Publ. Math. Debrecen
66/1-2 (2005), 25–39

The Lebesgue function for Lagrange interpolation on the augmented Chebyshev nodes

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Abstract. Given $f \in C[-1,1]$ and n points (nodes) in $[-1,1]$, the well-known Lagrange interpolation polynomial is the polynomial of minimum degree which agrees with f at each of the nodes. Properties of the Lebesgue function and Lebesgue constant associated with Lagrange interpolation on the Chebyshev nodes (the zeros of the nth Chebyshev polynomial of the first kind) have been studied by several authors. In this paper a study is made of Lagrange interpolation on the Chebyshev nodes augmented with -1 and 1. It is shown that, although the convergence properties of interpolation polynomials based on the Chebyshev and augmented Chebyshev nodes are similar, there are considerable differences in the behaviour of the Lebesgue function. In particular, the local maxima of the Lebesgue function for the augmented nodes are strictly increasing from the outside towards the middle of $[-1,1]$, whereas they are decreasing for the unaugmented nodes, and the Lebesgue constant for the augmented nodes is essentially double that for the unaugmented nodes.

1. Introduction

Suppose f is a continuous real-valued function defined on the interval $[-1,1]$, and let

$$X = \{x_{k,n} : k = 1,2,\dots,n;\ n = 1,2,3,\dots\}$$

Mathematics Subject Classification: 41A05, 41A10.
Key words and phrases: Lagrange interpolation, Lebesgue function, Lebesgue constant, augmented Chebyshev nodes.

be an infinite triangular matrix such that, for all n,

$$1 \geq x_{1,n} > x_{2,n} > \ldots > x_{n,n} \geq -1.$$

The well-known Lagrange interpolation polynomial of f is the polynomial $L_n(X,f)(x) = L_n(X,f,x)$ of degree at most $n-1$ which satisfies

$$L_n(X,f,x_{k,n}) = f(x_{k,n}), \qquad 1 \leq k \leq n.$$

It can be expressed as

$$L_n(X,f,x) = \sum_{k=1}^{n} f(x_{k,n})\ell_{k,n}(X,x),$$

where the fundamental polynomials $\ell_{k,n}(X,x)$ are the unique polynomials of degree at most $n-1$ which satisfy

$$\ell_{k,n}(X,x_{j,n}) = \delta_{k,j}, \qquad 1 \leq k,j \leq n,$$

where $\delta_{k,j}$ denotes the Kronecker delta.

Central to the study of the convergence properties of Lagrange interpolation polynomials is the behaviour of the Lebesgue functions

$$\lambda_n(X,x) = \sum_{k=1}^{n} |\ell_{k,n}(X,x)| \tag{1}$$

and the Lebesgue constants

$$\Lambda_n(X) = \max_{-1\leq x\leq 1} \lambda_n(X,x)$$

(see Rivlin [6]). For example ([6, Section 4.1]), if the modulus of continuity $\omega(\delta;f)$ of f is defined by

$$\omega(\delta;f) = \max\left\{|f(s)-f(t)| : -1 \leq s,\ t \leq 1,\ |s-t| \leq \delta\right\},$$

the polynomials $L_n(X,f,x)$ converge uniformly to f as $n \to \infty$ if

$$\lim_{n\to\infty} \Lambda_n(X)\,\omega(1/n;f) = 0. \tag{2}$$

As shown by Luttmann and Rivlin [5], for $n \geq 3$ the Lebesgue function $\lambda_n(X,x)$ is a piecewise polynomial that satisfies $\lambda_n(X,x) \geq 1$, with equality if and only if $x = x_{k,n}$ $(k = 1, 2, \ldots, n)$. Between each pair of consecutive nodes, $\lambda_n(X, x)$ is a polynomial with a single maximum, while in $(x_{1,n}, \infty)$ it is a strictly increasing, convex polynomial, and in $(-\infty, x_{n,n})$ it is a strictly decreasing, convex polynomial. Further, if the nodes are symmetrically arranged about 0 (i.e. $x_{k,n} = -x_{n-k+1,n}$ for $1 \le k \le n$), then $\lambda_n(X, x) = \lambda_n(X, -x)$, $x \in [-1, 1]$.

Properties of $\lambda_n(X, x)$ have been studied extensively for some particular choices of X. One such choice is the matrix of Chebyshev nodes

$$T = \left\{ x_{k,n} = \cos\left(\frac{2k-1}{2n}\pi\right) : k = 1, 2, \ldots, n;\ n = 1, 2, 3, \ldots \right\},$$

where, for fixed n, the $x_{k,n}$ are the zeros of the nth Chebyshev polynomial of the first kind, $T_n(x) = \cos(n \arccos x)$, $-1 \le x \le 1$. A reason for the focus on the Chebyshev nodes is that whereas $\Lambda_n(X) > (2/\pi)\log n + 1/2$ for any X, it is known that $\Lambda_n(T) \le (2/\pi)\log n + 1$ (see Rivlin [7, Section 1.3]). Thus the Chebyshev nodes are a simple node system that is near-optimal in terms of both its Lebesgue constants and, by (2), the convergence properties of its interpolation polynomials.

