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On polyslender context-free languages

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Abstract. A. Szilárd, S. Yu, K. Zhang, and J. Shallit showed that for any positive integer k, a regular language is k-polyslender if and only if it is a finite union of $(k+1)$-multiple loop languages. M. Latteux and G. Thierrin and later, independently, D. Raz proved that the family of polyslender context-free languages is bounded. Polyslender context-free languages are also characterized by L. Ilie, G. Rozenberg and A. Salomaa. In this paper, we continue this line of research.

1. Introduction

Combinatorial properties of words and languages play an important role in mathematics and thoretical computer science (see [1], [7], [10], [16], etc.). M. Kunze, H. J. Shyr and G. Thierrin [8], and later, independently, J. Shallit [13]–[15], and more later, also independently, G. Păun and A. Salomaa [11] proved that slender regular and USL-languages coincide. A. Szilárd, S. Yu, K. Zhang, and J. Shallit [17] characterized the k-polyslender regular languages as finite unions of $(k+1)$-multiple loop languages. The next characterization of slender context-free languages was proved by M. Latteux and G. Thierrin [9] and later, independently,

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L. Ilie [5] and D. Raz [12] showed that every slender context-free language is UPL and vice versa. (It was also conjectured by G. Păun and A. Salomaa [11].) The characterization of Parikh slender regular languages and Parikh slender context-free languages is given by J. Honkala [3]. M. Latteux and G. Thierrin [9] and later, independently, D. Raz [12] proved that the class of polyslender context-free languages is a (real) subclass of bounded languages. The first characterization of polyslender context-free languages is given by L. Ilie, G. Rozenberg and A. Salomaa [6]. In this paper we continue this line of research.

2. Preliminaries

A *word* (over Σ) is a finite sequence of elements of some finite nonempty set Σ. We call the set Σ an *alphabet,* the elements of Σ *letters.* If u and v are words over an alphabet Σ, then their *catenation* uv is also a word over Σ. Especially, for every word u over Σ, $u\lambda = \lambda u = u$, where λ denotes the *empty word.* Given a word u, we define $u^0 = \lambda$, $u^n = u^{n-1}u$, $n > 0$, $u^* = \{u^n : n \geq 0\}$ and $u^+ = u^* \setminus \{\lambda\}$.

The *length* $|w|$ of a word w is the number of letters in w, where each letter is counted as many times as it occurs. Thus $|\lambda| = 0$. By the *free monoid* Σ^* *generated by* Σ we mean the set of all words (including the *empty word* λ) having catenation as multiplication. We set $\Sigma^+ = \Sigma^* \setminus \{\lambda\}$, where the subsemigroup Σ^+ of Σ^* is said to be the *free semigroup generated by* Σ. Subsets of Σ^* are referred to as *languages* over Σ. Denote by $|H|$ the *cardinality* of H for every set H. A language L is said to be *length bounded* by a function $f : N \to N$ if we have $|\{w \in L : |w| = n\}| \leq f(n)$. Note that every language $L \subseteq \Sigma^*$ is length bounded by $f(n) = |\Sigma|^n$. A language that is length bounded by a polynomial of degree k is termed *k-polyslender.* Thus, for a positive integer k, a language L is called k-polyslender if the number of words of length n in L is of order $O(n^k)$. *Slender languages* coincide with 0-polyslender languages. A language is called *polyslender* iff it is k-polyslender for some k. A language of the form $L \subseteq w_1^* \dots w_k^*$ is called *k-bounded.* In addition, a language is *bounded* iff it is k-bounded for a positive integer k. A language $L \subseteq \Sigma^*$ is said to be *k-slender* if $|\{w \in L : |w| = n\}| \leq k$, for every $n \geq 0$. A language is *slender* if it is k-slender for some positive integer k. A 1-slender language is called a *thin* language. A language L is said to be a *union of single loops* (or, in short, USL) if for some positive integer k and words u_i, v_i, w_i, $1 \le i \le k$,

