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Interval-filling sequences with respect to a finite set of real coefficients

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Abstract. A generalization of interval-filling sequences given in [2] served as a powerful device for a representation of real numbers in canonical number systems. In this paper the coefficient set $\{0, 1, \dots, N\}$ is replaced by an arbitrary finite set of real numbers. Though the complete characterization of interval-filling sequences of this type is given only in special cases, some general results are also obtained. Finally, an example shows that generally no necessary and sufficient condition of the "usual" form exists.

Notation. Denote by P a fixed finite set of real numbers. Write $P = \{p_0, p_1, \dots, p_N\}$ where $N \in \mathbb{N}$ (i.e. N is a positive integer) and $p_{i-1} < p_i$ for $i = 1, 2, \dots, N$. Let Λ denote the set of real sequences $\lambda = (\lambda_n)$ for which

(i) $$|\lambda_n| > |\lambda_{n+1}| > 0 \text{ for every } n \in \mathbb{N} \text{ and}$$

(ii) $$\sum_{n=1}^{\infty} |\lambda_n| < \infty.$$

Set

$$\Lambda^+ = \{(\lambda_n) \in \Lambda \mid \lambda_n > 0 \text{ for every } n \in \mathbb{N}\}$$

and for $\lambda \in \Lambda$ define

$$S(P, \lambda) = \left\{ \sum_{n=1}^{\infty} \varepsilon_n \lambda_n \mid \varepsilon_n \in P \text{ for every } n \in \mathbb{N} \right\},$$

$$L_k^+ = \sum_{n=k+1}^{\infty} \lambda_n^+, \quad L_k^- = \sum_{n=k+1}^{\infty} \lambda_n^-, \quad L_k = \sum_{n=k+1}^{\infty} \lambda_n \quad (k = 0, 1, 2, \dots)$$

where $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$, and

$$I_k(P, \lambda) = [p_0 L_k^+ - p_N L_k^-,\ -p_0 L_k^- + p_N L_k^+] \qquad (k = 0, 1, 2, \dots).$$

Obviously $S(P,\lambda) \subseteq I_0(P,\lambda)$ and the endpoints of the interval $I_0(P,\lambda)$ are contained in $S(P,\lambda)$.

Definition. We call the sequence $\lambda \in \Lambda$ interval-filling of type P, if $S(P,\lambda) = I_0(P,\lambda)$.

Let $IF(P)$ denote the set of interval-filling sequence of type P. By definition, $IF(P) \subseteq \Lambda$.

Our first result is a sufficient condition for the interval-filling property of this type.

Theorem 1. *Put $\lambda = (\lambda_n) \in \Lambda$. If*

$$|\lambda_n| \le \frac{p_N - p_0}{d_P} \sum_{k=n+1}^{\infty} |\lambda_k| \tag{1}$$

with $d_P = \max\{p_j - p_{j-1} \mid j = 1, \dots, N\}$ holds for every $n \in \mathbb{N}$, then $\lambda \in IF(P)$.

Proof. First suppose $p_0 = 0$. Then (1) can be written in the form

$$d_P|\lambda_n| \le p_N(L_n^+ + L_n^-) \tag{1a}$$

and $I_k(P,\lambda) = [-p_N L_k^-,\ p_N L_k^+]$ $(k = 0,1,2,\dots)$. According to the remark before the definition we have to prove $I_0(P,\lambda) \subseteq S(P,\lambda)$. Choose $x \in I_0(P,\lambda)$ arbitrarily. Set $s_0(x) = 0$ and

$$s_n(x) = s_{n-1}(x) + \alpha_n(x)\lambda_n \tag{2}$$

for $n \ge 1$, supposed that $s_{n-1}(x)$ is already defined, $A_n(x) \neq \emptyset$ where

$$A_n(x) = \{p \in P \mid x \in s_{n-1}(x) + p\lambda_n + I_n(P,\lambda)\}$$

and $\alpha_n(x) \in A_n(x)$. First we show that such a construction can be continued i.e. $A_n(x)$ is a non-empty subset of P for all $n \in \mathbb{N}$, whenever (1a) is satisfied. Let us assume that $s_{m-1}(x)$ is defined but $A_m(x) = \emptyset$ for some $m \in \mathbb{N}$. Observe that $x \in s_0(x) + I_0(P,\lambda) = I_0(P,\lambda)$ while in case $1 \le n < m$ the assumption $A_n(x) \neq \emptyset$ implies

$$x \in s_{n-1}(x) + \alpha_n(x)\lambda_n + I_n(P,\lambda) = s_n(x) + I_n(P,\lambda).$$

