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On the stability of Mikusiński's equation

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Abstract. We combine two research directions: conditional Cauchy equations (with the condition dependent on the unknown function) with the stability question. Our main results concern the stability of Mikusiński's equation

$$f(x+y) \neq 0 \implies f(x+y) = f(x) + f(y).$$

1. Introduction

Our considerations may be treated as a combination of two, continually present in the literature, research directions. One of them is the stability of functional equations, which following D. H. Hyers (cf. [5]) and S. M. Ulam (cf. [9]), has been widely investigated (cf. e.g. [6]). The second direction concerns the question of conditional Cauchy equations – the idea based on the assumption that the Cauchy equation

$$f(x+y) = f(x) + f(y) \tag{1}$$

is valid for these arguments x, y that satisfy some additional condition (cf. e.g. [1], [3], [7]). Taking into account the form of the condition we divide conditional Cauchy equations into two complementary classes:

Class 1. Conditional Cauchy equations with the condition independent of the unknown function f.

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Class 2. Conditional Cauchy equations where the condition does depend on f.

Even though the idea to combine the above two research directions (the Hyers–Ulam stability of conditional Cauchy equations) is present in the literature, it concerns only the case where the condition does not depend on the function (cf. e.g. [4], [6], [8]). Stability of the conditional Cauchy equations of *Class 2* has not yet been investigated.

This paper is devoted to the stability of Mikusiński's equation – the fundamental member of *Class 2.*

Throught the paper $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{C}$ denote the sets of all positive integers, real numbers, nonnegative real numbers and complex numbers respectively.

2. Main results

Certain geometrical considerations have led J. Mikusiński to the functional equation

$$f(x+y)(f(x+y)-f(x)-f(y))=0, \tag{2}$$

with a continuous function f mapping real line into itself. Equation (2) is usually written in the conditional form

$$f(x+y)\neq 0 \implies f(x+y)=f(x)+f(y), \tag{3}$$

which enables us to consider its generalization to structures endowed with one operation. For the reader's convenience we quote the theorem of L. Dubikajtis, C. Ferens, R. Ger and M. Kuczma, which describes the general solution of Mikusiński's equation:

Theorem DFGK (cf. [2], Theorem 1)**.** *Let $(G,+)$, $(H,+)$ be groups. If group G has no normal subgroup of index 2, then the conditional equation* (3)*, for functions $f : G \to H$, is equivalent to the Cauchy equation* (1)*. If G has normal subgroups of index 2, then the family of all solutions $f : G \to H$ of* (3) *consists of all solutions of* (1) *and all functions f of the form*

$$f(x)=\begin{cases}0, & \text{for } x\in K,\\ c, & \text{for } x\in G\setminus K,\end{cases}$$

where $c \neq 0$ is an arbitrary element of H, and K is a normal subgroup of G of index 2.

In this section we will prove the stability of Mikusiński's equation (3). As a corollary we obtain the stability of equation (2).

Theorem 1. *Let $(G,+)$ be an abelian group and let $(X, \|\cdot\|)$ be a Banach space. If for some $\delta, \varepsilon \geq 0$ a function $f : G \to X$ satisfies*

$$\|f(x+y)\| > \delta \Longrightarrow \|f(x+y) - f(x) - f(y)\| \leq \varepsilon \quad \text{for } x, y \in G \tag{4}$$

then there exists a function $a : G \to X$ fulfilling Mikusiński's equation (3) such that

$$\|f(x) - a(x)\| \leq 2\varepsilon + 3\delta \quad \text{for } x \in G. \tag{5}$$

Moreover, if $\mathrm{Ker}_\delta f := \{x \in G \mid \|f(x)\| \leq \delta\}$ is not a subgroup of G of index 2, then the function a is additive. In the opposite case, the function a is a nonadditive solution of Mikusiński's equation.