$$L = \bigcup_{i=1}^{k} u_i v_i^* w_i. \qquad (*)$$

L is called a *union of paired loops* (or UPL, in short) if for some positive k and words u_i, v_i, w_i, x_i, y_i, $1 \le i \le k$,

$$L = \bigcup_{i=1}^{k} \{u_i v_i^n w_i x_i^n y_i \mid n \ge 0\}. \qquad (**)$$

For a USL (or UPL) language L the smallest k such that $(*)$ (or $(**)$) holds is referred to as the USL-index (or UPL-index) of L. A USL language L is said to be a *disjoint union of single loops* (DUSL, in short) if the sets in the union $(*)$ are pairwise disjoint. In this case the smallest k such that $(*)$ holds and the k sets are pairwise disjoint is referred to as the DUSL-index of L. The notions of a *disjoint union of paired loops* (DUPL) and DUPL-index are defined analogously considering $(**)$.

For slender regular languages, we have the following characterization, first proved by M. Kunze, H. J. Shyr and G. Thierrin [8], and later, independently, by J. Shallit [13]–[15], and more later, also independently, by G. Păun and A. Salomaa [11] ([14] and [15] are an extended abstract form and a revised form, respectively, of [13]).

Theorem 2.1. *The next conditions,* (i)–(iii), *are equivalent for a language L.*

(i) *L is regular and slender.*

(ii) *L is USL.*

(iii) *L is DUSL.*

Moreover, if L is regular and slender, then the USL- and DUSL-indices of L are effectively computable.

The following result is given by G. Păun and A. Salomaa [11].

Theorem 2.2. *Every UPL language is DUPL, slender, linear and unambiguous.*

The next characterization of slender context-free languages was proved by M. Latteux and G. Thierrin [9] and later, independently, by L. Ilie [5] and D. Raz [12]. It was also conjectured by G. Păun and A. Salomaa [11].

Theorem 2.3. *Every slender context-free language is UPL.*

The next characterization of polyslender languages by bounded languages is given by D. Raz [12].

Theorem 2.4. *Every* $(k+1)$*-bounded language is* k*-polyslender.*

The next statement was first proved by M. Latteux and G. Thierrin [9] and later, independently, by D. Raz [12].

Theorem 2.5. *Every polyslender context-free language is bounded.*

We shall use the following simple observation.

Proposition 2.6. *Let* L *be the union of the languages* $L_1, \ldots, L_k$ $(k \geq 1)$. *Then* L *is polyslender if and only if all of* $L_1, \ldots, L_k$ *are polyslender. In particular,* L *is slender if and only if all of* $L_1, \ldots, L_k$ *are slender.*

Following S. Ginsburg [2], for any pair of words $x, y \in \Sigma^*$ and $Z \subseteq \Sigma^*$ we put $(x, y) \star Z = \{x^n Z y^n : n \geq 0\}$. The next result is from S. Ginsburg [2].

Theorem 2.7. *The family of bounded context-free languages is the smallest family of languages containing all finite languages and closed with respect to the following operations: finite union, finite product,* $(x, y) \star Z$*, where* x *and* y *are words.*

Now we consider the following recursive definition. We say that a language $L \subseteq \Sigma^*$ is a *non-crossing* 1*-multiple paired loop language* iff it is of the form $L = \{uv^n wx^n y : n \geq 0\}$ for some words $u, v, w, x, y \in \Sigma^*$.[1]

Inductively, for every pair k, ℓ of positive integers, L is a *non-crossing* $(k+\ell)$*-multiple paired loop language* iff one of the following conditions holds:

[1] Observe that $vx = \lambda$ is possible. Thus every singleton language is a non-crossing 1-multiple paired loop language.

(i) $L = \{uv^nL'x^ny : n \geq 0\}$ for some non-crossing $(k+\ell-1)$-multiple paired loop language L' and words $u, v, x, y \in \Sigma^*$;

(ii) $L = L_1L_2$, where L_1 is a non-crossing k-multiple paired loop language and L_2 is a non-crossing ℓ-multiple paired loop language.

