_{\|i} = (\dot{\partial}_i\mathcal{L})_{\|j}$

(6) $\dot{\partial}_i\dot{\partial}_j T = -y^s\{\widetilde{g}_{ij\|s} - \widetilde{G}_i{}^r{}_{js}\mathcal{L}_r\}, \quad \widetilde{G}_i{}^r{}_{jk} := \dot{\partial}_k\widetilde{G}_i{}^r{}_j$.

Theorem 3.2. *The following conditions (1)–(3) are equivalent*

(1) $M = 0$,

(2) $\mathcal{L}_{\|i} = 0$ *and* $T = 0$

(3) $(\dot{\partial}_i\mathcal{L})_{\|j} = 0$ *and* $T = 0$.

The theorem is proved from Proposition 3.2.

Thus if $M = 0$ is satisfied, $\widetilde{B\Gamma}$ is metrical in the sense of (2), but the following Theorem 3.3 shows that in the sense of $g_{ij\|k} = 0$, $\widetilde{B\Gamma}$ is not metrical.

Theorem 3.3. *If $M = 0$ is satisfied, we have:* (4) $g_{ij\|k} = \widetilde{G}_i{}^r{}_{jk}\mathcal{L}_r$.

Finally an analogous consideration on a C^∞ power function $p(x)$ which never takes values 0 and 1 leads to

Theorem 3.4. $\mathcal{L}^{p(x)} \sim \mathcal{L}$ *iff (3.6) holds good*

$$(\mathcal{L}\log\mathcal{L})[\mathcal{L} + (p-1)\|\dot{\partial}\mathcal{L}\|^2]p^i = pK\mathcal{L}^i \text{ where} \tag{3.6}$$

$$K := (p-1)M + \frac{1}{p}J\mathcal{L}, \quad J := (1 + p\log\mathcal{L})p_0 + (p-1)\log\mathcal{L}(p^s\mathcal{L}_s)$$

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