Thus we have proved $x \in s_{m-1}(x) + I_{m-1}(P,\lambda)$. Recalling

$$I_{m-1}(P,\lambda) = [-p_N L_{m-1}^-,\ p_N L_{m-1}^+] = [-p_N(\lambda_m^- + L_m^-),\ p_N(\lambda_m^+ + L_m^+)]$$

we have

$$s_{m-1}(x) - p_N\lambda_m^- - p_N L_m^- \le x \le s_{m-1}(x) + p_N\lambda_m^+ + p_N L_m^+.$$

Since $\{-\lambda_m^-, \lambda_m^+\} = \{0, \lambda_m\}$, the above boundaries for x are in the intervals $s_{m-1}(x) + 0\lambda_m + I_m(P,\lambda)$ and $s_{m-1}(x) + p_N\lambda_m + I_m(P,\lambda)$. The assumption

$A_m(x) = \emptyset$ then means that there exists $j \in \{1, 2, \dots, N\}$ such that x has its value between the intervals $s_{m-1}(x) + p_{j-1}\lambda_m + I_m(P, \lambda)$ and $s_{m-1}(x) + p_j\lambda_m + I_m(P, \lambda)$. Hence

a) in case $\lambda_m > 0$ we have

$$s_{m-1}(x) + p_{j-1}\lambda_m + p_N L_m^+ < x < s_{m-1}(x) + p_j\lambda_m - p_N L_m^-$$

while

b) in case $\lambda_m < 0$ we have

$$s_{m-1}(x) + p_j\lambda_m + p_N L_m^+ < x < s_{m-1}(x) + p_{j-1}\lambda_m - p_N L_m^-.$$

Both imply

$$p_N(L_m^+ + L_m^-) < (p_j - p_{j-1})|\lambda_m| \tag{3}$$

in contradiction with (1a). Thus $A_m(x)$ cannot be empty. So the sequence $(s_n(x))$ can be defined as above and we have derived

$$x \in s_n(x) + I_n(P, \lambda)$$

and consequently

$$|x - s_n(x)| \le p_N \max\{L_n^-, L_n^+\} \le p_N \sum_{k=n+1}^{\infty} |\lambda_k|$$

for all $n \in \mathbb{N}$. Since $\Sigma\lambda_n$ is absolutely convergent, the right side of the above inequality is a null-sequence. Therefore

$$x = \lim_{n\to\infty} s_n(x) = \sum_{n=1}^{\infty} \alpha_n(x)\lambda_n,$$

in other words $x \in S(P, \lambda)$.

When $p_0 \neq 0$, introduce $p_j^0 = p_j - p_0$ $(j = 0, 1, \dots, N)$, $P^0 = \{p_j^0 \mid j = 0, 1, \dots, N\}$ and observe $p_0^0 = 0$, $p_{j-1}^0 < p_j^0$ $(j = 1, \dots, N)$, $d_{P^0} = d_P$, $p_N^0 - p_0^0 = p_N - p_0$, $I_0(P, \lambda) = p_0 L_0 + I_0(P^0, \lambda)$ and $S(P, \lambda) = p_0 L_0 + S(P^0, \lambda)$. Then the sequence λ satisfies the inequalities obtained from (1) by replacing the elements of P with those of P^0 and our above argument proves $I_0(P^0, \lambda) = S(P^0, \lambda)$, which implies $I_0(P, \lambda) = S(P, \lambda)$.

The sufficient condition given in the above theorem consists of an infinite number of inequalities. Now we prove that the first of these is necessary as well.

