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Power classes of recurrence sequences

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To the memory of Professor Béla Brindza

Abstract. A linear recursive sequence G of order k is defined by the integer initial terms $G_0, G_1, \dots, G_{k-1}$, integer constants $A_1, A_2, \dots, A_k$ and by the recursion $G_n = A_1 G_{n-1} + \dots + A_k G_{n-k}$ for $k \le n$. In the case $k = 2$, $G_0 = 0$, $G_1 = 1$ (when we denote the sequence by R) it is known that there are only finitely many perfect powers in such sequences. Ribenboim and McDaniel investigated the so called square-classes. We say that R_m and R_n is the same square-class if $R_m R_n = t^2$ for some integer t. They proved that every square-class is finite. For a general sequence we investigate a similar problem, we show that the equation $G_x^r G_y^{q-r} = w^q$, under some restrictions, has no (x, y, w, q) solutions if q is large enough depending on some parameters.

Let $R = R(A, B, R_0, R_1)$ be a second order linear recursive sequence defined by

$$R_n = AR_{n-1} + BR_{n-2} \qquad (n > 1),$$

where A, B, R_0 and R_1 are fixed rational integers. In the sequel we assume that the sequence is not a degenerate one, i.e. α/β is not a root of unity, where α and β denote the roots of the polynomial $x^2 - Ax - B$.

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The special cases $R(1,1,0,1)$ and $R(2,1,0,1)$ of the sequence R is called Fibonacci and Pell sequence, respectively.

Many results are known about relationship of the sequences R and pure powers. For the Fibonacci sequence Cohn [2] and Wylie [24] showed that a Fibonacci number F_n is a square only when $n = 0, 1, 2$ or 12. Pethő [13] and furthermore London and Finkelstein [10], [11] proved that F_n is full cube only if $n = 0, 1, 2$ or 6. From a result of Ljunggren [9] it follows that a Pell number is a square only if $n = 0, 1$ or 7 and Pethő [13] showed that these are the only perfect powers in the Pell sequence. Similar, but more general results was showed by McDaniel and Ribenboim [12], Robbins [20] [21], Cohn [3]–[5] and Pethő [16]. Shorey and Stewart [22] proved that any non degenerate binary recurrence sequence contains only finitely many pure powers which can be effectively determined. This results follows also from a result of Pethő [15].

Another type of problems was studied by Ribenboim and McDaniel. For a sequence R we say that the terms R_m, R_n are in the same square-class if there exist non zero integers x, y such that

$$R_m x^2 = R_n y^2.$$

or equivalently

$$R_m R_n = t^2,$$

where t is a positive rational integer.

A square-class is called trivial if it contains only one element. Ribenbiom [17] proved that in the Fibonacci sequence the square-class of a Fibonacci number F_m is trivial, if $m \neq 1, 2, 3, 6$ or 12 and for the Lucas sequence $L(1,1,2,1)$ the square-class of a Lucas number L_m is trivial if $m \neq 0, 1, 3$ or 6. For more general sequences $R(A,B,0,1)$, with $(A,B) = 1$, Ribenboim and McDaniel [18] obtained that each square class is finite and its elements can be effectively computable (see also Ribenboim [19]).

Further on we shall study more general recursive sequences.

Let $G = G(A_1, \dots, A_k, G_0, \dots, G_{k-1})$ be a k^{th} order linear recursive sequence of rational integers defined by

$$G_n = A_1 G_{n-1} + A_2 G_{n-2} + \cdots + A_k G_{n-k} \qquad (n > k-1),$$

where $A_1, \dots, A_k$ and $G_0, \dots, G_{k-1}$ are not all zero integers. Denote by $\alpha = \alpha_1, \alpha_2, \dots, \alpha_s$ the distinct zeros of the polynomial $x^k - A_1 x^{k-1} -$

$A_2x^{k-2} - \cdots - A_k$. Assume that $\alpha, \alpha_2, \ldots, \alpha_s$ has multiplicity $1, m_2, \ldots, m_s$ respectively and $|\alpha| > |\alpha_i|$ for $i = 2, \ldots, s$. In this case, as it is known, the terms of the sequence can be written in the form

$$G_n = a\alpha^n + r_2(n)\alpha_2^n + \cdots + r_s(n)\alpha_s^n \qquad (n \geq 0), \tag{1}$$

where r_i's $(i = 2, \ldots, s)$ are polynomials of degree $m_i - 1$ and the coefficients of the polynomials and a are elements of the algebraic number field $\mathbb{Q}(\alpha, \alpha_2, \ldots, \alpha_s)$. Shorey and Stewart [22] prowed that the sequence G does not contain q^{th} powers if q is large enough. This result follows also from [7] and [23], where more general theorems was showed.

