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On the irrationality of Cantor and Ahmes series

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To the memory of Béla Brindza

Abstract. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of integers with $b_n > 1$ for all n. We derive criteria for the (ir)rationality of the sum $\sum_{n=1}^{\infty} \frac{b_n}{a_1 \dots a_n}$ in terms of the sequence $(\frac{b_n}{a_n})_{n=1}^{\infty}$. We present refinements of criteria of Oppenheim, Erdős and Straus, and Tijdeman and Yuan. Furthermore we make some remarks on a similar approach to determine the (ir)rationality of sums $\sum_{n=1}^{\infty} \frac{1}{a_n}$.

1. Introduction

In Sections 2 and 3 we consider series $\sum_{n=1}^{\infty} \frac{b_n}{a_1 \dots a_n}$ where $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are sequences of integers with $a_n > 1$ for all n. We assume that (a_n) and (b_n) are explicitly given and that $b_n = o(a_{n-1}a_n)$ as $n \to \infty$. We wonder how we can use the behaviour of the sequence $(r_n)_{n=1}^{\infty}$, where $r_n = b_n/a_n$ for all n, to decide whether $\alpha := \sum_{n=1}^{\infty} \frac{b_n}{a_1 \dots a_n}$ is rational. Theorem 1 is a sharpening of results of Oppenheim [15] and Tijdeman and Yuan [19]. Theorem 2 is a simplification of a theorem of Erdős and Straus [11]. Together they make a useful test.

In Section 4 we wonder whether a similar approach works for convergent sequences $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$. Theorem 3 shows that even in case $b_n = 1$

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for all n the fact that $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$ is an irrational number θ does not guarantee the irrationality of the sum $\sum_{n=1}^{\infty}\frac{b_n}{a_n}$. We wonder whether the more severe condition $a_{n+1}-\theta a_n\to 0$ for some irrational number θ implies irrationality, but unfortunately we cannot answer the question.

The history of the developments is given in the various sections.

2. Limit points in case of Cantor series

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences of integers with $a_n>1$ for all n such that $\alpha:=\sum_{n=1}^{\infty}\frac{b_n}{a_1\ldots a_n}$ converges. Put

$$r_n=\frac{b_n}{a_n},\quad R_n=\sum_{i=0}^{\infty}\frac{b_{n+i}}{a_n\ldots a_{n+i}}\qquad(n=1,2,\ldots).$$

It is an easy observation that α is rational if $\frac{b_n}{a_n-1}$ is ultimately constant. In 1869 Cantor [6] showed that if $0\le b_n<a_n$ and for every positive integer q there is a positive integer n such that q divides $a_1a_2\ldots a_n$, then α is irrational if and only if $b_n>0$ infinitely often and $b_n<a_n-1$ infinitely often. In 1954 Oppenheim [15] extended this result to sequences satisfying $|b_n|<a_n$ for all n. It also follows from his results that if α is rational, then all the limit points of (r_n) are rational, and that, if $\alpha\in\mathbb{Q}$ and (r_n) has an integer limit point t (hence $t\in\{-1,0,1\}$), then $b_n=t(a_n-1)$ for all n larger than some n_0. Cantor and Oppenheim found the results in connection with their studies of expansions of real numbers as sums of infinite series of rational numbers which now bear their names. A Cantor expansion of α is a series $\alpha:=\sum_{n=1}^{\infty}\frac{b_n}{a_1\ldots a_n}$ with $0\le b_n<a_n$ for all n.

In 2002 Tijdeman and Yuan [19] showed that $\alpha\notin\mathbb{Q}$ under the more general condition that $b_n=O(a_n)$ as $n\to\infty$ and (r_n) has an irrational limit point. In the present paper we prove the following improvement. By a denominator of a rational number α we mean the smallest positive integer m such that $m\alpha$ is an integer. Furthermore $x \pmod 1$ denotes the number y with $y-x\in\mathbb{Z}$ and $0\le y<1$.

Theorem 2.1. *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of integers with $a_n>1$ for all n and $b_n=o(a_{n-1}a_n)$ as $n\to\infty$. Put $\alpha:=\sum_{n=1}^{\infty}\frac{b_n}{a_1\ldots a_n}$, $r_n=\frac{b_n}{a_n}$, $R_n=\sum_{i=0}^{\infty}\frac{b_{n+i}}{a_n\ldots a_{n+i}}$ as $n=1,2,\ldots$. If α is rational, then all* *the limit points of* $(r_n \pmod 1)_{n=1}^{\infty}$ *are rational numbers having the same denominator* u.