We also say that a language is a non-crossing multiple paired loop language if it is a non-crossing k-multiple paired loop language for some nonnegative integer k. The non-crossing 1-multiple paired loop languages are simply called *paired loop languages,* or in short, paired loops as before.[2]

Proposition 2.8. *The family of non-crossing multiple paired loop languages is the smallest family of languages containing all paired loop languages and closed with respect to the following operations: finite product, $(x, y) \star Z$, where x and y are words.*

$L \subseteq \Sigma^*$ with $k \geq 1$ is called a *k-multiple loop language* iff there exist $u_1, v_1, \ldots, u_k, v_k, u_{k+1} \in \Sigma^*$ such that, $L = u_1v_1^* \ldots u_kv_k^*u_{k+1}$. 1-multiple loop languages are simply called *single loop languages,* or in short, single loops as previously. The next result is given by A. Szilárd, S. Yu, K. Zhang, J. Shallit [17].

Theorem 2.9. *Given a nonnegative integer k, a regular language is k-polyslender if and only if it is a finite disjoint union of $(k+1)$-multiple loop languages.*

The following statement is obvious.

Proposition 2.10. *Every finite union of k-multiple loop languages can be given as a finite disjoint union of k-multiple loop languages.*

Of course, Theorem 2.1 is a consequence of Theorem 2.9 and Proposition 2.10.

Given a positive integer k, let D_k be the Dyck language of order k, i.e., the context-free language over the alphabet $\Delta_k = \{[_i\, ,]_i : 1 \leq i \leq k\}$ generated by the grammar $G = (\{S\}, \Delta_k, S, \{S \to \lambda\} \cup \{S \to [_iS]_iS : 1 \leq i \leq k\})$. Consider a word $z \in D_k$ with $z = z_1 \cdots z_{2k}$, $z_1, \ldots, z_{2k} \in \Delta_k$ which has exactly one occurrence of each bracket $[_i$ and $]_i$ (for $i = 1, 2, \ldots k$).

[2] It is clear that every non-crossing k-multiple loop language with $k \geq 0$ is a non-crossing $(k+1)$-multiple loop language.

Let Σ be an alphabet with $\Sigma \cap \Delta_k = \emptyset$. A language D over Σ is called a *k-Dyck loop* if

$$D = \{h_{n_1,\cdots,n_k}(w_0 z_1 w_1 z_2 w_2 \cdots z_{2k} w_{2k}) : n_i \geq 0,\ 1 \leq i \leq k\},$$

where $h_{n_1,\cdots,n_k}$: $(\Sigma \cup \Delta_k)^\star \to \Sigma^\star$ is the homomorphism defined by $x \mapsto x$ $(x \in \Sigma)$, $[_i \mapsto u_i^{n_i}$, and $]_i \mapsto v_i^{n_i}$ $(1 \leq i \leq k)$ for some fixed words $u_i, v_i, w_i \in \Sigma^\star$ and numbers n_i $(1 \leq i \leq k)$. A k-Dyck loop D is *degenerate* if, for each i $(1 \leq i \leq k)$, at most one of the words u_i and v_i is nonempty.

Observation 2.11. *A language is a k-Dyck loop if and only if it is a non-crossing k-multiple paired loop language. Moreover, a language is a degenerate k-Dyck loop if and only if it is a k-multiple loop language.*

The following characterization is given by L. Ilie, G. Rozenberg and A. Salomaa [6].

Theorem 2.12. *For any $k \geq 0$, a context-free language L is k-polyslender if and only if L equals a finite union of $(k+1)$-Dyck loops.*

As an immediate consequence of Theorem 2.12, the authors of the above result obtain the following statement which is equivalent to Theorem 2.9.

Theorem 2.13. *For any $k \geq 0$, a regular language L is k-polyslender if and only if L equals a finite union of degenerate $(k+1)$-Dyck loops.*

We will use the next three results of S. Ginsburg [2].

