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# On Cauchy-differences that are also quasisums

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*Dedicated to the memory of Professors Béla Brindza and Jenő Erdős*

**Abstract.** In this paper, we completely describe those Cauchy-differences that can also be written as a quasisum, i.e., we solve the functional equation

$$f(x) + f(y) - f(x+y) = a\big(b(x) + b(y)\big)$$

under strict monotonicity assumptions on the unknown functions $a, b$. As an application of the result obtained, we solve a functional equation arising in utility theory.

## 1. Introduction

Throughout this paper let $I$ denote an open real interval of the form $I = ]0, K[$, where $0 < K \leq \infty$. Denote by $\Delta$ the set $\big\{(x, y) \in \mathbb{R}^2 \mid x,\, y,\, x + y \in I\big\}$. A two variable function $F : \Delta \to \mathbb{R}$ is called a *Cauchy-difference* if there exists a function $f : I \to \mathbb{R}$ (called a generator of $F$)

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such that

$$F(x,y)=f(x)+f(y)-f(x+y) \qquad \big((x,y)\in\Delta\big).$$

The theory and application of Cauchy-differences form a rapidly growing field of functional equations. Cauchy-differences are trivially symmetric and satisfy the so-called *co-cycle equation*

$$F(x,y)+F(x+y,z)=F(x,y+z)+F(y,z) \qquad (x,y,z,x+y+z\in I).$$

Surprisingly, under appropriate conditions, the converse is also true, e.g., by classical results of Jessen, Karpf and Thorup [17] and J. Erdős [13], the symmetric solutions of the co-cycle equation are Cauchy-differences. The problem of the boundedness of $F$ belongs to the stability theory of Cauchy's equation. For developments in this direction, we refer to the papers [30] and [26]. Cauchy-differences of particular form have been investigated, too ([5], [12]).

Another essential notion in the theory of functional equations in several variables is that of *quasi-sum* introduced by Aczél [2]. A function $F:\Delta\to\mathbb{R}$ is called a (symmetric) quasi-sum if there exist two strictly monotone functions $b:I\to\mathbb{R}$ and $a:D\to\mathbb{R}$, where $D:=\big\{b(x)+b(y)\mid (x,y)\in\Delta\big\}$ such that

$$F(x,y)=a\big(b(x)+b(y)\big) \qquad \big((x,y)\in\Delta\big).$$

Quasi-sums play an important role in the characterization of associative operations (cf. [1], [11]), in the various characterizations of bisymmetry and quasi-arithmetic means (cf. [10], [20], [21], [22], [23], [24]).

Motivated by the problem detailed in the last section of this paper (see [4], [6], [16] for the origin), our aim is to determine those two variable real functions defined on $\Delta$ that can be represented as a Cauchy-difference and a quasi-sum simultaneously. In other words, we intend to solve the functional equation

$$f(x)+f(y)-f(x+y)=a\big(b(x)+b(y)\big) \qquad \big((x,y)\in\Delta\big) \tag{1.1}$$

for the unknown functions $f,b:I\to\mathbb{R}$ and $a:D\to\mathbb{R}$ under the only regularity assumption that $a$ and $b$ are strictly monotone functions. In the first step of the proof, we prove that $f$ is of the form $g+A$, where $g$ is a convex/concave function and $A$ is an additive function. Then, utilizing and extending the ideas of the papers [7], [8], [9] concerning composite functional equations, differentiability properties of the unknown functions $g, b$ and $a$ are obtained. In the next step, after differentiating (1.1) with respect to the variables, we eliminate the composite term and obtain a non-composite functional equation. Using the regularity theory developed in the books [14] and [15] for the equation so obtained, we derive $C^\infty$ properties of the unknown functions. Finally, differential equations are deduced and the unknown functions are completely determined.

## 2. Solution of (1.1)

We solve (1.1) in several steps. First we decompose $f$ into a regular and non-regular part using a result of Ng [25].

**Lemma 2.1.** *Assume that $f, b : I \to \mathbb{R}$ and $a : D \to \mathbb{R}$ satisfy* (1.1) *and $a$ and $b$ are strictly monotone functions. Then there exists an additive function $A : \mathbb{R} \to \mathbb{R}$ such that $g := f - A$ is strictly convex or concave, furthermore, $D$ is an open interval of positive length, $g$, $a$, $a^{-1}$, $b$ and $b^{-1}$ are continuous functions that satisfy the functional equation*

$$g(x) + g(y) - g(x+y) = a\big(b(x) + b(y)\big) \qquad \big((x, y) \in \Delta\big). \tag{2.1}$$