Let δ be an arbitrary nonnegative real number. For a fixed $x \in G$, we consider two complementary cases:

(i) $\|f(2^n x)\| > \delta$ for $n \in \mathbb{N}$;

(ii) there exists $k \in \mathbb{N}$ such that $\|f(2^k x)\| \leq \delta$.

In the proof of Theorem 1, the following lemma will be used:

Lemma 1. *Let $(G,+)$ be a group and let $(X, \|\cdot\|)$ be a normed space. If a function $f : G \to X$ satisfies condition (4) with given $\delta, \varepsilon \geq 0$ and $\mathrm{Ker}_\delta f$ is not a subgroup of G of index 2 then, for every $x \in G$ satisfying (ii), we have*

$$\|f(x)\| \leq 2\varepsilon + 3\delta. \tag{6}$$

Let $x \in G$ satisfying (ii) be arbitrarily chosen and let k be the smallest positive integer with $\|f(2^k x)\| \leq \delta$. At first let us observe that the inequality

$$\|f(2^{k-1}x)\| \leq 2\varepsilon + 3\delta \tag{7}$$

implies (6). Indeed, for $k = 1$ conditions (6) and (7) coincide and if $k \geq 2$ then one can easily show that

$$\left\| \frac{f(2^{k-1}x)}{2^{k-1}} - f(x) \right\| \leq \left(1 - \frac{1}{2^{k-1}}\right) \varepsilon,$$

which, along with (7), yields (6). Thus it is enough to prove (7). For convenience, let us denote $v := 2^{k-1}x$. Substituting x by $2v$ and y by $-v$ in (4) we have

$$\|f(v) - f(-v)\| \le \varepsilon + \delta, \tag{8}$$

on account of $\|f(2v)\| \le \delta$. Since $\mathrm{Ker}_\delta f$ is not a subgroup of G of index 2, exactly one of the following cases holds:

Case 1. $\mathrm{Ker}_\delta f$ is not a subgroup of G. Let us observe that $\mathrm{Ker}_\delta f \neq \emptyset$, as $2^k x \in \mathrm{Ker}_\delta f$. Hence, either there exists $y \in \mathrm{Ker}_\delta f$ such that $-y \in G \setminus \mathrm{Ker}_\delta f$ or there exist $y, z \in \mathrm{Ker}_\delta f$ with $y + z \in G \setminus \mathrm{Ker}_\delta f$.

If there is $y \in \mathrm{Ker}_\delta f$ such that $-y \in G \setminus \mathrm{Ker}_\delta f$ then, replacing x by $v - y$ in (4), we obtain

$$\|f(v) - f(v-y) - f(y)\| \le \varepsilon,$$

which implies (7), provided that $v - y \in \mathrm{Ker}_\delta f$. If this is not the case then, substituting $-y$ for x and v for y in (4), we get

$$\|f(-y+v) - f(-y) - f(v)\| \le \varepsilon. \tag{9}$$

Replacing x with $-y+v$ and y with $-v$ in (4), we have

$$\|f(-y) - f(-y+v) - f(-v)\| \le \varepsilon. \tag{10}$$

Adding inequalities (9) and (10), side by side, and making use of (8), we obtain (7).

If there exist $y, z \in \mathrm{Ker}_\delta f$ such that $y+z \in G \setminus \mathrm{Ker}_\delta f$ then, replacing x with y and y with z in (4), we have

$$\|f(y+z)\| \le \varepsilon + 2\delta. \tag{11}$$

Having applied (4) once again, with $y + z - v$ and v instead of x and y respectively, we obtain

$$\|f(y+z) - f(y+z-v) - f(v)\| \le \varepsilon. \tag{12}$$

If $y + z - v \in \mathrm{Ker}_\delta f$ then (7) results from (12) along with (11). In the opposite case, if $y + z - v \in G \setminus \mathrm{Ker}_\delta f$, then replacing x with $y + z$ and y with $-v$ in (4), we have

$$\|f(y+z-v) - f(y+z) - f(-v)\| \le \varepsilon.$$

Adding the inequality above and (12), side by side, and using (8) we finish the proof of (7) in *Case 1.*