Theorem 2. *If the sequence $\lambda = (\lambda_n) \in \Lambda$ is interval-filling of type P, then*

$$|\lambda_1| \le \frac{p_N - p_0}{d_P} \sum_{k=2}^{\infty} |\lambda_k|. \tag{4}$$

Proof. At the first step we assume $\lambda_1 > 0$ and $p_0 = 0$. In this case the inequality (4) can be written in the form

$$d_P \lambda_1 \le p_N (L_1^+ + L_1^-). \tag{4a}$$

If (4a) is not true, that is, there exists a $j \in \{1, \dots, N\}$ such that

$$(p_j - p_{j-1})\lambda_1 > p_N (L_1^+ + L_1^-),$$

we can calculate

$$p_j \lambda_1 - \sum_{n=2}^{\infty} p_N \lambda_n^- > p_{j-1}\lambda_1 + \sum_{n=2}^{\infty} p_N \lambda_n^+. \tag{5}$$

Each side in (5) is an element of $S(P,\lambda)$ and $S(P,\lambda) \subseteq I_0(P,\lambda)$. Choose a real number x between these two numbers, and coefficient sequences $(\delta_n), (\varepsilon_n) : \mathbb{N} \to P$ such that $\delta_1 \ge p_j$ and $\varepsilon_1 \le p_{j-1}$. Then $x \in I_0(P,\lambda)$ and

$$\sum_{n=1}^{\infty} \delta_n \lambda_n \ge p_j \lambda_1 - \sum_{n=2}^{\infty} p_N \lambda_n^- > x > p_{j-1}\lambda_1 + \sum_{n=2}^{\infty} p_N \lambda_n^+ \ge \sum_{n=1}^{\infty} \varepsilon_n \lambda_n. \tag{6}$$

If $(\sigma_n) : \mathbb{N} \to P$ is an arbitrary coefficient sequence, either $\sigma_1 \ge p_j$ or $\sigma_1 \le p_{j-1}$ holds, so $x \ne \Sigma_{n=1}^{\infty} \sigma_n \lambda_n$ according to (6), in other words $x \notin S(P,\lambda)$ in contradiction with the hypothesis $\lambda \in IF(P)$. This proves (4a).

To get rid off the assumption $p_0 = 0$ one can use the coefficient set P^0 introduced in the previous proof and follow the converse of the argument presented there.

In case $\lambda_1 < 0$ our considerations are valid for the sequence $-\lambda = (-\lambda_1, -\lambda_2, \dots)$. Since $S(P,-\lambda) = -S(P,\lambda)$ and $I_0(P,-\lambda) = -I_0(P,\lambda)$, $\lambda \in IF(P)$ implies $-\lambda \in IF(P)$ and then (4) holds for the sequence $-\lambda$, hence (4) is true.

In the special case when λ is a geometric sequence, the inequality (4) implies all the inequalities of the form (1), thus it is a necessary and sufficient condition for $\lambda \in IF(P)$. Applying a closed form for the infinite sum it is easy to derive the following

Corollary. *Let $q \in \mathbb{R}$, $0 < |q| < 1$. The sequence (q^n) is interval-filling of type P if and only if*

$$|q| \ge \frac{d_P}{d_P + p_N - p_0}. \tag{7}$$

Our following result is a necessary condition, which consists of infinitely many inequalities like Theorem 1.

Theorem 3. *If the sequence $\lambda = (\lambda_n) \in \Lambda$ is interval-filling of type P, then*

$$|\lambda_n| \leq \frac{p_N - p_0}{g_P} \sum_{k=n+1}^{\infty} |\lambda_k| \tag{8}$$

with $g_P = \min\{p_1 - p_0, p_N - p_{N-1}\}$ holds for every $n \in \mathbb{N}$. If moreover $\lambda \in \Lambda^+ \cap IF(P)$, then

$$\lambda_n \leq \frac{p_N - p_0}{g_P^*} \sum_{k=n+1}^{\infty} \lambda_k \tag{9}$$

with $g_P^* = \max\{p_1 - p_0, p_N - p_{N-1}\}$ also holds for every $n \in \mathbb{N}$.

Proof. Let us suppose that $\lambda \in IF(P)$ but there exists $r \in \mathbb{N}$ such that (8) does not hold for $n = r$. We can also assume $p_0 = 0$ and $\lambda_r > 0$ (as we pointed out for $r = 1$ in the proof of Theorem 2), thus we have

$$g_P \lambda_r > p_N \left(\sum_{k=r+1}^{\infty} \lambda_k^+ + \sum_{k=r+1}^{\infty} \lambda_k^- \right),$$

which, choosing an appropriate $x \in \mathbb{R}$ and applying the definition of q_P, can be written in the form

$$\begin{aligned} p_1 \lambda_r - p_N \sum_{k=1}^{\infty} \lambda_k^- &\geq g_P \lambda_r - p_N \sum_{k=1}^{\infty} \lambda_k^- \\ &> x > p_N \sum_{k=r+1}^{\infty} \lambda_k^+ - p_N \sum_{k=1}^{r} \lambda_k^- . \end{aligned} \tag{10}$$