Proof. Due to the strict monotonicity of $a$ and $b$, the right hand side of (1.1) is a strictly monotone function of $x$, hence, the left hand side is strictly monotone, too, i.e., for all fixed $y \in I$, we have that the function

$$x \mapsto f(x+y) - f(x) \qquad \big(x \in I \cap (I - y)\big) \tag{2.2}$$

is either strictly increasing or strictly decreasing. From this, we deduce that $f$ is either strictly Wright-convex or strictly Wright-concave on $I$. Indeed, let $u, v \in I$ with $u < v$ and $t \in \,]0, 1[$ and set $y := t(v - u)$. Then we have that $u$, $tu + (1-t)v \in I \cap (I - y)$ and $u < tu + (1-t)v$, hence, if the function in (2.2) is strictly increasing (resp. decreasing), we get that $f(u + y) - f(u) < f\big(tu + (1-t)v + y\big) - f\big(tu + (1-t)v\big)$, i.e.,

$$f\big(tu + (1-t)v\big) + f\big((1-t)u + tv\big) < f(u) + f(v).$$

Thus $f$ is strictly Wright-convex (resp. strictly Wright-concave) on $I$ (cf. [31]). Hence, by the result of Ng [25], $f$ is of the form

$$f(x) = g(x) + A(x) \qquad (x \in I), \tag{2.3}$$

where $g : I \to \mathbb{R}$ is either strictly convex or strictly concave and $A : \mathbb{R} \to \mathbb{R}$ is additive. Substituting $f = g + A$ into (1.1) and using the additivity of $A$, it follows that (2.1) holds. By well known regularity properties of convex/concave functions (cf. [18], [29]), we have that $g$ is continuous. Thus, the range of the Cauchy-difference $g(x) + g(y) - g(x + y)$ over $(x, y) \in \Delta$ is an interval. Hence, the range $a(D)$ is also an interval, which yields that $a$ is continuous. In order to prove the continuity of $b$, let $x_0$ be an arbitrary fixed point in $I$ (where the continuity of $b$ is to be proved). We rewrite (2.1) into the following form:

$$b(x) = a^{-1}\big(g(x) + g(y) - g(x + y)\big) - b(y) \qquad \big((x, y) \in \Delta\big). \tag{2.4}$$

Due to the strict convexity/concavity of $g$, the function $I \cap (I - x_0) \ni y \mapsto g(y) - g(x_0 + y)$ is strictly monotone, therefore the set

$$I_{x_0} := \big\{g(x_0) + g(y) - g(x_0 + y) \mid y \in I \cap (I - x_0)\big\} \tag{2.5}$$

is an open interval of positive length. Thus, a point $y_0 \in I \cap (I - x_0)$ can be chosen so that $a^{-1}$ be continuous at $g(x_0) + g(y_0) - g(x_0 + y_0)$. Putting $y = y_0$, the right hand side of (2.4) is a continuous function of $x$ at $x_0$, hence $b$ is also continuous at $x_0$. The continuity and strict monotonicity of $b$ result that $b^{-1}$ is also continuous. Consequently, the range of $b$ as well as the set $D$ are open intervals of positive length. Thus, the domain of $a$ is also an interval, which yields that $a^{-1}$ is continuous, as well. □

In the next result, using the differentiability properties of monotone and convex functions, we obtain one-sided differentiability of the unknown functions $a, b, g$ and eliminate the composite part of (2.1) by deducing a non-composite functional equation for the right-hand-side derivatives $g'_+$ and $b'_+$. In the sequel, the ordinary (two-sided), the left, and the right derivatives will be denoted by $(\cdot)'$, $(\cdot)'_+$, and $(\cdot)'_-$, respectively,

**Lemma 2.2.** *Assume that* $g, b : I \to \mathbb{R}$ *and* $a : D \to \mathbb{R}$ *satisfy* (2.1), $g$ *is a strictly convex or concave function, and* $a$, $b$ *are strictly monotone continuous functions. Then* $a$, $b$, *and* $g$ *are differentiable both from the left and from the right on their domain, the right-hand-side derivative* $b'_+$ *is nowhere zero and the functions* $G := g'_+$ *and* $h := 1/b'_+$ *satisfy the following functional equation*

$$G(x+y)\big(h(x)-h(y)\big) = h(x)G(x) - h(y)G(y) \quad (x, y \in I). \tag{2.6}$$

Proof. By known differentiability properties of convex/concave functions (cf. [18], [29]), we have that $g$ is differentiable both from the right and from the left at each point of $I$. The strict convexity/concavity of $g$ yields that $g'_+$ and $g'_-$ are strictly increasing/decreasing functions.

Let $x_0 \in I$ be fixed arbitrarily and let $I_{x_0}$ be defined by (2.5). As we have seen in the proof of Lemma 2.1, $I_{x_0}$ is an interval of positive length. By Lebesgue's differentiability theorem on monotone functions, $a^{-1}$ is differentiable almost everywhere in $I_{x_0}$. Therefore, we can find a point $y_0 \in I \cap (I - x_0)$ such that $a^{-1}$ is differentiable at $g(x_0) + g(y_0) - g(x_0 + y_0)$. Rewriting (2.1) in the form (2.4), replacing $y$ by $y_0$ therein, and using the Chain Rule, we can see that $b$ is differentiable both from the left and from the right at the point $x_0$.

