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Moment functions on polynomial hypergroups in several variables

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Dedicated to the memory of Béla Brindza

Abstract. In a former paper (see [7]) we described the general form of generalized moment functions on polynomial hypergroups in a single variable. Here we extend the results to polynomial hypergroups in several variables.

1. Polynomial hypergroups in several variables

The basic concepts of hypergroups, in particular polynomial hypergroups in several variables can be found in [5] and [2]. The volume [8] contains expository papers on hypergroups, in particular, [4] describes a method for finding moment functions on hypergroups, which is of special interest from the point of view of the present work. The role of polynomial hypergroups in the theory of orthogonal polynomials is studied in [3], [6], [9]. The role of moment functions in the probability theory on hypergroups is presented in [10] and [11]. The single variable case of moment functions on polynomial hypergroups has been considered in [7]. Here we summarize the necessary facts.

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Let K be a countable set equipped with the discrete topology and let d be a positive integer. We consider a set $(Q_x)_{x\in K}$ of polynomials in d complex variables. If for any nonnegative integer n the symbol K_n denotes the set of all elements x in K for which the degree of Q_x is not greater than n, then we suppose that the polynomials Q_x with x in K_n form a basis for all polynomials of degree not greater than n. In this case for every x, y in K the product $Q_x\,Q_y$ admits a unique representation

$$Q_x\,Q_y = \sum_{w\in K} c(x,y,w)Q_w \tag{1}$$

with some complex numbers $c(x,y,w)$. A hypergroup $(K,*)$ is called a *polynomial hypergroup in d variables* or *d-dimensional polynomial hypergroup* if there exists a family of polynomials $(Q_x)_{x\in K}$ in d complex variables satisfying the above condition and such that the convolution in K is defined by

$$\delta_x * \delta_y(\{w\}) = c(x,y,w)$$

for any x, y, w in K. We say that this polynomial hypergroup is *associated with the family of polynomials* $(Q_x)_{x\in K}$.

It is clear that the polynomial hypergroups in one variable, defined in [7] represent a special class of this concept. The above equation (1) is a generalization of the linearization formula in [7]. It is obvious that any sequence $(p_n)_{n\in\mathbb{N}}$ of polynomials in one variable having the property that for any nonnegative integer n the degree of p_n is exactly n satisfies the above condition in (1).

By the conditions on the sequence of polynomials $(Q_x)_{x\in K}$ it follows that there is exactly one element x in K for which Q_x is a nonzero constant. It is easy to see that necessarily $x = e$ is the identity of the hypergroup, and $Q_e = 1$. Sometimes it is convenient to identify the element x in K with the polynomial Q_x. Clearly K contains exactly d nonconstant linear polynomials which are linearly independent.

2. Basic functional equations

Let K be a discrete hypergroup with convolution $*$, involution $^\vee$, and identity e (see [2]). For any y in K let T_y denote the right translation operator on the space of all complex valued functions on K. We call the complex valued function a on K *additive*, if it satisfies

$$T_y a(x) = a(x) + a(y)$$

for all x, y in K. In more details this means that

$$\int_K a(t)\, d(\delta_x * \delta_y)(t) = a(x) + a(y)$$

holds for any x, y in K. The complex valued function m on K is called an *exponential*, if it is not identically zero, and

$$T_y m(x) = m(x)m(y)$$

holds for all x, y in K. In other words m satisfies the functional equation

$$\int_K m(t)\, d(\delta_x * \delta_y)(t) = m(x)m(y)\,.$$

In general, for $T_y f(x)$ we can use the more suggestive abbreviation $f(x*y)$. It is obvious that any linear combination of additive functions is additive again. However, in contrast with the case of groups, the product of exponentials is not necessarily an exponential. Obviously $a(e) = 0$ for any additive function a, and $m(e) = 1$ for any exponential m. The set of all exponential functions on K is called the *generalized dual* of K.