Theorem 2.14. *Each context-free language over one letter is a regular language.*

A language L is called *commutative* if $uv = vu$ for every pair $u, v \in L$.

Lemma 2.15. *A language $L \subseteq \Sigma^*$ is commutative if and only if there exists a word w such that $L \subseteq w^*$.*

Theorem 2.16. *For each context-free grammar $G = (V, \Sigma, P, S)$ and variable $X \in V$ let $L_X(G) = \{u \in \Sigma^* : X \stackrel{*}{\Rightarrow} uXv$ for some $v \in \Sigma^*\}$, and $R_X(G) = \{v \in \Sigma^* : X \stackrel{*}{\Rightarrow} uXv$ for some $u \in \Sigma^*\}$. Let G be reduced. A necessary and sufficient condition that $L(G) \neq \emptyset$ is bounded is that $L_X(G)$ and $R_X(G)$ both are commutative for each variable $X \in V$.*

The following result by N. J FINE and H. S. WILF [1], [4] will also be applied. (For a version of this statement see also H. J. SHYR [16].)

Theorem 2.17. *Let u and v be nonempty words, and, $p, q \geq 0$ integers. If u^p and v^q contain a common prefix or suffix of length $|u| + |v| - \gcd(|u|, |v|)$ (where $\gcd(|u|, |v|)$ denotes the greatest common divisor of $|u|$ and $|v|$) then $u = w^m$ and $v = w^n$, for some word w and positive integers m, n.*

The following observation shows that there exists no analogous statement of Proposition 2.10 for the finite union of non-crossing k-multiple paired loop languages.

Observation 2.18. *It is clear that $\{a^n b^n c^m : m,\ n \geq 0\} \cup \{a^m b^n c^n : m,\ n \geq 0\}$ is a finite union of non-crossing 2-multiple paired loop languages which cannot be given as a finite disjoint union of non-crossing k-multiple paired loop languages for some k.*

3. Results on bounded languages

The following statement is a consequence of the proof of Theorem 2.14 given in [2]. It also follows from the fact that the length set of a regular language is ultimately periodic.

Lemma 3.1. *Given a singleton alphabet $\{a\}$, every regular language $L \subseteq a^*$ can be represented as a disjoint finite union of languages having the form $a^m(a^n)^*, m, n \geq 0$.*

PROOF. Using Theorem 2.1, we show only that every regular language $L \subseteq a^*$ can be represented as a finite union of languages having the form $a^m(a^n)^*, m, n \geq 0$. For this statement, it is enough to prove that L is a finite union of languages having the form $a^m(a^n)^*, m, n \geq 0$ whenever $L \in \{a^{m_1}(a^{n_1})^* \cup a^{m_2}(a^{n_2})^*, a^{m_1}(a^{n_1})^* a^{m_2}(a^{n_2})^*,\ (a^m(a^n)^*)^*\}$ for appropriate non-negative integers m_1, n_1, m_2, n_2, m, n. Of course, the case $L = a^{m_1}(a^{n_1})^* \cup a^{m_2}(a^{n_2})^*$ is trivial. Using the identity

$$a^{m_1}(a^{n_1})^* a^{m_2}(a^{n_2})^* = a^{m_1+m_2} \left(\bigcup_{i=1}^{n_2-1} a^{in_1} \bigcup_{j=1}^{n_1-1} a^{jn_2} \bigcup (a^{n_1+n_2})^* \right),$$

we also have our statement for $L \in \{a^{m_1}(a^{n_1})^* a^{m_2}(a^{n_2})^*\}$. Finally, our proposition trivially holds for $L = (a^m(a^n)^*)^*$. □

The next statement is a direct consequence of Theorem 2.9.