For the Chebyshev nodes, Ehlich and Zeller [4] showed that

$$\Lambda_n(T) = \lambda_n(T, \pm 1) = \frac{1}{n}\sum_{k=1}^{n} \cot\frac{(2k-1)\pi}{4n}, \tag{3}$$

from which representation the asymptotic result

$$\Lambda_n(T) = \frac{2}{\pi}\log n + \frac{2}{\pi}\left(\gamma + \log\frac{8}{\pi}\right) + O\left(\frac{1}{n^2}\right), \quad \text{as } n \to \infty, \tag{4}$$

can be obtained, where γ denotes Euler's constant. (In fact, much more precise results than (4) are known — see the survey paper by Brutman [3] and the references therein.) Brutman [2] made a careful study of the behaviour of $\lambda_n(T, x)$ on $[-1, 1]$, and showed its local maxima are strictly decreasing from the outside towards the middle of the interval, and that the smallest local maximum, $\underline{\Lambda}_n(T)$, satisfies

$$\underline{\Lambda}_n(T) = \frac{2}{\pi}\log n + \frac{2}{\pi}\left(\gamma + \log\frac{4}{\pi}\right) + o(1), \quad \text{as } n \to \infty. \tag{5}$$

This asymptotic result was obtained from the observations that if n is even, then

$$\underline{\Lambda}_n(T) = \lambda_n(T, 0) = \Lambda_{n/2}(T),$$

while if n is odd, then

$$\underline{\Lambda}_n(T) \sim \lambda_n(T, \sin \pi/(2n)), \quad \text{as } n \to \infty.$$

Our aim in this paper is to study Lagrange interpolation on the Chebyshev nodes augmented with the endpoints -1 and 1. Denote the augmented nodes by

$$T_a = \{x_{k,n+2} : k = 0, 1, \ldots, n+1;\ n = 1, 2, 3, \ldots\},$$

where

$$\begin{cases} x_{0,n+2} = 1, \quad x_{n+1,n+2} = -1, \\ x_{k,n+2} = \cos\left(\dfrac{2k-1}{2n}\pi\right), \quad 1 \le k \le n. \end{cases} \tag{6}$$

Now, polynomial interpolation on T_a has been studied by several authors, but almost exclusively in the context of Hermite–Fejér interpolation, where the polynomial is required to not only interpolate the function at each node, but also to have zero derivative at each node. In a series of papers in the 1960s, D. L. Berman showed that adding ± 1 to the Chebyshev nodes may completely change the convergence behaviour of Hermite–Fejér polynomials. For instance, in [1, Theorem 2], Berman showed that the sequence of Hermite–Fejér interpolation polynomials to $f(x) = x^2$, based on the nodes T_a, diverges as $n \to \infty$ for all $x \in (-1, 1)$, even though the corresponding sequence of polynomials based on the nodes T converges uniformly to f on $[-1, 1]$. The behaviour of the Lebesgue function for Hermite–Fejér interpolation on T_a was discussed by Smith [9], who showed, for example, that the addition of ± 1 to the node system T causes the Lebesgue constant to increase dramatically, from 1 to $2n^2 + O(1)$.

In contrast to Hermite–Fejér interpolation, the Lagrange interpolation process on T_a has received only limited study. This is perhaps due to the fact that it fails to exhibit the extreme behaviour of the Hermite–Fejér process. Indeed, from

$$L_{n+2}(T_a, f, x) - L_n(T, f, x) = \frac{T_n(x)}{2}\big[(1+x)(f(1) - L_n(T, f, 1)) + (-1)^n(1-x)(f(-1) - L_n(T, f, -1))\big],$$

which is verified readily by noting that both sides are polynomials of degree at most $n+1$ that agree at the $n+2$ nodes $x_{k,n+2}$ $(0 \le k \le n+1)$, it follows that if $L_n(T, f, x)$ converges uniformly to f on $[-1, 1]$, then $L_{n+2}(T_a, f, x)$ also converges uniformly to f.