The first and the last arguments of this sequence of inequalities are elements of the set $S(P, \lambda)$ and, consequently, of the interval $I_0(P, \lambda)$. Hence $x \in I_0(P, \lambda)$, and then there exists $(\delta_n) : \mathbb{N} \to P$ such that $x = \Sigma_{n=1}^{\infty} \delta_n \lambda_n$. Now we can determine δ_m for $m \leq r$:

a) if $m \leq r$ and $\lambda_m > 0$ then $\delta_m = 0$, because $\delta_m \geq p_1$ would imply

$$x = \sum_{n=1}^{\infty} \delta_n \lambda_n \geq p_1 \lambda_m - p_N \sum_{k=1}^{\infty} \lambda_k^- \geq p_1 \lambda_r - p_N \sum_{k=1}^{\infty} \lambda_k^- > x,$$

which is a contradiction;

b) if $m \leq r$ and $\lambda_m < 0$ then $\delta_m = p_N$, because $\delta_m \leq p_{N-1}$ would imply

$$x=\sum_{n=1}^{\infty}\delta_n\lambda_n \geq -p_{N-1}\lambda_m^- - \sum_{n\in N\setminus\{m\}} p_N\lambda_n^- = (p_N-p_{N-1})\lambda_m^-$$
$$- p_N\sum_{n=1}^{\infty}\lambda_n^- \geq g_P|\lambda_m| - p_N\sum_{n=1}^{\infty}\lambda_n^- \geq g_P\lambda_r - p_N\sum_{n=1}^{\infty}\lambda_n^- > x,$$

which is a contradiction again.

Applying the above results and the last inequality of (10) we get

$$x=\sum_{n=1}^{\infty}\delta_n\lambda_n = \sum_{n=1}^{r} p_N(-\lambda_n^-) + \sum_{n=r+1}^{\infty}\delta_n\lambda_n$$
$$\leq -p_N\sum_{n=1}^{r}\lambda_n^- + p_N\sum_{n=r+1}^{\infty}\lambda_n^+ < x,$$

which is impossible. This contradiction proves the first proposition in the theorem.

We can also assume $p_0=0$ throughout the proof of the second proposition in the theorem. In case $g_P^*=p_1$ we wish to derive

$$p_1\lambda_n \leq p_N\sum_{k=n+1}^{\infty}\lambda_k \tag{9a}$$

for all $n\in\mathbb{N}$. Its negative would mean the existence of $r\in\mathbb{N}$ and $x\in\mathbb{R}$ such that

$$p_1\lambda_r > x > \sum_{k=r+1}^{\infty} p_N\lambda_k,$$

consequently $x\in I_0(P,\lambda)\setminus S(P,\lambda)$ in contradiction with the hypothesis $\lambda\in IF(P)$. When $g_P^*=p_N-p_{N-1}$ we can introduce the coefficients $p_i^{\dashv}=p_N-p_{N-i}$ $(i=0,1,\dots,N)$ and the set $P^{\dashv}=\{p_i^{\dashv}\mid i=0,1,\dots,N\}$. Then $p_0^{\dashv}=0$, $p_N^{\dashv}=p_N$. Yet $\lambda\in IF(P^{\dashv})$, because $x\in I_0(P^{\dashv},\lambda)=[0,p_NL_0]$ implies $p_NL_0-x\in[0,p_NL_0]=I_0(P,\lambda)=S(P,\lambda)$, and choosing a coefficient sequence $(\delta_n):\mathbb{N}\to P$ such that $p_NL_0-x=\Sigma_{n=1}^{\infty}\delta_n\lambda_n$, we get

$$x=\sum_{n=1}^{\infty}(p_N-\delta_n)\lambda_n,$$

where $p_N-\delta_n\in P^{\dashv}$, therefore $x\in S(P^{\dashv},\lambda)$. Observe also $p_N^{\dashv}-p_{N-1}^{\dashv}=p_1$, $g_{P^{\dashv}}^*=p_1^{\dashv}=p_N-p_{N-1}=g_P^*$. Now we can apply the above established inequality (9a) for the coefficient set $P^{\dashv}$ to obtain (9) in the case $g_P^* = p_N - p_{N-1}$.