It follows immediately from Theorem 2.1 that if α is rational then the sequence $(r_n \pmod 1)_{n=1}^{\infty}$ has only finitely many limit points. The following Cantor expansion shows that it can happen that all the possible values modulo 1 occur as limit points:

$$\frac{1}{5} = \frac{1}{7} + \frac{4}{7 \cdot 12} + \frac{13}{7 \cdot 12 \cdot 17} + \frac{13}{7 \cdot 12 \cdot 17 \cdot 22} + \frac{5}{7 \cdot 12 \cdot 17 \cdot 22 \cdot 27} + \dots$$

where $\frac{b_{4n}}{a_{4n}} \to \frac{3}{5}$, $\frac{b_{4n+1}}{a_{4n+1}} \to \frac{1}{5}$, $\frac{b_{4n+2}}{a_{4n+2}} \to \frac{2}{5}$, $\frac{b_{4n+3}}{a_{4n+3}} \to \frac{4}{5}$ as $n \to \infty$. More generally, if p is an odd prime and g is a primitive root of p, then the Cantor expansion of $\frac{1}{p}$ as

$$\frac{1}{p} = \sum_{n=1}^{\infty} \frac{b_n}{\prod_{i=1}^{n}(ip+g)}$$

has the property that $(r_n)_{n=1}^{\infty}$ has $\frac{1}{p}, \frac{2}{p}, \dots, \frac{p-1}{p}$ as limit points. On the other hand, the examples

$$\frac{1}{5} = \frac{1}{6} + \frac{2}{6 \cdot 11} + \frac{3}{6 \cdot 11 \cdot 16} + \frac{4}{6 \cdot 11 \cdot 16 \cdot 21} + \frac{5}{6 \cdot 11 \cdot 16 \cdot 21 \cdot 26} + \dots$$

and

$$\frac{1}{6} = \frac{1}{8} + \frac{4}{8 \cdot 11} - \frac{5}{8 \cdot 11 \cdot 14} + \frac{6}{8 \cdot 11 \cdot 14 \cdot 17} - \frac{7}{8 \cdot 11 \cdot 14 \cdot 17 \cdot 20} + \dots$$

show that it can also happen that $(r_n \pmod 1)$ does not assume all possible values and that u does not equal the denominator of α. In the latter case $q = 6$, $u = 3$.

Proof of Theorem 2.1. Suppose α is rational with denominator q. Then for $n = 1, 2, \dots$

$$\begin{aligned} qR_n &= q \sum_{i=0}^{\infty} \frac{b_{n+i}}{a_n \dots a_{n+i}} \\ &= q\alpha a_1 \dots a_{n-1} - q \sum_{i=1}^{n-1} b_i a_{i+1} \dots a_{n-1} \in \mathbb{Z}. \end{aligned} \tag{1}$$

By applying this argument to R_n in place of α, we find that if q_n denotes the denominator of R_n for $n = 1, 2, \dots,$ then $q_m \mid q_n$ whenever $n < m$. Hence $(q_n)_{n=1}^{\infty}$ is a non-increasing sequence of integers with limit u, say.

Let $0 < \varepsilon < 1/(4q)$. Let n_0 be so large that $uR_n \in \mathbb{Z}$ and $\left|\frac{b_n}{a_{n-1}a_n}\right| < \varepsilon$ for $n > n_0$. Then, for $n > n_0$, u is the denominator of R_n and

$$\left|R_n - \frac{b_n}{a_n}\right| < \varepsilon + \frac{\varepsilon}{a_n} + \frac{\varepsilon}{a_n a_{n+1}} \cdots \le 2\varepsilon < \frac{1}{2q}. \tag{2}$$

Thus all the limit points of the sequence $(r_n \pmod 1)_{n=1}^{\infty}$ are of the form t/u where t is an integer coprime to u. □

Remarks. 1. The condition $b_n = o(a_{n-1}a_n)$ in Theorem 2.1 can be replaced with the condition $b_n = O(a_n)$ as $n \to \infty$. The proof requires a slight modification of formula (2).

2. If $d_n := \gcd(q_n, a_{n+1})$ then $q_{n+1} = q_n/d_n$. Therefore we have

$$u = q \,/\, \big(\lim_{n\to\infty} \gcd(q, a_1 a_2 \dots a_n)\big).$$

In particular, if for every integer q there exists an n such that $q \mid a_1 a_2 \dots a_n$ then we are certain that $u = 1$, hence that all the limit points of the sequence $(r_n)_{n=1}^{\infty}$ are integers. This special case has been investigated thoroughly by Cantor [6] and Oppenheim [15].