Moments of probability measures on hypergroups can be introduced in terms of moment functions. The notion of moment functions has been formalized in [11] (see also [2]). For any nonnegative integer N the function $\varphi : K \to \mathbb{C}$ is called a *moment function of order* N, if there are complex valued functions $\varphi_k : K \to \mathbb{C}$ for $k = 0, 1, \ldots, N$ such that $\varphi_0 = 1$, $\varphi_N = \varphi$, and the *binomial functional equation*

$$\varphi_k(x * y) = \sum_{j=0}^{k} \binom{k}{j} \varphi_j(x)\varphi_{k-j}(y)$$

holds for $k = 0, 1, \ldots, N$ and for all x, y in K. In this case we say that the functions φ_k $(k = 0, 1, \ldots, N)$ form *a moment sequence of order* N. Hence the study of moment functions on hypergroups leads to the study of systems of functional equations. An extensive study of these types of functional equations can be found in [1]. Systems of functional equations characterizing moment functions and sequences of moment functions are closely related to exponential functions and additive functions. In particular, moment functions of order 1 are exactly the additive functions. In [4] the general form of moment functions of order $N = 1$ and $N = 2$ have been determined in the case of polynomial hypergroups.

We can generalize the concept of moment functions by omitting the hypothesis $\varphi_0 = 1$. In this case φ_0 is an exponential function and we say that φ_0 *generates the generalized moment sequence of order* N and φ_k is a *generalized moment function of order* k *with respect to* φ_0 $(k = 0, 1, \ldots, N)$. For instance, generalized moment functions of order 1 with respect to the exponential φ_0 are solutions of the *sine functional equation*

$$\varphi_1(x * y) = \varphi_0(x)\varphi_1(y) + \varphi_0(y)\varphi_1(x)$$

for any x, y in K.

In [7] the general form of generalized moment functions has been presented in the case of polynomial hypergroups in a single variable. In this work we extend those results to polynomial hypergroups in several variables.

3. Exponential and additive functions on multivariate polynomial hypergroups

For the description of moment functions on polynomial hypergroups we shall need the general form of exponential functions. The following characterization of the exponential functions on K can be found in [2].

Theorem 1. *Let K be a d dimensional polynomial hypergroup generated by the family of polynomials $(Q_x)_{x \in K}$. The function $m : K \to \mathbb{C}$ is an exponential if and only if there exists a λ in $\mathbb{C}^d$ such that*

$$m(x) = Q_x(\lambda) \tag{2}$$

holds for each x in K.

This theorem implies that the generalized dual of the d dimensional polynomial hypergroup can be identified with $\mathbb{C}^d$ (see [2]). Consequently, every polynomial hypergroup admits a normalization in the sense that there exists a λ_0 in $\mathbb{C}^d$ such that $Q_x(\lambda_0) = 1$ holds for any x in K. Indeed, λ_0 is the unique element in $\mathbb{C}^d$ which corresponds to the exponential identically 1. We call λ_0 the *normalizing point* of the hypergroup K. In the case of the polynomial hypergroups of one variable we studied in [7] the normalizing point was 1.

The next theorem describes additive functions on multivariate polynomial hypergroups.

Theorem 2. *Let K be a d dimensional polynomial hypergroup generated by the family of polynomials $(Q_x)_{x\in K}$ with normalizing point λ_0. The function $a : K \to \mathbb{C}$ is an additive function if and only if there exist complex numbers c_j for $j = 1, 2, \ldots, d$ such that*

$$a(x) = \sum_{i=1}^{d} c_i \partial_i Q_x(\lambda_0) \tag{3}$$

holds for each x in K.

Proof. By the linearization formula (1)

$$Q_x(\lambda)Q_y(\lambda) = \sum_{w\in K} c(x, y, w)Q_w(\lambda)$$

holds for any x, y in K and for any λ in $\mathbb{C}^d$. Applying ∂_i on both sides of this equation and then substituting $\lambda = \lambda_0$ we have for $i = 1, 2, \ldots, d$

$$\partial_i Q_x(\lambda_0) + \partial_i Q_y(\lambda_0) = \sum_{w\in K} c(x, y, w)\partial_i Q_w(\lambda_0)\,,$$

which means that the functions $x \mapsto \partial_i Q_x(\lambda_0)$ are additive for $i = 1, 2, \ldots, d$, hence the function a given in (3) is additive for any complex numbers $c_1, c_2, \ldots, c_d$.

