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Self-inversive polynomials whose zeros are on the unit circle

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Dedicated to the memory of Béla Brindza and Jenő Erdős

Abstract. We prove that all zeros of the self-inversive polynomial $P_m(z) = \sum_{k=0}^{m} A_k z^k \in \mathbb{C}[z]$ of degree $m \geq 1$ are on the unit circle if

$$|A_m| \geq \frac{1}{2} \sum_{k=1}^{m-1} |A_k| \, .$$

If this inequality is strict the zeros e^{iu_l} $(l = 1, \dots, m)$ are simple and can be arranged such that

$$\frac{2((l-1)\pi - \beta_m)}{m} < u_l < \frac{2(l\pi - \beta_m)}{m} \quad (l = 1, \dots, m)$$

where $\beta_m = \arg A_m \left(\frac{\bar{A}_0}{A_m}\right)^{\frac{1}{2}}$.

If equality holds in the above inequality then double zeros may arise, we discuss when this happens.

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1. Introduction

The first author [3] proved that all zeros of the real *reciprocal polynomial*

$$P_m(z) = \sum_{k=0}^{m} A_k z^k \in \mathbb{R}[z] \tag{1}$$

of degree $m \geq 2$ (i.e. $A_k \in \mathbb{R}$, $A_m \neq 0$ and $A_k = A_{m-k}$ for all $k=0, \dots, \left[\frac{m}{2}\right]$) are on the unit circle, provided that

$$|A_m| \geq \sum_{k=1}^{m-1} |A_k - A_m|. \tag{2}$$

Moreover if (2) holds, then all zeros e^{iu_j} $(j = 1, 2, \dots, m)$ of P_m can be arranged such that

$$\left|\varepsilon_j - e^{iu_j}\right| < \frac{\pi}{m+1} \quad (j = 1, \dots, m)$$

where $\varepsilon_j = e^{i\frac{j}{m+1}2\pi}$ $(j = 1, 2, \dots, m)$ are the $(m+1)$st roots of unity except 1.

A. Schinzel [8] generalized (the first part of) this result for self-inversive polynomials. He proved that all zeros of the polynomial

$$P_m(z) = \sum_{k=0}^{m} A_k z^k \in \mathbb{C}[z]$$

satisfying

$$\begin{gathered} A_k \in \mathbb{C}, A_m \neq 0, A_{m-k} = \varepsilon \overline{A}_k \quad (k = 0, \dots, m) \\ \text{with a fixed } \varepsilon \in \mathbb{C}, |\varepsilon| = 1 \end{gathered} \tag{3}$$

are on the unit circle if

$$|A_m| \geq \inf_{\substack{c,d \in \mathbb{C} \\ |d|=1}} \sum_{k=0}^{m} \left| cA_k - d^{m-k} A_m \right|.$$

If the inequality is strict the zeros are simple.

Polynomials satisfying (3) are called *self-inversive* see e.g. [1], [2], [6].

Lakatos and Losonczi [4] proved that for real reciprocal polynomials of odd degree Lakatos' result remains valid even if

$$|A_m| \geq \cos^2 \frac{\pi}{2(m+1)} \sum_{k=1}^{m-1} |A_k - A_m|$$

and *conjectured* that Schinzel's result can also be extended similarly (by inserting the corresponding cos factor, not $\cos^2$, in front of the inf sign in Schinzels' condition), i.e. for self-inversive polynomials of odd degree

$$|A_m| \geq \cos \frac{\pi}{2(m+1)} \inf_{\substack{c,d\in\mathbb{C} \\ d|=1}} \sum_{k=0}^{m} |cA_k - d^{m-k}A_m|$$

is sufficient for all zeros to lie on the unit circle.

This conjecture was recently proved by Losonczi and Schinzel [5].