Kiss [6] generalize the square-class notion of Ribenboim and McDaniel. Let q and r be fixed natural numbers with the condition $0 < r < q$ and $q \geq 2$. For a sequence G we say that the terms G_m and G_n are in the same (q, r) power-class if there is an integer w such that

$$G_n^r G_m^{q-r} = w^q.$$

It can be easily seen that this relation is an equivalence relation; it is reflexive, symmetric and transitive. In the above mentioned paper Kiss proved that the equation

$$G_n^r G_x^{q-r} = w^q$$

has no solutions x, w, q, r if $x > n$ and $q > q_0(n, G)$. In the followings we shall show a more general result.

Theorem. *Let G be a k^{th} order linear recursive sequence satisfying the above conditions. $\alpha \notin \mathbb{Z}$. Moreover we assume that $\frac{1}{K} < \frac{x}{y} < K$, $(q, r) = 1$ and $\delta q \leq r < q$, where $K > 1$ and $0 < \delta < \frac{1}{2}$ are fixed numbers. Then there exists a number q_0, depending on G, K, δ, such that the equation*

$$G_x^r G_y^{q-r} = w^q \tag{2}$$

in positive integer x, y, w, q, r has no solution with $x \neq y$, $w > 1$ and $q > q_0$.

We use the following results in the proof.

Lemma 1 (A. Baker [1]). *Let $\gamma_1, \ldots, \gamma_v$ be non-zero algebraic numbers. Let $M_1, \ldots, M_v$ be upper bounds for the heights of $\gamma_1, \ldots, \gamma_v$, respectively. We assume that M_v is at least 4. Further let $b_1, \ldots, b_{v-1}$ be* *rational integers with absolute values at most B and let b_v be a non-zero rational integer with absolute value at most B'. We assume that B' is at least three. Let L defined by*

$$L = b_1 \log \gamma_1 + \dots + b_v \log \gamma_v,$$

where the logarithms are assumed to have their principal values. If $L \neq 0$, then

$$|L| > \exp(-C(\log B' \log M_v + B/B')),$$

where C is an effectively computable positive number depending only on the numbers $M_1, \dots, M_{v-1}$, and v (see Theorem 1 of [1] with $\delta = 1/B'$).

Lemma 2 (P. Kiss [8])**.** *Let G be a linear recurrence defined above satisfying the condition $G_n \neq a\alpha^n$ for $n \geq n_0$ If*

$$G_x^r G_y^{q-r} = w^q$$

for positive integers x, y, q and r with the condition $(q, r) = 1$ and $y < n_1$, then $q < q_1$, where q_1 is a constant depending on G, n_0 and n_1, but does not depend on r.

Proof. Proof the Theorem Lemma 2 implies the assertion of the Theorem if x or y is bounded. We can assume, without loss of generality, that the terms G_n are positive and the sequence is increasing. Since r and $q-r$ can be inverted in the Theorem and the symmetry is valid we can assume that $x > y$. Let $c_1, c_2, \dots$ be positive numbers which depend on G, K and δ. Because of (1), G_n can be written in the form:

$$G_n = a\alpha^n \left(1 + \frac{1}{a} r_2(n) \left(\frac{al_2}{\alpha}\right)^n + \frac{1}{a} r_3(n) \left(\frac{\alpha_3}{\alpha}\right)^n + \dots + \frac{1}{a} r_s(n) \left(\frac{\alpha_s}{\alpha}\right)^n\right) = a\alpha^n(1 + \varepsilon_n) \tag{3}$$

where $\lim_{n\to\infty} \varepsilon_n = 0$ since $|\alpha_i| < |\alpha|$ for $2 \leq i \leq s$.