Theorem 3.2. *Consider words $x_1, y_1, \dots, x_k, y_k, x_{k+1} \in \Sigma^*$, a regular language $L \subseteq x_1 y_1^* \dots x_k y_k^* x_{k+1}$. Then L can be represented as a finite disjoint union of languages having the form $x_1 y_1^{m_1}(y_1^{n_1})^* \dots x_k y_k^{m_k}(y_k^{n_k})^* x_{k+1}$.*

By Theorem 2.5, every polyslender context-free language is bounded. Therefore, the polyslender regular languages are also bounded. On the other side, it is clear that every language L with $L \subseteq x_1 y_1^* \dots x_{k+1} y_{k+1}^* x_{k+2}$, $x_1,\ y_1, \dots, x_{k+1},\ y_{k+1},\ x_{k+2} \in \Sigma^*$ is k-polyslender in consequence of Theorem 2.4. Thus we can also get Theorem 2.9 by using Theorem 3.2, Theorem 2.5, and Theorem 2.4.

Given a context-free grammar $G = (V, \Sigma, S, H)$, put $L(W) = \{w \in \Sigma^* : W \stackrel{*}{\Rightarrow} w\}$ for every sentence form $W \in \{V \cup \Sigma\}^*$. We shall use the following two lemmas.

Lemma 3.3. *Given a reduced context-free grammar $G = (V, \Sigma, P, S)$, let $L(G)$ be bounded. For every variables $A, B \in V$ there exist words $w, z \in \Sigma^*$ such that for every sentential forms $W, Z, W_i, Z_i, \in (V \cup \Sigma)^*$, $i = 1, 2, 3, 4$ we have the following statements.*

(i) $A \stackrel{*}{\Rightarrow} WAZ$ *implies* $L(W) \subseteq w^*$, $L(Z) \subseteq z^*$

(ii) $A \stackrel{*}{\Rightarrow} W_1 A Z_1$, $A \stackrel{*}{\Rightarrow} W_2 B Z_2$, $B \stackrel{*}{\Rightarrow} W_3 B Z_3$, $B \stackrel{*}{\Rightarrow} W_4 A Z_4$ *imply* $L(W_1^*), L((W_2 W_3^* W_4)^*) \subseteq w^*$, $L(Z_1^*), L((Z_4 Z_3^* Z_2)^*) \subseteq z^*$.

Proof. Theorem 2.16 and Lemma 2.15 imply directly (i). If $W_1^* = \{\lambda\}$ or $(W_2 W_3^* W_4)^* = \{\lambda\}$ then we have (i) and (ii) immediately. Otherwise, we can get (i) and (ii) by an inductive application of Theorem 2.17. □

Lemma 3.4. *Given a reduced context-free grammar $G = (V, \Sigma, P, S)$, let $L(G)$ be bounded. For every variable $A \in V$, $L(A)$ is a finite union of languages of the form $\{(w^i)^*(w^j)^n w' L(A') z' (z^k)^n (z^\ell)^* : w',\ z' \in \Sigma^*,\ n \geq 0,\ A' \in V,\ A' \Rightarrow W B' Z$ implies $B' \neq A\}$.*

Proof. Consider an arbitrary variable $A \in V$. By Lemma 3.3, $L(A)$ is a finite union of languages of the form $\{L(W)^n w' L(B) z' (L(Z))^n : w',\ z' \in \Sigma^*,$

$n \geq 0\}$ such that $B \in V$, $L(W) \subseteq w^*$, $L(Z) \subseteq z^*$ for some $w, z \in \Sigma^*$, and simultaneously, $B \stackrel{*}{\Rightarrow} W''AZ''$ implies $w'L(W'') \subseteq w^*, L(Z'')z' \subseteq z^*$. Therefore, we obtain that $L(A)$ is a finite union of languages $\{L(W)^n w' L(A') z' (L(Z))^n \colon w',\ z' \in \Sigma^*,\ n \geq 0\}$ such that $A' \in V$, $L(W) \subseteq w^*$, $L(Z) \subseteq z^*$ for some $w, z \in \Sigma^*$, and moreover, $A' \Rightarrow WB'Z$ implies $B' \neq A$. On the other hand, $L(W) \subseteq w^*$, $L(Z) \subseteq z^*$ are context-free languages. Thus, by Theorem 2.14, they are regular languages. But then, using Lemma 3.1, they are a (disjoint) finite union of languages of the form $(w^i)^* w^j$, $z^k (z^\ell)^*$. □

Now we are ready to prove our main result.