3. The order in which limit points are visited

For the rationality of α it does not suffice that all the limit points of the sequence $(r_n \pmod 1)_{n=1}^{\infty}$ are rationals with the same denominator. For example $e = \sum_{n=1}^{\infty} \frac{1}{n!}$ is irrational but the only limit point of $(r_n \pmod 1)_{n=1}^{\infty}$ is 0. Erdős and Straus [11] Lemma 2.29 = Erdős and Straus [12] Theorem 2.1 gave the following criterion for the rationality of α: *there exists a positive integer B and a sequence of integers $(t_n)_{n=1}^{\infty}$ so that for all large n we have*

$$Bb_n = t_n a_n - t_{n+1}, \qquad |t_{n+1}| < a_n/2. \tag{3}$$

Suppose α is rational with denominator q. It follows from their proof that one may choose $B = q$ and t_n as the integer nearest to $q\frac{b_n}{a_n}$. According to

(2) the integer t_n equals qR_n for $n > n_0$. Hence the recurrence relation $Bb_n = t_n a_n - t_{n+1}$ becomes $b_n = R_n a_n - R_{n+1}$, but this relation follows immediately from the definition of $(R_n)_{n=1}^{\infty}$. The theorem of Erdős and Straus implies that the relation is crucial for the rationality of α. The condition $|t_{n+1}| < a_n/2$ is optional. We specify their theorem to a more practical criterion, omit the optional condition and adjust their proof.

Theorem 3.1. *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences of integers with $a_n > 1$ for all n and $b_n = o(a_{n-1}a_n)$ as $n \to \infty$. Then*

$$\sum_{n=1}^{\infty} \frac{b_n}{a_1 \dots a_n} \tag{4}$$

is rational if and only if for some positive integer q and all large n the integers t_n nearest to $q\frac{b_n}{a_n}$ satisfy

$$qb_n = t_n a_n - t_{n+1}. \tag{5}$$

Proof. Assume that (5) holds for $n \geq \mathbb{N}$. Then

$$qa_1 \dots a_{N-1} \sum_{n=1}^{\infty} \frac{b_n}{a_1 \dots a_n} \in \mathbb{Z} + \sum_{n=N}^{\infty} \frac{t_n a_n - t_{n+1}}{a_N \dots a_n} = \mathbb{Z} + t_N = \mathbb{Z}.$$

So condition (5) is sufficient for the rationality of the series (4).

To prove the necessity we recall from the proof of Theorem 2.1 that $|R_n - \frac{b_n}{a_n}| < \frac{1}{2q}$ for $n > n_0$. From the definition of R_n we immediately see that $a_n R_n = b_n + R_{n+1}$ for all n. Choosing t_n as the integer nearest to $q\frac{b_n}{a_n}$ we obtain that $t_n = qR_n$ and that (5) holds for $n \geq n_0$. □

Theorems 2.1 and 3.1 provide the following test for the rationality of α given the values of $b_n/a_n = r_n$.

1. Determine the limit points of $(r_n \pmod 1)_{n=1}^{\infty}$ and check whether they all have the same denominator u. If not, then $\alpha \notin \mathbb{Q}$.

2. Let s_n denote the integer nearest to ur_n for all n. Check whether $ub_n = a_n s_n - s_{n+1}$ for all large n. If not, then $\alpha \notin \mathbb{Q}$. Otherwise $\alpha \in \mathbb{Q}$.

In the test s_n/u is the simplified fraction t_n/q for all large n.

Application 1. Suppose a, b, c, d, e, f are integers with $a > 0, b > 0$ such that $a_n = an^2 + cn + e, b_n = bn^2 + dn + f$ for all n. We wonder when

$\alpha \in \mathbb{Q}$. Since $\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{b}{a}$, we can choose $u = a$. The nearest integer to $a\frac{b_n}{a_n}$ is b for large n. Hence $\alpha \in \mathbb{Q}$ if and only if $a(bn^2 + dn + f) = b(an^2 + cn + e) - b$ for all large n, that is, if $bc = ad$ and $b(e-1) = af$.

Application 2. Suppose a, b, c, d are integers with $a > 0, b > 0$ such that $a_n = an^2 + c$, $b_n = bn^3 + d$ for all n. Since $\frac{b_n}{a_n} = \frac{bn}{a} + o(1)$, the limit points of ($r_n \pmod 1$) are the multiples of $\frac{b}{a}$ considered modulo 1. If $a > 1$, then not all denominators are equal and α is irrational. If $a = 1$, then $s_n = bn$ for all large n. Hence $\alpha \in \mathbb{Q}$ if and only if $bn^3 + d = bn(n^2 + c) - b(n+1)$ for all large n, that is, if $c = 1$ and $d = -b$.