Case 2. $\mathrm{Ker}_\delta f$ is a subgroup of G of index different from 2. Thus there are $y, z \in G \setminus \mathrm{Ker}_\delta f$ with $y - z \in G \setminus \mathrm{Ker}_\delta f$. Observe that either $y - v \in G \setminus \mathrm{Ker}_\delta f$ or $z - v \in G \setminus \mathrm{Ker}_\delta f$, since $\mathrm{Ker}_\delta f$ is a group. Since the case $y - v \in G \setminus \mathrm{Ker}_\delta f$ is analogous, we assume that $z - v \in G \setminus \mathrm{Ker}_\delta f$. Let us use (4) with z and $-v$ in place of x and y respectively, to obtain

$$\|f(z-v) - f(z) - f(-v)\| \le \varepsilon. \tag{13}$$

On the other hand, substituting $z - v$ in place of x and v in place of y in (4), we have

$$\|f(z) - f(z-v) - f(v)\| \le \varepsilon.$$

Now (7) results easily from the above two inequalities and from (8). $\square$

The proof of Theorem 1. – Part I. In this part, we assume that $\mathrm{Ker}_\delta f$ is not a subgroup of G of index 2.

Step 1. We will show the following inequality

$$\left\| \frac{f(2^n x)}{2^n} - f(x) \right\| \le \left(1 - \frac{1}{2^n}\right)(4\varepsilon + 7\delta) \quad \text{for } x \in G,\ n \in \mathbb{N}. \tag{14}$$

If x satisfies condition (i) then it is easy to obtain

$$\left\| \frac{f(2^n x)}{2^n} - f(x) \right\| \le \left(1 - \frac{1}{2^n}\right)\varepsilon \quad \text{for } n \in \mathbb{N}. \tag{15}$$

Thus let $x \in G$ satisfying condition (ii) be arbitrarily chosen. From (4), with y replaced with x, we have $\|f(2x) - 2f(x)\| \le \varepsilon$, provided that $\|f(2x)\| > \delta$. In the other case, if $\|f(2x)\| \le \delta$, then using Lemma 1, we get $\|f(2x) - 2f(x)\| \le 4\varepsilon + 7\delta$. In both cases

$$\|f(2x) - 2f(x)\| \le 4\varepsilon + 7\delta.$$

Using the inequality above one can easily derive (14) by induction.

By (14) one can show that $\left(\frac{f(2^n x)}{2^n}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence for an arbitrary $x \in G$. Thus, the map $a : G \to X$ given by

$$a(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \quad \text{for } x \in G$$

is well defined.

Step 2. We will prove the additivity of a. Given $x, y \in G$ with $a(x+y) \neq 0$, we observe that for a sufficiently large $n \in \mathbb{N}$ we have $\|f(2^n(x+y))\| > \delta$. Thus, using the definition of a and (4), we obtain $a(x+y) = a(x) + a(y)$. Consequently, function a satisfies Mikusiński's equation (3). Since, additionally, $a(2x) = 2a(x)$ for all $x \in G$, a is additive on account of Theorem DFGK.

Step 3. We will prove inequality (5). Letting $n \to \infty$ in (14) we obtain

$$\|f(x) - a(x)\| \leq 4\varepsilon + 7\delta \quad \text{for } x \in G. \tag{16}$$