Theorem 3.5. *Every finite union* $\bigcup_{i=1}^m L_i$ *of non-crossing* k*-multiple paired loop languages* $L_1, \ldots, L_m$ *is a* $(2k+1)m$*-bounded context-free language. Conversely, every* k*-bounded context-free language can be represented as a finite union of non-crossing* k*-multiple paired loop languages.*

Proof. It is clear that a finite union $\bigcup_{i=1}^m L_i$ of non-crossing k-multiple paired loop languages $L_1, \ldots, L_m$ is $(2k+1)m$-bounded. On the other hand, it is easy to prove that a non-crossing multiple paired loop language can be generated by a context-free grammar. Therefore, a finite union of non-crossing multiple paired loop languages is a bounded context-free language. Conversely, consider a k-bounded context-free language L and a reduced context-free grammar $G = (V, \Sigma, P, S)$ with $L = L(G)$. By an inductive application of Lemma 3.4 we conclude that $L(S) (= L(G))$ is a finite union of non-crossing multiple paired loop languages. On the other hand, given a k-bounded language L, every sublanguage of L is k-bounded by definition. Therefore, $L(G)$ is a finite union of non-crossing k-multiple paired loop languages. (Of course, it is possible that L can be given as a finite union of non-crossing ℓ-multiple paired loop languages such that $\ell < k$.) □

Of course, we can consider the multiple loop languages as special types of non-crossing multiple loop languages. The following simple observation shows that the above result cannot be strenghtened in general.

Observation 3.6. *It is clear that every* k*-multiple loop language can be considered as a* $(2k+1)$*-bounded language. Moreover, a* k*-multiple loop language can be found which is not* $2k$*-bounded. (For example,*

$(ab^*)^k a, a, b \in \Sigma$ is such a language.) On the other hand, for every m there are k-multiple loop languages $L_1, \ldots, L_m$ such that $\bigcup_{i=1}^m L_i$ is $(k+1)m$-bounded but it is not $((k+1)m-1)$-bounded language. (For example, let $L_i = (a(b^i c)^*)^k d,\ a, b, c, d \in \Sigma,\ i = 1, \ldots, m$. Then, using $L = \bigcup_{i=1}^m L_i \subseteq (a^*(bc)^*)^k d^*(a^*(b^2c)^*)^k d^* \ldots (a^*(bc^m)^*)^k d^*$, L is $(2k+1)m$-bounded but it is not $((k+1)m-1)$-bounded.)

4. Polyslender context-free languages

First we show the following

Proposition 4.1. *Every non-crossing $(k+1)$-multiple paired loop language is k-polyslender.*

Proof. It is clear that $w_1 L_1 \ldots w_m L_m w_{m+1}, w_1, \ldots, w_{m+1} \in \Sigma^*, L_1, \ldots, L_m$ are k-polyslender if and only if $L_1 \ldots L_m$ is k-poly-slender. Therefore, using Theorem 2.4, it follows that all $(k+1)$-multiple loop languages are k-polyslender. On the other hand, by an easy computation we obtain that $\{L_1 u^n L_2 v^n L_3 : n \geq 0,\ u, v \in \Sigma^*,\ uv \neq \lambda,\ L_1, L_2, L_3 \subseteq \Sigma^*\}$ is k-polyslender if and only if $L_1 u^* L_2 L_3$ and $L_1 L_2 v^* L_3$ are k-polyslender. □

The following statement is obvious.