We show that $b'_+(x_0) \neq 0$ for all $x_0 \in I$. Assume, on the contrary, that $b'_+(x_0) = 0$ for some $x_0 \in I$. Choose $y_0 \in I \cap (I - x_0)$ so that $a$ be differentiable at $b(x_0) + b(y_0)$. (This is possible by Lebesgue's theorem.) Differentiating (2.1) from the right, we get that

$$g'_+(x_0) - g'_+(x_0 + y_0) = a'\big(b(x_0) + b(y_0)\big) b'_+(x_0) = 0,$$

i.e., $g'_+(x_0) = g'_+(x_0 + y_0)$ which contradicts the strict monotonicity of $g'_+$. Thus $b'_+(x) \neq 0$ on $I$. It follows from this that $b^{-1}$ is also differentiable from the right at each point of $b(I)$. (A similar argument shows that $b'_-(x) \neq 0$ for all $x \in I$ and that $b^{-1}$ is differentiable from the left on $b(I)$, too.)

Now rewrite (2.1) with the substitution $x := b^{-1}(u)$ and $y := b^{-1}(t-u)$ as

$$a(t) = g \circ b^{-1}(u) + g \circ b^{-1}(t-u) - g\big(b^{-1}(u) + b^{-1}(t-u)\big)$$
$$\big(t \in D,\ u \in b(I) \cap (t - b(I))\big).$$

By the Chain Rule, the right hand side of this equation is differentiable with respect to $t$ both from the right and from the left, therefore, the one-sided derivatives of $a$ exist everywhere.

Differentiating (2.1) with respect to $x$ and $y$ from the right, we get, for all $(x,y)\in\Delta$, that

$$g'_+(x)-g'_+(x+y)=a'_\pm\big(b(x)+b(y)\big)b'_+(x),$$

$$g'_+(y)-g'_+(x+y)=a'_\pm\big(b(x)+b(y)\big)b'_+(y),$$

where $a'_\pm$ denotes $a'_+$ if $b$ is increasing and $a'_-$ if $b$ is decreasing. Utilizing that $b'_+$ is nowhere zero, it follows from these equations that

$$\frac{g'_+(x)-g'_+(x+y)}{b'_+(x)}=\frac{g'_+(y)-g'_+(x+y)}{b'_+(y)}\qquad\big((x,y)\in\Delta\big).$$

Introducing the notations $G:=g'_+$ and $h:=1/b'_+$, we get that (2.6) is satisfied. □

The functional equation (2.6) is non-composite and the functions involved are measurable, therefore, the general results of the regularity theory of functional equations can be applied to it. Using Theorem 1.26 of the book [15] (see also [14, Theorem 1.7]), we deduce the $C^\infty$ properties of the unknown functions $G$ and $h$.

**Lemma 2.3.** *Let $G:I\to\mathbb{R}$ be a strictly monotone and $h:I\to\mathbb{R}$ be a nonzero function such that* (2.6) *is satisfied. Then $G$ and $h$ are infinitely many times differentiable on $I$ and there exists a nonzero constant $k$ and a polynomial $P$ of at most second degree such that*

$$G'(2x)h'(x)^2=k\quad\big(x\in\tfrac{1}{2}I\big)\tag{2.7}$$

*and*

$$h'(x)^2=P(h(x))\quad\big(x\in\tfrac{1}{2}I\big).\tag{2.8}$$

Proof. If $(x,y)\in\Delta$ and $h(x)=h(y)$, then the left hand side of (2.6) is zero. Therefore $h(x)G(x)=h(y)G(y)$. Since $h$ is nonzero, thus $G(x)=G(y)$, which, due to the strict monotonicity of $G$, yields that $x=y$. In other words,

$$\text{if } (x,y)\in\Delta \text{ and } x\neq y \text{ then, } h(x)\neq h(y).\tag{2.9}$$

Let $p \in I$ be fixed. Then, by (2.6), we have that

$$G(t+p)\big(h(t)-h(p)\big) = h(t)G(t) - h(p)G(p) \qquad \big(t \in I \cap (I-p)\big).$$

Hence

$$h(t) = h(p)\frac{G(t+p)-G(p)}{G(t+p)-G(t)} \qquad \big(t \in I \cap (I-p)\big). \tag{2.10}$$

(Here, due to the strict monotonicity of $G$, we have that $G(t+p)-G(t) \neq 0$ for all $t \in I \cap (I-p)$.) Using this expression for $h$, if $(x,y) \in \Delta$, $x, y \in I \cap (I-p)$ and $x \neq y$ then we get that

$$G(x+y) = \frac{h(x)G(x)-h(y)G(y)}{h(x)-h(y)}$$

$$= \frac{h(p)\dfrac{G(x+p)-G(p)}{G(x+p)-G(x)}G(x) - h(p)\dfrac{G(y+p)-G(p)}{G(y+p)-G(y)}G(y)}{h(p)\dfrac{G(x+p)-G(p)}{G(x+p)-G(x)} - h(p)\dfrac{G(y+p)-G(p)}{G(y+p)-G(y)}}$$

$$= \frac{(G(x+p)-G(p))(G(y+p)-G(y))G(x)-(G(y+p)-G(p))(G(x+p)-G(x))G(y)}{(G(x+p)-G(p))(G(y+p)-G(y))-(G(y+p)-G(p))(G(x+p)-G(x))}$$