We remark that by a classical result of Cohn [2] p. 121, see also Theorem 2.1.6 of [6], *a necessary and sufficient condition for all zeros of a complex polynomial P_m of degree m to lie on the unit circle is that P_m is self-inversive and all zeros of P_m' lie on the closed unit disc.*

The aim of this note is to prove a more natural sufficient condition for a self-inversive polynomial to have all of its zeros on the unit circle. We also discuss the location and multiplicity of these zeros. In view of the mentioned Cohn's result, our sufficient conditions can be reformulated as sufficient conditions for all zeros of the derivative of a self-inversive polynomial to lie on the closed unit disc.

2. The main result

Theorem 1. (i) *If all zeros of the polynomial $P_m(z) = \sum_{k=0}^{m} A_k z^k \in \mathbb{C}[z]$ of degree $m \geq 1$ are on the unit circle then P_m is self-inversive.*

(ii)-1 *If P_m is self-inversive and*

$$|A_m| \geq \frac{1}{2} \sum_{k=1}^{m-1} |A_k| \tag{4}$$

holds then all zeros of P_m are on the unit circle.

Let

$$\beta_{m-l} = \arg A_{m-l}\left(\frac{\bar{A}_0}{A_m}\right)^{\frac{1}{2}} \qquad \left(l = 0, \dots, \left[\frac{m}{2}\right]\right),$$

$$\varphi_l = \frac{2(l\pi - \beta_m)}{m} \qquad (l = 0, \dots, m),$$

where $\left[\frac{m}{2}\right]$ *denotes the integer part of* $\frac{m}{2}$.

(ii)-2 *If the inequality* (4) *is strict then there are numbers* u_l $(l = 1, \dots, m)$ *such that*

$$\varphi_{l-1} < u_l < \varphi_l \quad (l = 1, \dots, m) \tag{5}$$

and e^{iu_l} $(l = 1, \dots, m)$ *are simple zeros of* P_m.

(ii)-3 *If* (4) *holds with equality then double zeros may arise. If* (4) *holds with equality then* $e^{i\varphi_l}$ $(1 \le l \le m)$ *is a zero of* P_m *if and only if the coefficients of* P_m *satisfy the conditions*

$$\begin{gathered}\cos\left(\beta_{m-k} + \left(\frac{m}{2} - k\right)\varphi_l\right) = (-1)^{l+1} \\ \text{for all } k = 1, \dots, \left[\frac{m}{2}\right] \text{ for which } A_k \ne 0.\end{gathered} \tag{6}$$

If (6) *holds then* $e^{i\varphi_l}$ *is necessarily a double zero of* P_m.

Proof. (i) follows from Cohn's theorem cited in the introduction.

(ii)-1 If $A_1 = A_2 = \dots = A_{m-1} = 0$ then (4) is strict and

$$P_m(z) = A_m z^m + A_0 = A_m\left(z^m - \frac{A_0}{A_m}\right) \quad \text{where} \quad \left|\frac{A_0}{A_m}\right| = 1$$

by the self-inversiveness condition $A_m = \varepsilon\overline{A}_0$. This shows that all zeros of P_m are on the unit circle and they are simple.

If *there is a subscript* $k \in \{1, 2, \dots, m-1\}$ *such that* $A_k \ne 0$ then we prove that P_m has no zero inside the unit circle. Then P_m being self-inversive cannot have any zero outside the unit circle, hence all zeros are on the unit circle.

Assume on the contrary that P_m has a zero $|z_0| < 1$. Then by another result of Cohn [2] p. 113, Theorem IV, see also [1], Theorem I the polynomial

$$z^{m-1}P'_m\left(z^{-1}\right)=\sum_{k=1}^{m}kA_kz^{m-k}$$

has the same number of zeros inside the unit circle as P_m. Thus there is a $|z_1|<1$ such that

$$\sum_{k=1}^{m}kA_kz_1^{m-k}=0.$$

Rearranging, and using (3) we get

$$\begin{aligned}2m\left|A_m\right| &= \left|2\sum_{k=1}^{m-1}kA_kz_1^{m-k}\right|=\left|\sum_{k=1}^{m-1}kA_kz_1^{m-k}+\sum_{k=1}^{m-1}k\varepsilon\overline{A}_{m-k}z_1^{m-k}\right|\\ &\le\sum_{k=1}^{m-1}k\left|A_k\right|\,\left|z_1\right|^{m-k}+\sum_{k=1}^{m-1}(m-k)\left|A_k\right|\,\left|z_1\right|^{k}<m\sum_{k=1}^{m-1}\left|A_k\right|,\end{aligned}$$

hence

$$2\left|A_m\right|<\sum_{k=1}^{m-1}\left|A_k\right|$$

which contradicts to (4).