Let us mention here that for non sign-preserving sequences g_P cannot be always replaced by g_P^* in (8). For example consider $P = \{0, 1, 2, 4\}$, $\lambda_1 = \frac{3}{2}$, $\lambda_2 = -1$ and $\lambda_n = \frac{1}{4}\left(\frac{1}{3}\right)^{n-3}$ for $n \geq 3$. Besides, if $m \in \mathbb{N} \cup \{0\}$, we use the notation $T^m\lambda$ for the sequence called the mth tail of the sequence λ, whose nth element is λ_{m+n}. In our example $T^2\lambda$ satisfies the sufficient condition given in Theorem 1, consequently $T^2\lambda \in IF(P)$. A simple calculation shows that for any $x \in I_0(P, \lambda) = [-4, 7.5]$ we can find $\varepsilon_1(x)$, $\varepsilon_2(x) \in P$ such that $x - \varepsilon_1(x)\lambda_1 - \varepsilon_2(x)\lambda_2 \in [0, 1.5] = I_2(P, \lambda) = I_0(P, T^2\lambda) = S(P, T^2\lambda)$, therefore $x \in S(P, \lambda)$, hence $\lambda \in IF(P)$. On the other hand,

$$|\lambda_2| > \frac{p_N - p_0}{g_P^*} \sum_{k=3}^{\infty} |\lambda_k|.$$

Observe, that in case $d_P = g_P$ the combination of Theorem 1 and the first part of Theorem 3 gives the characterization of interval-filling sequences of type P. The assumption $d_P = g_P$ can be written in the form

$$p_N - p_{N-1} = p_1 - p_0 \geq p_i - p_{i-1} \qquad (i = 2, \ldots, N-1). \tag{11}$$

Since the coefficient set $P = \{0, 1, 2, \ldots, N\}$ satisfies (11), Theorem 1 in [2] is a special case of our results. In case $d_P = g_P^*$, i.e. when $d_P \in \{p_1 - p_0, p_N - p_{N-1}\}$, we can characterize the sequences $\lambda \in IF(P) \cap \Lambda^+$ by combaining Theorem 1 with the second part of Theorem 3, obtaining a generalization of Satz. 2.1 in [1].

One would naturally wish to characterize the interval-filling sequences of type P for an arbitrary coefficient set P. But consider our previous example again. As it was presented, $\lambda \in IF(P)$, consequently (8) holds for every $n \in \mathbb{N}$ by Theorem 3. Therefore $T^1\lambda$ also satisfies (8) for every $n \in \mathbb{N}$. On the other hand (1) does not hold for $n = 2$, i.e. $T^1\lambda$ does not satisfy (4), hence $T^1\lambda$ is not interval-filling of type P by Theorem 2. Thus the sufficient condition given in Theorem 1 is not necessary while the necessary condition given in Theorem 3 is not sufficient. Furthermore, let us suppose that there exists a number $h(P)$ depending only on the coefficient set P such that a sequence $\lambda \in \Lambda$ is interval-filling of type P if and only if

$$|\lambda_n| \leq h(P) \sum_{k=n+1}^{\infty} |\lambda_k| \tag{12}$$

holds for every $n \in \mathbb{N}$. Consider any sequence $\lambda \in IF(P)$. Then (12) holds for all $n \in \mathbb{N}$. Therefore $T^1\lambda$ also satisfies (12) for all $n \in \mathbb{N}$, hence $T^1\lambda \in IF(P)$. In other words, $\lambda \in IF(P)$ should imply $T^1\lambda \in IF(P)$.

But in the above example $\lambda \in IF(P)$ and $T^1\lambda \notin IF(P)$. Thus no such characterization is available in general. The situation $\lambda \in IF(P)$ with $T^1\lambda \notin IF(P)$ occurs even if we are restricted to the sequences in Λ^+ (see $P = \{0,1,3,4\}$, $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_n = \frac{1}{4}\left(\frac{1}{3}\right)^{n-3}$ for $n \geq 3$).

Finally let us give a general result:

Theorem 4. *Put $\lambda = (\lambda_n) \in \Lambda$. All the tails $T^m\lambda$ $(m = 0,1,2,\dots)$ of the sequence λ are interval-filling of type P if and only if* (1) *holds for every $n \in \mathbb{N}$.*

Proof. It is an immediate consequence of Theorem 1 and Theorem 2.

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