$$= H\big(G(x+p), G(y+p), G(x), G(y), G(p)\big).$$

Obviously, the function $H$ is analytic. Substituting $x = t-y$, we obtain

$$G(t) = H\big(G(t-y+p), G(y+p), G(t-y), G(y), G(p)\big)$$

for all $(y,p,t)$ satisfying

$$t,\ t-y+p,\ y+p,\ t-y,\ y,\ p \in I \quad \text{with} \quad t \neq 2y.$$

The function $G$ is strictly monotonic, hence it is differentiable almost everywhere. Thus, by Theorem 1.26 of [15] (cf. [14, Theorem 1.7]), we have that $G$ is infinitely many times differentiable on $I$. Using (2.10), we can also see that $h$ is infinitely many times differentiable on $I \cap (I-p)$, for all $p \in I$. Thus, $G$ and $h$ are infinitely many times differentiable on the entire interval $I$.

Finally, differentiating (2.6) in various ways, we deduce (2.7) and (2.8). First apply the differential operator $\partial_x\partial_y$ to both sides of (2.6). Then we obtain

$$G''(x+y)\big(h(x)-h(y)\big)+G'(x+y)\big(h'(x)-h'(y)\big) = 0 \quad \big((x,y) \in \Delta\big). \tag{2.11}$$

Dividing by $y - x$ and taking the limit $y \to x$, we get

$$G''(2x)h'(x) + G'(2x)h''(x) = 0 \qquad (x \in \tfrac{1}{2}I).$$

Therefore,

$$\big(G'(2x)h'(x)^2\big)' = 0 \qquad (x \in \tfrac{1}{2}I),$$

i.e., there exists a constant $k \in \mathbb{R}$ such that (2.7) holds. We show that $k$ cannot be zero. Assume, on the contrary, that $k = 0$. The function $G$ being strictly monotone, its derivative is nonzero in an open subinterval $J$ of $\frac{1}{2}I$. Then, by (2.7), the function $h$ is constant in $J$, which contradicts the injectivity property (2.9) of $h$.

Thus, $G'$ and $h'$ are nowhere zero in $I$ and in $\frac{1}{2}I$, respectively.

Rearranging (2.11), we have that

$$-\frac{G''(x+y)}{G'(x+y)} = \frac{h'(x) - h'(y)}{h(x) - h(y)} \qquad \big((x,y) \in \Delta,\ x \neq y\big).$$

Applying the differential operator $\partial_y - \partial_x$ to this equation, after some calculations, we get that

$$\big(h''(x) + h''(y)\big)\big(h(x) - h(y)\big) = h'(x)^2 - h'(y)^2 \quad \big((x,y) \in \Delta\big). \tag{2.12}$$

Define the function $P : h(\frac{1}{2}I) \to \mathbb{R}$ by

$$P(u) := h'\big(h^{-1}(u)\big)^2 \qquad \big(u \in h\big(\tfrac{1}{2}I\big)\big). \tag{2.13}$$

Then, $P$ is infinitely many times differentiable since $h'(x) \neq 0$ for $x \in \frac{1}{2}I$. Using the equality $P\big(h(x)\big) = h'(x)^2$, it follows from (2.12) that

$$\frac{P'\big(h(x)\big) + P'\big(h(y)\big)}{2}\big(h(x) - h(y)\big) = P\big(h(x)\big) - P\big(h(y)\big) \quad \big((x,y) \in \Delta\big),$$

whence,

$$\big(P'(u) + P'(v)\big)(u - v) = 2\big(P(u) - P(v)\big) \qquad \big(u, v \in h\big(\tfrac{1}{2}I\big)\big),$$

Differentiating with respect to $u$ twice, we get that

$$P'''(u) = 0 \quad \big(u \in h\big(\tfrac{1}{2}I\big)\big).$$

Thus, $P$ is the restriction of a polynomial of at most second degree. In view of (2.13) we also have that (2.8) holds. □

Combining the results of Lemma 2.1–Lemma 2.3 and solving the differential equations obtained in Lemma 2.3, we can describe the exact form of $f$.

**Theorem 2.4.** *Assume that* $f, b : I \to \mathbb{R}$ *and* $a : D \to \mathbb{R}$ *satisfy* (1.1) *and* $a$ *and* $b$ *are strictly monotone functions. Then there exist an additive function* $A : \mathbb{R} \to \mathbb{R}$ *and four real constants* $\alpha$, $\beta$, $\gamma$, $\delta$ *such that* $\alpha\beta \neq 0$ *and* $f$ *is one of the following forms:*

(I) $f(x) = \alpha \log(\cosh(\beta x + \gamma)) + A(x) + \delta$;

(II) $f(x) = \alpha \log\big(\sinh(\beta x + \gamma)\big) + A(x) + \delta$, (*here* $\gamma \geq 0$ *and* $\beta I + \gamma \subset\, ]0, \infty[\,$);

(III) $f(x) = \alpha \log(\sin(\beta x + \gamma)) + A(x) + \delta$, (*here* $\gamma \in [0, \pi]$ *and* $\beta I + \gamma \subset\, ]0, \pi[\,$);

(IV) $f(x) = \alpha e^{\beta x} + A(x) + \delta$;

(V) $f(x) = \alpha \log |\, x + \gamma| + A(x) + \delta$, (*here* $\gamma \notin (-I)$);

(VI) $f(x) = \alpha x^2 + A(x) + \delta$

*for all* $x \in I$.