(ii)-2 With $\varepsilon=\frac{A_m}{A_0}$, $B_k=\varepsilon^{-\frac{1}{2}}A_k$, we have $B_k=\overline{B}_{m-k}$ $(k=0,\dots,m)$. If $m=2n+1$ is odd, $z=e^{i\varphi}$ then

$$\begin{aligned}\varepsilon^{-\frac{1}{2}}z^{-\frac{m}{2}}P_m(z) &= \sum_{k=0}^{n}\left(\overline{B_{m-k}z^{\frac{m}{2}-k}}+B_{m-k}z^{\frac{m}{2}-k}\right)\\ &= \sum_{k=0}^{n}2|B_{m-k}|\cos\left(\beta_{m-k}+\left(\frac{m}{2}-k\right)\varphi\right),\end{aligned}$$

where $B_k=|B_k|e^{i\beta_k}$ $(k=0,\dots,m)$.

For even $m=2n$ we have

$$\varepsilon^{-\frac{1}{2}}z^{-\frac{m}{2}}P_m(z)=\sum_{k=0}^{n-1}2|B_{m-k}|\cos\left(\beta_{m-k}+\left(\frac{m}{2}-k\right)\varphi\right)+|B_n|\cos\beta_n,$$

where $\beta_n=0$ or π as B_n is real.

Denoting $\frac{\varepsilon^{-\frac{1}{2}}}{2|B_m|}z^{-\frac{m}{2}}P_m(z)$ at $z = e^{i\varphi}$ by $F_m(\varphi)$ we have

$$F_m(\varphi) = \cos\left(\beta_m + \frac{m}{2}\varphi\right) + f_m(\varphi),$$

where

$$f_m(\varphi) = \begin{cases} \sum_{k=1}^{n}\left|\frac{B_k}{B_m}\right|\cos\left(\beta_{m-k} + \left(\frac{m}{2} - k\right)\varphi\right) & \text{if } m = 2n+1, \\ \sum_{k=1}^{n-1}\left|\frac{B_k}{B_m}\right|\cos\left(\beta_{m-k} + \left(\frac{m}{2} - k\right)\varphi\right) + \left|\frac{B_n}{2B_m}\right|\cos\beta_n & \\ & \text{if } m = 2n. \end{cases}$$

Parallel to this rewrite (4) as

$$1 \geq \begin{cases} \sum_{k=1}^{n}\left|\frac{B_k}{B_m}\right| & \text{if } m = 2n+1, \\ \sum_{k=1}^{n-1}\left|\frac{B_k}{B_m}\right| + \left|\frac{B_n}{2B_m}\right| & \text{if } m = 2n. \end{cases} \tag{7}$$

It follows from (4) that $|f_m(\varphi)| \leq 1$ with strict inequality if (4) holds with strict inequality.

Let us consider $m+1$ numbers

$$\varphi_l = \frac{2(l\pi - \beta_m)}{m} \quad (l = 0, \ldots, m)$$

satisfying

$$\cos\left(\beta_m + \frac{m}{2}\varphi_l\right) = (-1)^l \quad (l = 0, \ldots, m).$$

If (4) holds with strict inequality then $|f_m(\varphi)| < 1$ hence for $l = 0, \ldots, m$

$$F_m(\varphi_l) = (-1)^l + f_m(\varphi_l) \begin{array}{ll} > 0 & \text{if } l \text{ is even}, \\ < 0 & \text{if } l \text{ is odd}. \end{array}$$