For the converse first we observe that the vectors

$$\big(\partial_1 Q_x(\lambda_0), \partial_2 Q_x(\lambda_0), \ldots, \partial_d Q_x(\lambda_0)\big)$$

for x in K_1 and $x \neq e$ are linearly independent, because the polynomials Q_x for x in K_1 form a basis for the linear polynomials in d variables. This implies that the system of linear equations

$$a(x) = \sum_{i=1}^{d} c_i \partial_i Q_x(\lambda_0) \tag{4}$$

for x in K_1 with $x \neq e$ has a unique solution $c_1, c_2, \dots, c_d$. Then (4) obviously holds also for $x = e$. We show by induction on n that (4) holds for any x in K_n and for any n in $\mathbb{N}$. Supposing that this holds for some n let x be in K_{n+1}. We know that Q_x has a representation in the form

$$Q_x(\lambda) = \sum_{j=1}^{s} a_j Q_{x_j}(\lambda) Q_{y_j}(\lambda) \tag{5}$$

for any λ in $\mathbb{C}^d$ with some complex numbers a_j and with some x_j in K_1 and y_j in K_n $(j = 1, 2, \dots, s)$, where s is a positive integer, which means that

$$\delta_x = \sum_{j=1}^{s} a_j \delta_{x_j} * \delta_{y_j}$$

holds. On the other hand, applying ∂_i on (5) and substituting $\lambda = \lambda_0$ we have for $i = 1, 2, \dots, d$

$$\partial_i Q_x(\lambda_0) = \sum_{j=1}^{s} a_j \big(\partial_i Q_{x_j}(\lambda_0) + \partial_i Q_{y_j}(\lambda_0)\big)\,.$$

Finally we obtain

$$\begin{aligned} a(x) &= \int_K a\, d\delta_x = \sum_{j=1}^{s} a_j \int_K a(t)\, d(\delta_{x_j} * \delta_{y_j})(t) \\ &= \sum_{j=1}^{s} a_j \big(a(x_j) + a(y_j)\big) = \sum_{j=1}^{s} a_j \sum_{i=1}^{d} c_i \big(\partial_i Q_{x_j}(\lambda_0) + \partial_i Q_{y_j}(\lambda_0)\big) \\ &= \sum_{i=1}^{d} c_i \sum_{j=1}^{s} a_j \big(\partial_i Q_{x_j}(\lambda_0) + \partial_i Q_{y_j}(\lambda_0)\big) = \sum_{i=1}^{d} c_i \partial_i Q_x(\lambda_0)\,, \end{aligned}$$

and our theorem is proved. □

4. Moment functions on multivariate polynomial hypergroups

In this section we generalize the results in [7] by characterizing moment functions on multivariate polynomial hypergroups. In particular, this is a solution to the problem raised in [4] about the general form of moment functions on polynomial hypergroups.

Theorem 3. *Let K be a d dimensional polynomial hypergroup generated by the family of polynomials $(Q_x)_{x\in K}$. The functions $\varphi_0, \varphi_1, \dots, \varphi_N : K \to \mathbb{C}$ form a generalized moment sequence of order N on K if and only if*

$$\varphi_k(x) = (Q_x \circ f)^{(k)}(0) \tag{6}$$

holds for all X in k and for $k = 0, 1, \dots, N$, where $f = (f_1, f_2, \dots, f_d) : \mathbb{R} \to \mathbb{C}^n$ and $f_i : \mathbb{R} \to \mathbb{C}$ is a polynomial of degree at most N ($i = 1, 2, \dots, d$).

Proof. Let φ_k denote the function defined by (6) for $k = 0, 1, \dots, N$ with some function $f = (f_1, f_2, \dots, f_d) : \mathbb{R} \to \mathbb{C}^n$, where $f_i : \mathbb{R} \to \mathbb{C}$ is any N-times differentiable function ($i = 1, 2, \dots, d$). By the linearization formula we have

$$(Q_x \circ f)(t)(Q_y \circ f)(t) = \sum_{w\in K} c(x, y, w)(Q_w \circ f)(t)$$

for each t in $\mathbb{R}$ and for all x, y in K. Differentiating both sides k times with respect to t and substituting $t = 0$ we have for $k = 0, 1, \dots, N$ and for any x, y in K

$$\begin{aligned} \sum_{j=0}^{k} \binom{k}{j} \varphi_j(x)\varphi_{k-j}(y) &= \sum_{j=0}^{k} \binom{k}{j} (Q_x \circ f)^{(j)}(0)(Q_y \circ f)^{(k-j)}(0) \\ &= \sum_{w\in K} c(x, y, w)(Q_w \circ f)^{(k)}(0) \\ &= \sum_{w\in K} c(x, y, w)\varphi_k(w) = \varphi_k(x * y)\,, \end{aligned}$$

which means that the functions $\varphi_0, \varphi_1, \dots, \varphi_N : K \to \mathbb{C}$ given above form a generalized moment sequence of order N on K for any complex numbers $c_{i,j}$ ($i = 1, 2, \dots, d$; $j = 0, 1, \dots, N$).