Proof. Combining the results of Lemma 2.1–Lemma 2.3, we get that $b$ is differentiable from the right on $I$ and its right derivative $h = b'_+$ infinitely many times differentiable and satisfies (2.8), where $P$ is a polynomial of at most second degree. Since $h'$ is non-vanishing, hence it satisfies one of the following differential equations:

$$h'(x) = \sqrt{P\big(h(x)\big)} \quad \text{or} \quad h'(x) = -\sqrt{P\big(h(x)\big)}$$

for $x \in \frac{1}{2}I$. Integrating these differential equations (and distinguishing five cases: (1) $P$ has no real roots; (2) $P$ has two real roots and it is convex; (3) $P$ has two real roots and it is concave; (4) $P$ is of second degree with a single real root; (5) $P$ is of first degree), it follows that there exist real constants $B$, $C$, $D$, and $E$ such that $h$ is one of the following forms:

(i) $h(x) = B \sinh(Cx + D) + E$, $\big($here $BC \neq 0\big)$;

(ii) $h(x) = B \cosh(Cx + D) + E$, $\big($here $BC \neq 0$ and $\frac{1}{2}CI + D \subset\, ]0, \infty[\big)$;

(iii) $h(x) = B \cos(Cx + D) + E$, $\big($here $BC \neq 0$ and $\frac{1}{2}CI + D \subset\, ]0, \pi[\big)$;

(iv) $h(x) = Be^{Cx} + E$, $\big($here $BC \neq 0\big)$;

(v) $h(x) = Bx^2 + Cx + E$, (here $0 \notin BI + C$)

for all $x \in \frac{1}{2}I$. The conditions $BC \neq 0$, etc., ensure that $h'$ is strictly monotone on $I$. In order that $h$ be non-vanishing, the parameter $E$ has to be chosen properly. This is not specified here since $E$ does not play any role in the sequel.

On the other hand, $f$ is of the form $g + A$, where $A$ is an additive function and $g$ is a strictly convex function whose right derivative $G = g'_+$ is infinitely many times differentiable and there exists a nonzero constant $k$ such that (2.7) holds, i.e.,

$$G'(x) = \frac{k}{h'(x/2)^2} \quad (x \in I).$$

Using the above forms of $h$, first $G'$ and, after integration, $G$ can be determined. Since $g$ is locally Lipschitz, $g$ can be obtained after integration from $G = g'_+$. Thus, the statement easily follows. □

At this point, we can only state that $f$ has to be one of the forms listed in Theorem 2.4. It is not obvious that all the functions obtained are indeed solutions of (1.1) with some strictly monotone functions $a$ and $b$. However, due to the addition theorems of the functions given by (I)–(VI) in Theorem 2.4, we can prove that all the Cauchy-differences with generators (I)–(VI) are also quasisums, moreover, the following theorem states a bit more.

**Theorem 2.5.** *Assume that $f, b : I \to \mathbb{R}$, $a : D \to \mathbb{R}$ and $a$ and $b$ are strictly monotone functions. Then the triple $(f, b, a)$ satisfies the functional equation* (1.1) *if and only if*

(S1) *either $f$ is given by* (I) *and*

$$b(x) = \frac{p}{\cosh\gamma}\log\frac{\sinh|\beta|x}{\cosh(\beta x+\gamma)} + q, \quad a(\xi) = \delta - \alpha\log\frac{e^{\frac{\xi-2q}{p}\cosh\gamma}+1}{\cosh\gamma},$$

(S2) *or $f$ is given by* (II) *with $\gamma > 0$, $\beta I + \gamma \subset ]0, \infty[$ and*

$$b(x) = \frac{p}{\sinh\gamma}\log\frac{\sinh|\beta|x}{\sinh(\beta x+\gamma)} + q, \quad a(\xi) = \delta - \alpha\log\left|\frac{e^{\frac{\xi-2q}{p}\sinh\gamma}-1}{\sinh\gamma}\right|,$$

($S2_0$) *or $f$ is given by* (II) *with $\gamma = 0$, $\beta > 0$ and*

$$b(x) = p\coth\beta x + q, \quad a(\xi) = \delta - \alpha\log\frac{\xi - 2q}{p},$$

(S3) *or* $f$ *is given by* (III) *with* $\gamma > 0$, $\beta I + \gamma \subset ]0,\pi[$, $|\beta| I \subset ]0,\pi[$ *and*

$$b(x) = \frac{p}{\sin\gamma} \log \frac{\sin|\beta| x}{\sin(\beta x + \gamma)} + q, \quad a(\xi) = \delta - \alpha \log \left| \frac{e^{\frac{\xi - 2q}{p}} \sin\gamma - 1}{\sin\gamma} \right|,$$