By the intermediate value theorem each open interval $]\varphi_{l-1}, \varphi_l[$ $(l = 1, \ldots, m)$ contains at least one zero u_l of F_m, which means that e^{iu_l} $(l = 1, \ldots, m)$ are zeros of P_m. Clearly each open interval can contain only one simple zero, otherwise the sum of the multiplicities of all zeros would exceed m. This proves (ii)-2 as

$$\beta_m = \arg B_m = \arg A_m \left(\frac{\bar{A}_0}{A_m}\right)^{\frac{1}{2}}.$$

(ii)-3 Assume that equality holds in (4). Then equality holds in (7) too. By (7) the equation

$$F_m(\varphi_l) = (-1)^l + f_m(\varphi_l) = 0 \tag{8}$$

can be written as

$$F_m(\varphi_l) = \sum_{k=1}^{\left[\frac{m}{2}\right]} \left|\frac{B_k}{B_m}\right| \left[\cos\left(\beta_{m-k} + \left(\frac{m}{2} - k\right)\varphi_l\right) + (-1)^l\right] = 0, \tag{9}$$

provided that $m = 2n+1$ is odd. For the even case $m = 2n$ the calculations are similar and the conclusions are the same, thus in the sequel we restrict ourselves to the case of odd degree. Since

$$\cos\left(\beta_{m-k} + \left(\frac{m}{2} - k\right)\varphi_l\right) + (-1)^l \quad \begin{array}{ll} \geq 0 & \text{if } l \text{ is even,} \\ \\ \leq 0 & \text{if } l \text{ is odd,} \end{array}$$

we conclude that φ_l $(1 \leq l \leq m)$ is a zero of F_m (or $e^{i\varphi_l}$ $(1 \leq l \leq m)$ is a zero of P_m, or (8), or (9) holds) if and only if

$$\begin{gathered} \cos\left(\beta_{m-k} + \left(\frac{m}{2} - k\right)\varphi_l\right) = (-1)^{l+1} \quad \text{for all } k = 1, \dots, \left[\frac{m}{2}\right] \\ \text{for which } B_k \neq 0 \text{ or } A_k \neq 0. \end{gathered} \tag{10}$$

If (10) holds then

$$\begin{aligned} F_m'(\varphi_l) = &-\frac{m}{2}\sin\left(\beta_m + \frac{m}{2}\varphi_l\right) \\ &- \sum_{k=1}^{n} \left|\frac{B_k}{B_m}\right| \left(\frac{m}{2} - k\right) \sin\left(\beta_{m-k} + \left(\frac{m}{2} - k\right)\varphi_l\right) = 0, \end{aligned}$$

$$\begin{aligned} F_m''(\varphi_l) &= -\left(\frac{m}{2}\right)^2 \cos\left(\beta_m + \frac{m}{2}\varphi_l\right) \\ &\quad - \sum_{k=1}^{n} \left|\frac{B_k}{B_m}\right| \left(\frac{m}{2} - k\right)^2 \cos\left(\beta_{m-k} + \left(\frac{m}{2} - k\right)\varphi_l\right) \\ &= -\left(\frac{m}{2}\right)^2 (-1)^l - \sum_{k=1}^{n} \left|\frac{B_k}{B_m}\right| \left(\frac{m}{2} - k\right)^2 (-1)^{l+1} \\ &= (-1)^{l+1} \sum_{k=1}^{n} \left|\frac{B_k}{B_m}\right| \left[\left(\frac{m}{2}\right)^2 - \left(\frac{m}{2} - k\right)^2\right] \\ &= (-1)^{l+1} \sum_{k=1}^{n} \left|\frac{B_k}{B_m}\right| (m-k)k \neq 0, \end{aligned}$$

as by (7) there is at least one subscript $k \in \{1, \dots, \left[\frac{m}{2}\right]\}$ with $B_k \neq 0$. This implies that the multiplicity of the zero φ_l $(1 \le l \le m)$ of F_m is two. This completes the proof of (ii)-3. □

The next corollary of Theorem 1, statement (ii)-1 was communicated to us by A. Schinzel and it appears here with his permission. For real polynomials this is known, see [7], Lemma 14.