Remark. If $b_n = o(a_{n-1}a_n)$ is not satisfied, then the above method may still be applicable. If there exist integers c_n such that $R_n = \frac{c_n}{a_n} + o(1)$, then it suffices to replace b_n in the test with c_n and r_n with $\frac{c_n}{a_n}$ for all n. See Hančl and Tijdeman [13], [14] and Tijdeman and Yuan [19] for conditions in case $b_n = o(a_{n-1}a_n)$ is not satisfied.

4. Limits of a_{n+1}/a_n for Ahmes series

One may wonder whether the limit point approach is also applicable for series of the form $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$. To simplify matters we shall study the so-called Ahmes series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ where the a_n's are positive integers such that the series converges. There are several irrationality results in case (a_n) grows doubly exponential, see e.g. Erdős and Straus [10], Sándor [18], Badea [2], and Duverney [7], [8]. We would like to have irrationality results in case of simple exponential growth. Such results have been given for special sequences, e.g. by André-Jeannin [1], Borwein [3], [4], Borwein and Zhou [5], Duverney, Nishioka, Nishioka and Shiokawa [9] and Prévost [16], [17].

We wondered whether $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \theta$ with $\theta \notin \mathbb{Q}$ would imply irrationality of α. The opposite is true as is shown by the following theorem.

Theorem 4.1. *Let $\alpha, \theta \in \mathbb{R}$, $\alpha > 0$, $\theta > 1$. Then there exists an increasing sequence of positive integers $a_1, a_2, \dots$ such that $\sum_{n=1}^{\infty} \frac{1}{a_n} = \alpha$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \theta$.*

PROOF. Put $\Delta_0 = \alpha$, $a_1 = \left[\frac{1}{\alpha}\right] + 1$, $\Delta_1 = \alpha - \frac{1}{a_1}$. Then $0 < \Delta_1 < \Delta_0$. We proceed by induction. Let a_n and Δ_n be defined with $\Delta_n > 0$. Put

$$a_{n+1} = \max\left(\left[\frac{1}{\Delta_n}\right] + 1, a_n + 1\right), \quad \Delta_{n+1} = \Delta_n - \frac{1}{a_{n+1}}.$$

Then $0 < \Delta_{n+1} < \Delta_n$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, there are infinitely many integers n with $a_{n+1} - a_n > 1$. Then $a_{n+1} - 1 \le \frac{1}{\Delta_n} < a_{n+1}$ and $\Delta_n \le 1$, hence

$$\Delta_{n+1} = \Delta_n - \frac{1}{a_{n+1}} \le \Delta_n - \frac{\Delta_n}{\Delta_n + 1} = \Delta_n \frac{\Delta_n}{\Delta_n + 1} \le \frac{1}{2}\Delta_n.$$

Thus there is an N with $\Delta_N < \min(\frac{1}{\theta-1}, 1)$.

From N on we change the choice of a_n by choosing $a_{n+1} = \left[\frac{\theta}{\Delta_n(\theta-1)}\right]$, $\Delta_{n+1} = \Delta_n - \frac{1}{a_{n+1}}$. Since $\Delta_n < \frac{1}{\theta-1}$, we have $\frac{\theta}{\Delta_n(\theta-1)} - 1 > \frac{1}{\Delta_n}$ whence $a_{n+1} > \frac{1}{\Delta_n}$ and $\alpha - \sum_{j=1}^{n+1} \frac{1}{\alpha_j} = \Delta_{n+1} > 0$. Therefore a_{n+1} is a well-defined integer and $\sum_{n=1}^{\infty} \frac{1}{a_n} \le \alpha$.

Next we show that $\sum_{n=1}^{\infty} \frac{1}{a_n} = \alpha$. We have, for $n \ge N$,

$$\Delta_{n+1} = \Delta_n - \frac{1}{a_{n+1}} \le \Delta_n - \frac{\Delta_n(\theta - 1)}{\theta} = \frac{\Delta_n}{\theta}. \tag{6}$$

Since $\theta > 1$, this shows that $\alpha - \sum_{j=1}^{n} \frac{1}{a_j} = \Delta_n \to 0$ as $n \to \infty$ and therefore $\sum_{n=1}^{\infty} \frac{1}{a_n} = \alpha$.