($S3_0$) *or* $f$ *is given by* (III) *with* $\gamma = 0$, $\beta > 0$, $\beta I \subset ]0,\pi[$ *and*

$$b(x) = p \cot \beta x + q, \quad a(\xi) = \delta - \alpha \log \frac{\xi - 2q}{p},$$

(S4) *or* $f$ *is given by* (IV) *and*

$$b(x) = p \log|e^{\beta x} - 1| + q, \quad a(\xi) = \delta - \alpha \Big( e^{\frac{\xi - 2q}{p}} - 1 \Big),$$

(S5) *or* $f$ *is given by* (V) *with* $\gamma \notin (-I) \cup \{0\}$ *and*

$$b(x) = \frac{p}{\gamma} \log \left| 1 + \frac{\gamma}{x} \right| + q, \quad a(\xi) = \delta - \alpha \log \left| \frac{e^{-\frac{\xi - 2q}{p}\gamma} - 1}{\gamma} \right|,$$

($S5_0$) *or* $f$ *is given by* (V) *with* $\gamma = 0$ *and*

$$b(x) = \frac{p}{x} + q, \quad a(\xi) = \delta - \alpha \log \frac{\xi - 2q}{p},$$

(S6) *or* $f$ *is given by* (VI) *and*

$$b(x) = p \log x + q, \quad a(\xi) = \delta - 2\alpha e^{\frac{\xi - 2q}{p}},$$

*where* $\alpha, \beta, \gamma, \delta, p, q \in \mathbb{R}$ *with* $\alpha\beta p \neq 0$ *and* $A$ *is an additive function.*

PROOF. First observe that if $b, b_0 : I \to \mathbb{R}$ and $a : D \to \mathbb{R}$, $a_0 : D_0 := \big\{ b_0(x) + b_0(y) \mid (x,y) \in \Delta \big\} \to \mathbb{R}$ are strictly monotone and continuous functions, then

$$a_0\big(b_0(x) + b_0(y)\big) = a\big(b(x) + b(y)\big) \qquad \big((x,y) \in \Delta\big) \tag{2.14}$$

holds if and only if there exist $0 \neq p \in \mathbb{R}$ and $q \in \mathbb{R}$ such that

$$a(\xi) = a_0\left(\frac{\xi - 2q}{p}\right) \qquad (\xi \in D) \tag{2.15}$$

and

$$b(x) = p b_0(x) + q \qquad (x \in I). \tag{2.16}$$

Indeed, let $\Delta^* = \big\{(u,v) \in b(I) \times b(I) \mid \big(b_0^{-1}(u), b_0^{-1}(v)\big) \in \Delta\big\}$. Then $\Delta^*$ is an open and connected set and (2.14) implies that

$$b \circ b_0^{-1}(u) + b \circ b_0^{-1}(v) = a^{-1} \circ a_0(u+v) \quad \big((u,v) \in \Delta^*\big).$$

Therefore, by a result of Radó and Baker [28], we get that

$$\begin{aligned} b \circ b_0^{-1}(u) &= pu + q \qquad && \big(u \in b_0(I)\big) \qquad \text{and} \\ a^{-1} \circ a_0(t) &= pt + 2q \qquad && \big(t \in b_0(I) + b_0(I)\big) \end{aligned}$$

with some $0 \neq p \in \mathbb{R}$ and $q \in \mathbb{R}$. Thus (2.15) and (2.16) hold. The converse can be proved by simple calculation.

Suppose now that the triple $(f, b, a)$ in the theorem satisfies (1.1). Then, by Theorem 2.4, we have the possible forms of $f$. According to the argument above it is enough to determine one pair $(b_0, a_0)$ for which the triple $(f, b_0, a_0)$ satisfies (1.1) since all the other such pairs $(b, a)$ can be obtained from equations (2.15) and (2.16).

We prove the theorem only in the case (S1), the other cases can be handled similarly. Denote by $F$ the Cauchy-difference generated by $f$. Then, for all $(x, y) \in \Delta$, we have

$$\begin{aligned} F(x,y) &= \alpha \log \frac{\cosh(\beta x + \gamma)\cosh(\beta y + \gamma)}{\cosh(\beta(x+y) + \gamma)} + \delta \\ &= \delta - \alpha \log \frac{\cosh(\beta x + \gamma + \beta y + \gamma - \gamma)}{\cosh(\beta x + \gamma)\cosh(\beta y + \gamma)} \\ &= \delta - \alpha \log \frac{\cosh(\beta x + \gamma + \beta y + \gamma)\cosh\gamma - \sinh(\beta x + \gamma + \beta y + \gamma)\sinh\gamma}{\cosh(\beta x + \gamma)\cosh(by + \gamma)} \\ &= \delta - \alpha \log \big[(1 + \tanh(\beta x + \gamma)\tanh(\beta y + \gamma))\cosh\gamma \\ &\quad - (\tanh(\beta x + \gamma) + \tanh(\beta y + \gamma))\sinh\gamma\big] \\ &= \delta - \alpha \log \left[\frac{\big(\cosh\gamma\tanh(\beta x + \gamma) - \sinh\gamma\big)(\cosh\gamma\tanh(\beta y + \gamma) - \sinh\gamma) + 1}{\cosh\gamma}\right] \\ &= \delta - \alpha \log \left[\frac{1}{\cosh\gamma}\left(\frac{\sinh\beta x}{\cosh(\beta x + \gamma)} \cdot \frac{\sinh\beta y}{\cosh(\beta y + \gamma)} + 1\right)\right]. \end{aligned}$$