To prove the converse statement we suppose now that the functions $\varphi_0, \varphi_1, \dots, \varphi_N : K \to \mathbb{C}$ form a generalized moment sequence of order N on K. As φ_0 is an exponential, we have that $\varphi_0(x) = Q_x(\lambda)$ holds for each x in K with some λ in $\mathbb{C}^d$, where $\lambda = (c_{1,0}, c_{2,0}, \dots, c_{d,0})$. We have seen in the proof of the previous theorem, that the vectors

$$\big(\partial_1 Q_x(\lambda), \partial_2 Q_x(\lambda), \dots, \partial_d Q_x(\lambda)\big)$$

for x in K_1 and $x \neq e$ are linearly independent, consequently, for any fixed $j = 1, 2, \dots, N$ the system of linear equations

$$\varphi_j(x) = \sum_{i=1}^{d} c_{i,j} \partial_i Q_x(\lambda)$$

for x in K_1 with $x \neq e$ has a unique solution $c_{i,j}$ $(i = 1, 2, \dots, d)$. Then we define $f = (f_1, f_2, \dots, f_d)$ by

$$f_i(t) = \sum_{j=0}^{N} \frac{c_{i,j}}{j!} t^j$$

for each t in $\mathbb{R}$ and for $i = 1, 2, \dots, d$. Further let

$$\psi_k(x) = \varphi_k(x) - (Q_x \circ f)^{(k)}(0)$$

for $k = 0, 1, \dots, N$ and for each x in K. We show that the functions $\psi_0, \psi_1, \dots, \psi_N$ vanish identically on K. For $k = 0$ we have obviously $\psi_0(x) = \varphi_0(x) - Q_x(f(0))$ for all n in $\mathbb{N}$. However, as $f(0) = \lambda$, it follows immediately from the choice of λ that $\varphi_0(x) = Q_x(f(0))$, hence $\psi_0(x) = 0$ for each x in K.

From the equation of the moment functions it follows by induction on k that $\varphi_k(e) = 0$ for $k = 1, 2, \dots, N$, consequently, we have that $\psi_k(e) = 0$ for $k = 0, 1, \dots, N$. On the other hand, for any x in K_1 the polynomial Q_x is linear, hence

$$(Q_x \circ f)^{(k)}(0) = \sum_{i=1}^{d} \partial_i Q_x(f(0)) f_i^{(k)}(0) = \sum_{i=1}^{d} \partial_i Q_x(\lambda) c_{i,k} = \varphi_k(x)$$

holds for $k = 1, 2, \dots, N$, whenever $x \neq e$. This means that $\psi_k(x) = 0$ for any x in K_1 and for $k = 0, 1, \dots, N$.

Now we proceed by induction. Suppose that we have proved $\psi_k(x) = 0$ for $k = 0, 1, \ldots, N$ and for any x in K_n and let x be arbitrary in K_{n+1}. By the same argument that we used in the proof of the previous theorems we have that Q_x has a representation in the form (5) for all λ in $\mathbb{C}^d$ with some complex numbers a_j and with some x_j in K_1 and y_j in K_n ($j = 1, 2, \ldots, s$), where s is a positive integer, which means that

$$\delta_x = \sum_{j=1}^{s} a_j \delta_{x_j} * \delta_{y_j}$$

holds. Consequently, we have

$$\varphi_k(x) = \sum_{j=1}^{s} a_j \varphi_k(x_j * y_j)$$

for $k = 1, 2, \ldots, N$. On the other hand, differentiating (5) k times and substituting $t = 0$ we have

$$\big(Q_x \circ f\big)^{(k)}(0) = \sum_{j=1}^{s} a_j \sum_{l=0}^{k} \binom{k}{l} \big(Q_{x_j} \circ f\big)^{(l)}(0) \big(Q_{y_j} \circ f\big)^{(k-l)}(0)$$

$$= \sum_{j=1}^{s} a_j \sum_{l=0}^{k} \binom{k}{l} \varphi_l(x_j) \varphi_{k-l}(y_j) = \sum_{j=1}^{s} a_j \varphi_k(x_j * y_j) = \varphi_k(x)\,,$$

which means that $\psi_k(x) = 0$ for $k = 0, 1, \ldots, N$. This completes the proof. □

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