Corollary 1 (A. Schinzel)**.** *If* $P(z) = \sum_{k=0}^{m} A_k z^k \in \mathbb{C}[z] \setminus \{0\}$ *and*

$$|A_m| + |A_0| \ge \sum_{k=1}^{m-1} |A_k| \tag{11}$$

then all common zeros of $P(z)$ *and* $Q(z) = \sum_{k=0}^{m} \overline{A}_{m-k} z^k$ *have modulus 1.*

Proof. Without loss of generality we may assume that $A_0 A_m \neq 0$. Taking

$$\varepsilon = \frac{|A_0|\, A_m}{\overline{A}_0\, |A_m|}$$

and applying Theorem 1 to the self-inversive polynomial $P(z) + \varepsilon Q(z) =$

$\sum_{k=0}^{m}\left(A_k+\varepsilon\overline{A}_{m-k}\right)z^k$ we have by (11)

$$
\begin{aligned}
2\left|A_m+\varepsilon\overline{A}_0\right| &= 2\left|A_m\right|+2\left|A_0\right| \\
&\geq \sum_{k=1}^{m-1}\left(\left|A_k\right|+\left|A_{m-k}\right|\right) \geq \sum_{k=1}^{m-1}\left|A_k+\varepsilon\overline{A}_{m-k}\right|
\end{aligned}
$$

hence by Theorem 1, $P(z)+\varepsilon Q(z)$ has all zeros on the unit circle and thus all common zeros of P and Q have the same property. □

Remark 1. It is easy to check that for $m=2$ condition (4) is not just sufficient but also *necessary* for all zeros to lie on the unit circle.

Remark 2. The sufficient condition (4) is best possible in the following sense. If $m\geq 2$,$a_m=a_0>0$, $a_1,\ldots,a_{m-1}\leq 0$ are given real numbers such that

$$
|a_m|<\frac{1}{2}\sum_{k=1}^{m-1}|a_k|
$$

then for the polynomial $p_m(z)=\sum_{k=0}^{m}a_kz^k$ we have $p_m(0)>0$, $p_m(1)<0$ thus p_m does not have all of its zeros on the unit circle.

3. Multiple zeros

From Theorem 1 it follows that multiple zeros are possible only if equality holds in (4).

Lemma 1. *Assume that* (4) *holds with equality. If* $e^{i\varphi_l}$ $(1\leq l\leq m)$ *is a (necessarily double) zero of* P_m *then* $e^{i\varphi_{l\pm 1}}$ *cannot be a (necessarily double) zero (where* $\varphi_{m+1}=\varphi_1$ *should be taken).*

Proof. If $m=2$ then P_m can have at most one double zero so our claim clearly holds. Assuming that $m>2$ and (10) holds for l, $l\pm 1$ we get

$$
\cos\left(\beta_{m-k}+\left(\frac{m}{2}-k\right)\varphi_l\right)=(-1)^{l+1},
$$

$$
\cos\left(\beta_{m-k}+\left(\frac{m}{2}-k\right)\varphi_{l\pm 1}\right)=(-1)^{l}
$$

at least for one $1 \le k \le \left[\frac{m}{2}\right]$. Hence

$$\begin{aligned}(-1)^l &= \cos\left(\beta_{m-k} + \left(\frac{m}{2} - k\right)\varphi_{l\pm 1}\right)\\ &= \cos\left(\beta_{m-k} + \left(\frac{m}{2} - k\right)\varphi_l\right)\cos\frac{2\pi}{m} = (-1)^{l+1}\cos\frac{2\pi}{m}\end{aligned}$$

i.e. $\cos\frac{2\pi}{m} = -1$ which contradicts to $m > 2$, completing the proof. □

Lemma 2. *Assume that* (4) *holds with equality. If* $e^{i\varphi_l}$, $e^{i\varphi_{l\pm 1}}$ $(1 \le l \le m)$ *are not (double) zeros of* P_m *then there is at least one zero* e^{iu_l} *of* P_m *such that* u_l *is strictly between* φ_l *and* $\varphi_{l\pm 1}$.