Now it is not difficult to notice that the equality

$$F(x,y) = a_0\big(b_0(x) + b_0(y)\big)$$

holds, too, with

$$b_0(x) = \frac{1}{\cosh\gamma} \log \frac{\sinh|\beta|x}{\cosh(\beta x + \gamma)} \qquad (x \in I)$$

and

$$a_0(\xi) = \delta - \alpha \log \frac{e^{\xi \cosh\gamma} + 1}{\cosh\gamma} \qquad \big(\xi \in b_0(I) + b_0(I)\big).$$

Thus, by equations (2.15) and (2.16), we obtain (S1). □

*Remark 2.6.* Observe that if $\gamma$ tends to 0 in (S2), (S5) and to $\pi(k+1/2)$ in (S3), then the solutions described therein tend to those in (S2$_0$), (S5$_0$), and (S3$_0$), respectively.

## 3. An application

In utility theory, an uncertain alternative or binary gamble is a triple $(a, C, b)$, where $C$ is a chance event, $a$ and $b$ are consequences if $C$ or non-$C$ happens, respectively. Looking for a representation of the utility of gambles, given the weights of events and the utility of consequences, several plausible assumptions about gambles are made. One of them leads to the functional equation

$$\begin{aligned}\varphi\Big(\varphi^{-1}\big(\varphi(xw) + \varphi(y) - \varphi(yw)\big)z\Big) - \varphi(yz)&\\ = \varphi\Big(\varphi^{-1}\big(\varphi(xz) + \varphi(y) - \varphi(yz)\big)w\Big) - \varphi(yw).&\end{aligned} \tag{3.1}$$

Here, $\varphi : [0, K[ \to [0, +\infty[$ (where $0 < K \le +\infty$) is the unknown function and (3.1) is supposed to hold for all $0 \le y \le x < K$ and $z, w \in [0, 1]$ (see [19], [6]). In order to make (3.1) sense, the injectivity of $\varphi$ and

$$\varphi(xz) + \varphi(y) - \varphi(yz) \in \operatorname{range}(\varphi) \quad \big(0 \le y \le x < K,\ z \in [0, 1]\big) \tag{3.2}$$

have to be assumed. Under the conditions (3.2), $\varphi(0) = 0$, $\varphi$ is twice differentiable on $]0, K[$ and $\varphi'(y) \ne 0$ for all $y \in ]0, K[$, equation (3.1) was solved in [6]. Here we show that the differentiability conditions can be cancelled if we use the stronger condition

$$\varphi(xz) + \varphi(y) - \varphi(yz) < \varphi(x) \quad \big(0 < y < x < K,\ z \in [0, 1]\big) \tag{3.3}$$

instead of (3.2). First, we prove the following

**Lemma 3.1.** *Let* $\varphi : [0, K[ \to [0, +\infty[$, *let* $x_0 \in ]0, K[$ *and define the function* $f$ *on* $\mathbb{R}_+$ *by*

$$f(t) := \varphi\big(x_0 e^{-t}\big). \tag{3.4}$$

*Then* $f$ *is a strictly decreasing and strictly convex function, consequently,* $\varphi$ *is strictly increasing and continuous on* $]0, K[$.

Proof. The condition (3.3) with $z = 0$ implies that $\varphi$ is strictly increasing, thus, by (3.4), $f$ is a strictly decreasing. On the other hand, let $0 < t < s$ and $0 < u$ and substitute $x = x_0 e^{-t}$, $y = x_0 e^{-s}$, and $z = e^{-u}$ into (3.3). Using the definition of $f$, it follows that

$$f(t + u) - f(t) < f(s + u) - f(s)$$

which, as in the proof of Lemma 2.1, implies that $f$ is strictly Wright convex, hence, it is also Jensen-convex. Because it is also monotone, $f$ must be strictly convex. Consequently, $f$ and $\varphi$ are continuous on $\mathbb{R}_+$ and $]0, K[$, respectively. □

This lemma shows that condition (3.3) implies (3.2) and the injectivity of $\varphi$. Thus, supposing (3.3), equation (3.1) makes sense.

Our main observation concerning the solution of (3.1) is the following.

**Lemma 3.2.** *Suppose that* $\varphi : [0, K[ \to [0, +\infty[$ *is solution of* (3.1) *and let* $x_0 \in ]0, K[$ *be arbitrary. Then the Cauchy-difference of the function* $f$ *defined by* (3.4) *is a quasisum, i.e.,* (1.1) *holds for some strictly monotone and continuous functions* $a$ *and* $b$.