If $x \in G$ satisfies condition (i) then, similarly, using (15), we have

$$\|f(x) - a(x)\| \leq \varepsilon,$$

which yields (5). Thus, let us take $x \in G$ satisfying condition (ii). Since a is additive, then, by (16), we have

$$a(x) = \lim_{n\to\infty} \frac{f(nx)}{n}.$$

If $a(x) = 0$ then (5) results from Lemma 1. In the opposite case, if $a(x) \neq 0$, then for a sufficiently large $n \in \mathbb{N}$ and all $p \in \{1, \dots, 2^{n-1}\}$, we have

$$\|f((2^n - p + 1)x)\| > \delta.$$

Consequently, replacing x sequentially by $(2^n - 1)x, (2^n - 2)x, \dots, 2^{n-1}x$ and y by x in (4) we obtain

$$\begin{gathered}
\|f(2^n x) - f((2^n - 1)x) - f(x)\| \leq \varepsilon, \\
\|f((2^n - 1)x) - f((2^n - 2)x) - f(x)\| \leq \varepsilon, \\
\vdots \\
\|f((2^{n-1} + 1)x) - f(2^{n-1}x) - f(x)\| \leq \varepsilon,
\end{gathered}$$

respectively. Adding the above inequalities up, side by side, we have

$$\|f(2^n x) - f(2^{n-1}x) - 2^{n-1} f(x)\| \leq 2^{n-1}\varepsilon. \tag{17}$$

Moreover, for a sufficiently large $n \in \mathbb{N}$ it is $\|f(2^n x)\| > \delta$, hence applying (4) with x and y replaced by $2^{n-1}x$ we get

$$\|f(2^n x) - 2f(2^{n-1}x)\| \leq \varepsilon.$$

Using the inequality above and (17), we obtain

$$\left\| \frac{f(2^{n-1}x)}{2^{n-1}} - f(x) \right\| \leq \left(1 + \frac{1}{2^{n-1}}\right)\varepsilon,$$

and then, letting $n \to \infty$, we have

$$\|f(x) - a(x)\| \leq \varepsilon,$$

which implies (5).

Part II. Suppose that $\mathrm{Ker}_\delta f$ is a subgroup of G of index 2. Then, for arbitrary $y, z \in G \setminus \mathrm{Ker}_\delta f$, we have $y - z \in \mathrm{Ker}_\delta f$. Replacing x with $y - z$ and y with z in (4) we obtain

$$\|f(y) - f(z)\| \leq \varepsilon + \delta \quad \text{for } y, z \in G \setminus \mathrm{Ker}_\delta f. \tag{18}$$

Let us fix $x_0 \in G \setminus \mathrm{Ker}_\delta f$ and define the function $a : G \to X$ by

$$a(x) := \begin{cases} f(x_0) & \text{for } x \in G \setminus \mathrm{Ker}_\delta f, \\ 0 & \text{for } x \in \mathrm{Ker}_\delta f. \end{cases}$$

It is easy to check that the function a satisfies Mikusiński's equation (3) and approximates f on the whole space with a constant of approximation equal to $\varepsilon + \delta$. □

Remark 1. The stability we have just studied, with the (δ, ε)-perturbation in a near-solution (4) of (3), is more restrictive than the simple adaptation of the stability notion used for *Class 1* (cf. e.g. [4]). Putting $\delta = 0$ in Theorem 1 we obtain the stability result which is conformable to this original notion.

As a corollary we have the stability of Mikusiński's equation (2).

Theorem 2. *Let $(G, +)$ be an abelian group. If for some $\varepsilon \geq 0$ a function $f : G \to \mathbb{C}$ satisfies*

$$|f(x + y)(f(x + y) - f(x) - f(y))| \leq \varepsilon \quad \text{for } x, y \in G, \tag{19}$$

then there exists a function $a : G \to \mathbb{C}$ satisfying Mikusiński's equaiton such that

$$|f(x) - a(x)| \leq 2\sqrt{6\varepsilon} \quad \text{for } x \in G.$$

It is enough to use Theorem 1 with δ and ε replaced by $\sqrt{\frac{2}{3}\varepsilon}$ and $\sqrt{\frac{3}{2}\varepsilon}$, respectively. □

Remark 2. The proof of Theorem 2, although obvious, shows an advantage of the (δ, ε)-approach to the stability question we have introduced, namely, we have the following conclusion: if an arbitrary conditional Cauchy equation is stable (in this sense), then its multiplicative version is also stable.

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