Let x, y, w, q, r be positive integers satisfying (2) with the above conditions. From (2)we get the equation

$$G_x^q \left(\frac{G_y}{G_x}\right)^{q-r} = w^q.$$

By $x > y$ it is obvious that $G_x^q > w^q$ and so $G_x > w$. Using the previous inequality and (3) we have $\log w < c_1 x$.

Similarly it follows that $G_y^q < w^q$ and $\log w > c_2 y > c_3 x$. In this way we obtain that

$$c_3 x < \log w < c_1 x. \tag{4}$$

The equation (2) can be written in the form

$$\left(\frac{G_x}{G_y}\right)^r = \left(\frac{w}{G_y}\right)^q. \tag{5}$$

Since $(q, r) = 1$ we obtain from (5) that

$$\frac{G_x}{G_y} = \left(\frac{v}{z}\right)^q \qquad v, z \in \mathbb{Z}, \tag{6}$$

where

$$\frac{v}{z} = \left(\frac{w}{G_y}\right)^{\frac{1}{r}}. \tag{7}$$

Using equations (3) and (6) we get

$$\frac{\alpha^{x-y} z^q}{v^q} = \frac{1+\varepsilon_y}{1+\varepsilon_x}. \tag{8}$$

Recalling that $|\log(1+x)| \leq x$ and $|\log(1-x)| \leq 2x$ for $0 \leq x < \frac{1}{2}$ and using our assumption that $\frac{x}{y} < K$ we find that

$$\left|\frac{1+\varepsilon_y}{1+\varepsilon_x}\right| < \exp(-c_4 x). \tag{9}$$

if x, y are sufficiently large.

Put

$$L = \left|\log \frac{\alpha^{x-y} z^q}{v^q}\right| = \left|(x-y)\log\alpha - q\log\frac{v}{z}\right|$$

and employ the Lemma with $v = 2$, $B' = q$, $B = x - y$ and $M_2 = w^{\frac{1}{r}}$ since it follows from (7) that $\frac{v}{z} > 1$ and $v = w^{\frac{1}{r}}$.

We suppose that

$$\left(\frac{v}{z}\right)^q = \alpha^{x-y},$$

moreover, we may assume that $\alpha \notin \mathbb{Z}$. Let $\alpha' \neq \alpha$ be any conjugate of α and let φ be an automorphism of $\overline{\mathbb{Q}}$ with $\varphi(\alpha) = \alpha'$. Then $|\alpha'| < |\alpha|$, since α is a dominating root. We have

$$\alpha^{x-y} = \varphi(\alpha^{x-y}).$$

Which is obviously impossible.

Hence $L \neq 0$ since $x \neq y$. Thus, by the lemma

$$L > \exp\left(-c_5\left(\log q \log w^{\frac{1}{r}} + \frac{x-y}{q}\right)\right). \tag{10}$$

Using (4), (9) and (10) we get the inequalities

$$\begin{aligned} c_4 x &< \frac{c_5}{r}\log q \log w + c_5\frac{x-y}{q} < \frac{c_5}{r}\log q\log w + c_6\frac{x}{q} \\ &< \frac{c_5}{r}\log q\log w + c_7\frac{\log w}{q} < \frac{c_8}{r}\log q \log w. \end{aligned} \tag{11}$$

By (4) and (11) we obtain

$$c_9 r \log w < \log q \log w$$

and by $r > \delta q$ we have

$$c_{10} q < \log q,$$

which is impossible if $q > q_0$. □

Remark. In the theorem we suppose that $\alpha \notin \mathbb{Z}$. This condition is necessary. Imre Ruzsa gave the following example. If

$$G_{2n-1} = 2^{2n-1} + 1, \quad G_{2n} = 2^{2n} + 2^n,$$

then the characteristic polynomial is $(x-2)(x^2-2)(x^2-1)$ and $\alpha = 2$. We have

$$\frac{G_{4n-2}}{G_{2n-1}} = 2^{2n-1}$$

that is there are infinitely many q-th power in this case.

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