Proposition 4.2. *Given a pair of positive integers k, ℓ with $k < \ell$, let L be an ℓ-multiple loop language (a non-crossing ℓ-multiple paired loop language). If L is a finite union of k-multiple loop languages (finite union of non-crossing k-multiple paired loop languages) then L is a k-multiple loop language (a non-crossing k-multiple paired loop language).*

Proposition 4.3. *Given a pair k, ℓ of positive integers with $k \leq \ell$, every k-poly-slender non-crossing ℓ-multiple loop language is a non-crossing $(k+1)$-multiple loop language.*

Proof. First we observe that for every positive integer i, $\{L_1 u^n L_2 v^n L_3 \mid n \geq 0\}$, $u, v \in \Sigma^*$, $L_1, L_2, L_3 \subseteq \Sigma^*$ is i-polyslender if and only if $L_1(uv)^* L_2 L_3$ is i-polyslender. Therefore, to prove our statement, we can restrict to k-polyslender ℓ-multiple loop languages. But in consequence of Theorem 2.9, a k-polyslender multiple loop language should be a finite union of $(k+1)$-multiple loop languages. □

By our Observation 2.11, the next statement is essentially the same as Theorem 2.12. Thus, by the proof of the next statement, we reach another proof of Theorem 2.12 given by .

By Observation 2.11, Theorem 2.12 (given by L. ILIE, G. ROZENBERG and A. SALOMAA [6]) can be written in the following form.

Theorem 4.4. *A context-free language is k-polyslender if and only if it is a finite union of non-crossing $(k+1)$-multiple paired loop languages.*

By our results, now we give a new proof of the above statement.

PROOF. Non-crossing $(k+1)$-multiple paired languages are obviously context-free. In addition, Lemma 4.1 shows that non-crossing $(k+1)$-multiple paired loop languages are k-polyslender, and then their finite unions also have this property. Conversely, let L be a k-polyslender context-free language. Then applying Theorem 2.5, L is ℓ-bounded for some ℓ. In consequence of Theorem 3.5 we have that L is a finite union of non-crossing ℓ-multiple paired loop languages. On the other hand, L is k-polyslender. Thus every sub-language of L inherits this property. By Proposition 4.3 the proof is complete. □

5. Polyslender languages and trajectories

Firstly, we define the shuffle of words on trajectories.

Let $V = \{r, u\}$ be the set of *versors* in the plane: r stands for the *right* direction, whereas, u stands for the *up* direction. A *trajectory* is said to be an element t, $t \in V^*$.

Let Σ be an alphabet and let t be a trajectory, let d be a versor, $d \in V$, let α, β be two (finite) words over Σ. The *shuffle* of α with β on the trajectory dt, denoted $\alpha \sqcup\!\sqcup_{dt} \beta$, is recursively defined as follows:

if $\alpha = ax$ and $\beta = by$, where $a, b \in \Sigma$ and $x, y \in \Sigma^*$, then:

$$ax \sqcup\!\sqcup_{dt} by = \begin{cases} a(x \sqcup\!\sqcup_t by), & \text{if } d = r, \\ b(ax \sqcup\!\sqcup_t y), & \text{if } d = u. \end{cases}$$

if $\alpha = ax$ and $\beta = \lambda$, where $a \in \Sigma$ and $x \in \Sigma^*$, then:

$$ax \sqcup\!\sqcup_{dt} \lambda = \begin{cases} a(x \sqcup\!\sqcup_t \lambda), & \text{if } d = r, \\ \emptyset, & \text{if } d = u. \end{cases}$$

if $\alpha = \lambda$ and $\beta = by$, where $b \in \Sigma$ and $y \in \Sigma^*$, then:

$$\lambda \sqcup\!\sqcup_{dt} by = \begin{cases} \emptyset, & \text{if } d = r, \\ b(\lambda \sqcup\!\sqcup_t y), & \text{if } d = u. \end{cases}$$

Finally,

$$\lambda \sqcup\!\sqcup_t \lambda = \begin{cases} \lambda, & \text{if } t = \lambda, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Comment. Note that if $|\alpha| \neq |t|_r$ or $|\beta| \neq |t|_u$, then $\alpha \sqcup\!\sqcup_t \beta = \emptyset$.