We now check that $a_{n+1} > a_n$ for $n \ge N$. It suffices to show that $\frac{\theta}{\Delta_n(\theta-1)} - \frac{\theta}{\Delta_{n-1}(\theta-1)} \ge 1$. Indeed we have, by (6) and $\Delta_n \le 1$,

$$\frac{1}{\Delta_n} - \frac{1}{\Delta_{n-1}} \ge \frac{1}{\Delta_n} - \frac{1}{\theta\Delta_n} \ge \frac{\theta - 1}{\theta}.$$

Finally we prove that $\frac{\Delta_{n+1}}{\Delta_n} \to \theta$ as $n \to \infty$. We know that $\Delta_n \to 0$ as $n \to \infty$. Hence

$$\frac{a_{n+1}}{a_n} = \frac{\frac{\theta}{(\theta-1)\Delta_n} + O(1)}{\frac{\theta}{(\theta-1)\Delta_{n-1}} + O(1)} = \frac{\Delta_{n-1}(1 + o(1))}{\Delta_n(1 + o(1))} \ge \theta(1 + o(1)). \tag{7}$$

On the other hand,

$$\Delta_{n+1} = \Delta_n - \frac{1}{a_{n+1}} > \Delta_n - \frac{1}{\frac{\theta}{\Delta_n(\theta-1)} - 1}$$

$$= \Delta_n - \frac{\Delta_n(\theta - 1)}{\theta}(1 + o(1))$$
$$= \frac{\Delta_n}{\theta}(1 + o(1))$$

so that

$$\frac{a_{n+1}}{a_n} = \frac{\Delta_{n-1}}{\Delta_n}(1 + o(1)) \leq \theta(1 + o(1)).$$

Thus $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \theta$. □

A much more restrictive requirement also leading to simple exponential growth is that $a_{n+1} - \theta a_n \to 0$ as $n \to \infty$ for some irrational number θ.

Open problem: *Is it true that if* $(a_n)_{n=1}^{\infty}$ *is a sequence of positive integers such that* $\alpha := \sum_{n=1}^{\infty} \frac{1}{a_n}$ *is a rational number and* $a_{n+1} - \theta a_n \to 0$ *as* $n \to \infty$ *for some number* θ*, then* θ *is an integer and* $a_{n+1} = \theta a_n$ *for all large* n*?*

It is obvious that $\alpha \in \mathbb{Q}$ if $\frac{a_{n+1}}{a_n} = \theta \in \mathbb{Z}_{>1}$ for all large n. On the other hand there are many numbers θ for which no sequence of positive integers $(a_n)_{n=1}^{\infty}$ exists such that $a_{n+1} - \theta a_n \to 0$ (independently of the arithmetic character of α). If $\theta \in \mathbb{Q} \setminus \mathbb{Z}$, then such a sequence cannot exist, since not all a_n's can be integers, and if θa_n is not an integer, it cannot be close to an integer. Hence, also roots of rational numbers are excluded, even roots of integers which are not rational integers themselves such as $\sqrt{2}$ and $\sqrt[3]{3}$.

On the other hand, there exist algebraic numbers θ which admit such sequences. The set of numbers θ admitting an integer sequence $(a_n)_{n=1}^{\infty}$ with $a_{n+1} - \theta a_n \to 0$ as $n \to \infty$ comprises the Pisot numbers. A Pisot number is an algebraic integer $\gamma = \gamma_1$ all of whose conjugates $\gamma_2, \ldots, \gamma_k$ are less than 1 in absolute value. For example, by the theorem on symmetric functions the numbers $a_n := \gamma_1^n + \gamma_2^n + \ldots \gamma_k^n$ are rational integers. Furthermore, $a_{n+1} - \gamma a_n = O(\max_{i>1} |\gamma_i|^n)$ which tends to 0 exponentially fast. The most famous sequences with a Pisot number limit ratio are the Fibonacci sequence (F_n) and the Lucas sequence (L_n). One has $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{L_{n+1}}{L_n} = \frac{1}{2} + \frac{1}{2}\sqrt{5}$. André-Jeannin [1] showed in 1989 that $\sum_{n=1}^{\infty} \frac{1}{F_n}$ and $\sum_{n=1}^{\infty} \frac{1}{L_n}$ are irrational. Duverney, Nishioka, Nishioka and Shiokawa [9] proved by another method that $\sum_{n=1}^{\infty} \frac{1}{F_n^{2s}}$

and $\sum_{n=1}^{\infty} \frac{1}{L_n^{2s}}$ are irrational for any positive integer s. In fact, they proved more general results on binary recurrence sequences. The result of ANDRÉ-JEANNIN was further generalised by PRÉVOST [16] and [17]. However, all these results concern only binary recurrence sequences with integer coefficients and therefore only quadratic irrational θ. What about Pisot numbers of degree > 2? The situation for transcendental numbers θ is totally obscure for us.

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