Proof. Namely in this case $F_m(\varphi_l) \ne 0, F_m(\varphi_{l\pm 1}) \ne 0$ thus

$$F_m(\varphi_l) \ne 0 \quad \text{and} \quad (-1)^l F_m(\varphi_l) = 1 + (-1)^l f_m(\varphi_l) \ge 0$$

imply that $(-1)^l F_m(\varphi_l) > 0$. Arguing similarly we get that $(-1)^{l\pm 1} F_m(\varphi_{l\pm 1}) > 0$ therefore

$$\operatorname{sgn} F_m(\varphi_l) \ne \operatorname{sgn} F_m(\varphi_{l\pm 1}).$$

By the intermediate value theorem there is at least one zero u_l of F_m strictly between φ_l and $\varphi_{l\pm 1}$ and e^{iu_l} is a zero of P_m. □

Let J^* denote *the set of all subscripts* l $(1 \le l \le m)$ for which (10) holds i.e. for which $e^{i\varphi_l}$ $(1 \le l \le m)$ is a (double) zero of P_m. Multiple zeros can arise only if equality holds in (4) and J^* in nonempty. In this case two neighboring simple zeros pull together and become a double zero at $e^{i\varphi_l}$ with $l \in J^*$. To be more precise we have

Theorem 2. *Assume that* $P_m(z) = \sum_{k=0}^{m} A_k z^k \in \mathbb{C}[z]$ *is a self-inversive polynomial of degree* $m \ge 1$ *for which* (4) *holds with equality.*

(j) *If* $J^* = \emptyset$ *then the zeros* e^{iu_l} $(l = 1, \dots, m)$ *of* P_m *are simple and can be arranged such that*

$$\varphi_{l-1} < u_l < \varphi_l \quad (l = 1, \dots, m).$$

(jj) *If* $J^* = \{l_1 < l_2 < \dots < l_p\}$ $(p \ge 1)$ *is nonempty then for* $p \ge 2$ *we have* $l_{j+1} - l_j \ge 2$ $(1 \le j \le p-1)$, *if* $l_1 = 1$ *then* $l_p \ne m$, *if* $l_p = m$ *then* $l_1 \ne 1$.

Each closed arc $\mathrm{Arc}_j = \{\, e^{i\varphi} \mid \varphi \in [\varphi_{l_j-1}, \varphi_{l_j+1}] \,\}$ $(j = 1, \dots, p)$ *of the unit circle contains the double zero* $e^{i\varphi_{l_j}}$ $(j = 1, \dots, p)$ *as its midpoint.*

All those open arcs $\{\, e^{i\varphi} \mid \varphi \in]\varphi_{j-1}, \varphi_j[\,\}$ $(j = 1, \dots, m)$ *which have no common points with any arcs* Arc_j *contain one simple zero* e^{iu_j}.

There are no other zeros of P_m *than those listed above.*

Proof. (j) follows from Lemma 2 taking $e^{i\varphi_{l-1}}$, $e^{i\varphi_l}$ $(1 \le l \le m)$ which are not (double) zeros of P_m.

The first statement of (jj) follows from Lemma 1, the existence of the zeros e^{iu_j} follows from Lemma 2 as $e^{i\varphi_{j-1}}, e^{i\varphi_j}$ $(j = 1, \dots, m)$ cannot be zeros of P_m.

Each open arc contains just one simple zero e^{iu_j}, otherwise the sum of the multiplicities of all zeros would exceed m. Since the sum of the multiplicities of the double and simple zeros described in (jj) is m, no other zeros are possible. $\square$

Remark 3. We can get a new proof of Theorem 1 (ii)-1 in the following way. Assume that (4) holds. Then the proof of (ii)-2, (ii)-3 (of Theorem 1) and (j), (jj) (of Theorem 2) remains valid. Counting the zeros (with multiplicities) on the unit circle we get m, thus all zeros of P_m have to be on the unit circle.

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