Proof. First, define a function $\psi : ]0, 1] \to \mathbb{R}$ by

$$\psi(t) := \varphi(x_0 t). \tag{3.5}$$

Then, by Lemma 3.1, $\psi$ is strictly increasing, continuous, and maps $]0, 1]$ onto $]0, \varphi(x_0)]$. Furthermore, for all $s, t \in ]0, 1[$, the strict increasingness, (3.5), and (3.3) imply that

$$\begin{aligned} 0 < \psi(s) < \psi(s) + \psi(t) - \psi(st) \\ = \varphi(x_0 s) + \varphi(x_0 t) - \varphi((x_0 s)t) < \varphi(x_0) = \psi(1). \end{aligned}$$

Hence, by the continuity of $\psi$, $\psi(s)+\psi(t)-\psi(st)$ lies in the codomain of $\psi$. Thus, the function $\Phi : ]0,1[^2 \to \mathbb{R}$ is correctly defined by the following expression:

$$\Phi(s,t)=\psi^{-1}\big(\psi(s)+\psi(t)-\psi(st)\big) \quad \big(s,t\in\,]0,1[\big). \tag{3.6}$$

Clearly, $\Phi$ is continuous and maps into $]0,1[$. Now we prove that $\Phi$ is strictly monotone in each variable. Due to the symmetry, it suffices to show the monotonicity concerning the first variable. This is equivalent to the strict monotonicity of the function $]0,1[\,\ni s\mapsto \psi(s)-\psi(st)$. Taking $0<s_1<s_2<1$ and applying (3.3) with $x=x_0s_2$, $y=x_0s_1$, and $z=t$, the statement follows at once.

Finally, we prove that $\Phi$ is an associative function, i.e., it satisfies

$$\Phi\big(\Phi(s,t),u\big)=\Phi\big(s,\Phi(t,u)\big) \qquad \big(s,t,u\in\,]0,1[\big). \tag{3.7}$$

According to (3.6), this is equivalent to

$$\psi\big(\Phi(s,t)u\big)-\psi(tu)=\psi\big(s\Phi(t,u)\big)-\psi(st),$$

which follows from (3.5) and (3.1) with $x=x_0$, $y=tx_0$, $z=s$, $w=u$.

At this point, we are in the position to apply a theorem of Aczél [3] (see also [11]), which says that $\Phi$ is of the form

$$\Phi(s,t)=\alpha^{-1}\big(\alpha(s)+\alpha(t)\big) \qquad \big(s,t\in\,]0,1[\big),$$

where $\alpha$ is a strictly monotone continuous function. This, and (3.6) imply that

$$\psi(s)+\psi(t)-\psi(st)=\psi\circ\alpha^{-1}\big(\alpha(s)+\alpha(t)\big) \qquad \big(s,t\in\,]0,1[\big), \tag{3.8}$$

which, with the definitions $a:=\psi\circ\alpha^{-1}$, $b(t):=\alpha(e^{-t})$ shows that the Cauchy-difference of $f$ is a quasisum on $\mathbb{R}_+\times\mathbb{R}_+$. □

Applying now the main result of the previous section, we have that $f$ is infinitely many times differentiable. Thus, by (3.4), so is $\varphi$ (being $x_0$ arbitrary in $]0,K[$). On the other hand,

$$\varphi'(x)=-\frac{1}{x}f'\big(-\log(x/x_0)\big) \qquad \big(x\in\,]0,x_0[\big).$$

By Lemma 3.1, we have that $\varphi$ is strictly increasing, hence $\varphi'(x) \geq 0$ for all $x \in I$. In view of Lemma 2.2, $f' = g'$+constant, hence, $f'$ is strictly monotone. Since, by Lemma 3.1, $f$ is strictly decreasing, we have that $f'$ is nowhere zero on $\mathbb{R}_+$. Therefore, $\varphi'$ is strictly positive on $]0, K[$. Hence, we can apply the result of [6] to have the following result.

**Corollary 3.3.** *Suppose that* $\varphi : [0, K[ \to [0, +\infty[$ *is a solution of* (3.1) *satisfying also* (3.3). *Then either*

$$\varphi(x) = \alpha x^q + \beta \qquad \big(x \in ]0, K[\big)$$

*with real constants* $q > 0$, $\alpha > 0$, $\beta \geq 0$, *or*

$$\varphi(x) = \gamma \log(\alpha x^q + \beta) \qquad \big(x \in ]0, K[\big)$$

*with real constants* $\gamma > 0$, $q > 0$, $\alpha > 0$, $\beta \geq 1$, *or with* $\gamma < 0$, $q > 0$, $-K^{-q} \leq \alpha < 0$, $-\alpha K^q \leq \beta \leq 1$ *in the case* $K < +\infty$.

*Remark 3.4.* Another possible condition for (3.2) to hold is

$$\varphi(xz) + \varphi(y) - \varphi(yz) > \varphi(x) \qquad \big(0 < y < x < K,\ z \in [0, 1]\big).$$

In this case, $\varphi$ and $\psi$ defined in (3.5) are strictly decreasing functions, while $f$ defined by (3.4) is strictly increasing and concave. Everything goes as before and finally one can get those nonnegative solutions of (3.1) that are strictly decreasing over $]0, K[$ (cf. [6, Theorem 1]).

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