The above operation is extended in the natural way to languages and sets of trajectories.

Remark 5.1. Here we show that all customary operations for the parallel composition of words are particular cases of the operation of shuffle on trajectories.

1. Let T be the set $T = \{r, u\}^*$. Observe that $\sqcup\!\sqcup_T = \sqcup\!\sqcup$, the shuffle operation. In order to prove this let Σ be an alphabet and consider two words x and y, $x, y \in \Sigma^*$ Assume that $w = x_1y_1x_2y_2 \ldots x_ny_n$ is an element of $x \sqcup\!\sqcup y$, where some x_i, y_j may be the empty word.

   Note that in T there is the trajectory $t = r^{i_1}u^{j_1} \ldots r^{i_n}u^{j_n}$ where $|x_{i_p}| = i_p$ and $|y_{j_q}| = j_q$, $1 \leq p.q \leq n$.

   Moreover, note that $x \sqcup\!\sqcup_t y = w$.

   The converse is trivial.

2. Assume that $T = r^*u^*$. It follows that $\sqcup\!\sqcup_T = \cdot$, the catenation operation.

   Let Σ be an alphabet and consider two words x and y, $x, y \in \Sigma^*$. Assume that $|x| = p$ and $|y| = q$. Note that in T there is only one trajectory $t = r^pu^q$ and moreover $x \sqcup\!\sqcup_t y = xy$. For the converse, consider x and y, $x, y \in \Sigma^*$. and assume that $|x| = p$ and $|y| = q$. Note that xy is equal with $x \sqcup\!\sqcup_t y$, where $t = r^pu^q$.

   All the next equalities can be proved by similar methods.

3. Define $T = r^*u^*r^*$ and note that $\sqcup\!\sqcup_T = \longleftarrow$, the insertion operation.
4. Let T be the set $T = \{r^i u^{2j} r^i \mid i,\ j \geq 0\}$. In this case $\sqcup\!\sqcup_T$ is the balanced insertion operation, $\sqcup\!\sqcup_T = \longleftarrow_{bal}$.
5. Consider $T = (ru)^*$ and observe that $\sqcup\!\sqcup_T = \sqcup\!\sqcup_{blit}$, the balanced literal shuffle.
6. Assume that $T = (ru)^*(r^* \cup u^*)$. Note that in this case $\sqcup\!\sqcup_T = \sqcup\!\sqcup_{lit}$, the literal shuffle.
7. Let T be the set $T = r^*u^* \cup u^*r^*$. In this case $\sqcup\!\sqcup_T = \odot$, i.e., it is the bi-catenation operation.
8. Consider $T = u^*r^*$ and observe that $\sqcup\!\sqcup_T = \bullet$, the anti-catenation operation.

We are now in position to state the following:

Theorem 5.2. *If L_1, L_2 are polyslender languages and if T is a set of trajectories such that T is polyslender, then $L_1 \sqcup\!\sqcup_T L_2$ is a polyslender language.*

Proof. Assume that L_i are polyslender with the polynomyals P_i, $i = 1, 2$. Also assume that T is polyslender for the polynomial P_T.

Note that for a trajectory $t \in T$, such that $|t|_r = k_1$ and $|t|_u = k_2$, then $L_1 \sqcup\!\sqcup_t L_2$ contains $P_1(k_1)P_2(k_2)$ words of length $k_1 + k_2$. Hence, $L_1 \sqcup\!\sqcup_T L_2$ contains $P_1(k_1)P_2(k_2)P_T(k_1 + k_2)$ words of length $k_1 + k_2$.

Therefore $L_1 \sqcup\!\sqcup_T L_2$ is a polyslender Language with the polynomial $P_1P_2P_T$. □

From the above theorem we conclude the following:

Corollary 5.3. *The family of polyslender languages is closed under: catenation, bicatenation, literal shuffle, balanced insertion, anti-catenation and insertion.*

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