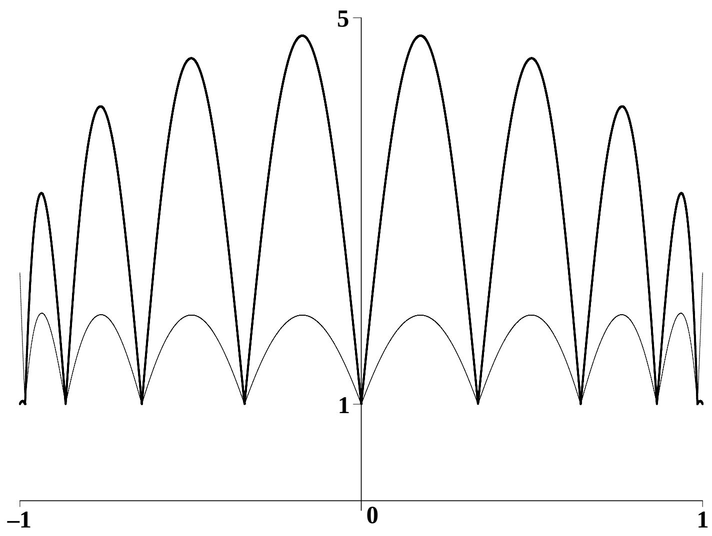

Figure 1. Graphs of the Lebesgue functions $\lambda_9(T, x)$ (thin line) and $\lambda_{11}(T_a, x)$

We will investigate properties of the Lebesgue function $\lambda_{n+2}(T_a, x)$. Now, Figure 1 suggests that in contrast to the behaviour of $\lambda_n(T, x)$, the local maxima of $\lambda_{n+2}(T_a, x)$ *increase* from the outside towards the middle of $[-1, 1]$. Further, the graphs suggest that the magnitude of $\Lambda_{n+2}(T_a)$ is considerably greater than that of $\Lambda_n(T)$. These observations are confirmed by the following two theorems, which will be proved in Sections 2 and 3.

Theorem 1. *Let $\Lambda^j_{n+2}(T_a)$ denote the local maximum value of the Lebesgue function $\lambda_{n+2}(T_a,x)$ in the interval (x_{j+1},x_j), and let $[n/2]$ denote the integer part of $n/2$. Then*

$$\Lambda^j_{n+2}(T_a) < \Lambda^{j+1}_{n+2}(T_a), \qquad j = 0,1,\ldots,[n/2]-1.$$

Theorem 2. *The Lebesgue constant $\Lambda_{n+2}(T_a)$ has the asymptotic expansion*

$$\Lambda_{n+2}(T_a) = \frac{4}{\pi}\log n + \frac{4}{\pi}\left(\gamma + \log\frac{4}{\pi}\right) + 1 + O\left(\frac{1}{n^2}\right), \quad \text{as } n \to \infty. \tag{7}$$

2. Proof of Theorem 1

For simplicity, write x_k for $x_{k,n+2}$, where the $x_{k,n+2}$ are given by (6). By using basic properties of the Chebyshev polynomial $T_n(x)$ (see Rivlin [7, Chapter 1]), it is easily verified that the fundamental polynomials for Lagrange interpolation on the nodes T_a are given by

$$\begin{cases} \ell_{0,n+2}(T_a,x) = \dfrac{1+x}{2}\,T_n(x), \quad \ell_{n+1,n+2}(T_a,x) = (-1)^n\dfrac{1-x}{2}\,T_n(x), \\ \ell_{k,n+2}(T_a,x) = \dfrac{1-x^2}{1-x_k^2}\,\ell_{k,n}(T,x) = (-1)^{k-1}\dfrac{(1-x^2)}{n\sqrt{1-x_k^2}}\,\dfrac{T_n(x)}{x-x_k}, \ 1 \le k \le n. \end{cases}$$

Therefore, if $x_{j+1} < x < x_j$, it follows from (1) and $(-1)^jT_n(x) > 0$ that the Lebesgue function is

$$\begin{aligned} \lambda_{n+2}(T_a,x) = (-1)^jT_n(x)\Bigg[1 &- \sum_{k=1}^{j}\frac{(1-x^2)}{n\sqrt{1-x_k^2}}\,\frac{1}{x-x_k} \\ &+ \sum_{k=j+1}^{n}\frac{(1-x^2)}{n\sqrt{1-x_k^2}}\,\frac{1}{x-x_k}\Bigg]. \end{aligned} \tag{8}$$

To simplify notation, we write $\lambda_{n+2}(T_a,x)$ as $\lambda_{n+2}(x)$, and define

$$\theta_0 = 0, \quad \theta_{n+1} = \pi, \quad \theta_k = \frac{2k-1}{2n}\pi \quad (1 \le k \le n).$$

Because of the symmetry of the nodes about 0, the theorem follows immediately from the following lemma.

Lemma 3. *Suppose* $x = \cos\theta$ *where* $\theta_j < \theta < \theta_{j+1}$ *and* $0 \le j \le [n/2] - 1$. *Let* $x_* = \cos\theta_*$, *where* $\theta_* = \theta + \pi/n$. *Then*

$$\lambda_{n+2}(x_*) - \lambda_{n+2}(x) > 0.$$

PROOF. Since $T_n(x_*) = -T_n(x)$, it follows from (8) that

$$n|T_n(x)|^{-1}\left[\lambda_{n+2}(x_*) - \lambda_{n+2}(x)\right] = \sum_{k=j+2}^{n} \frac{(1-x_*^2)}{\sqrt{1-x_k^2}} \frac{1}{x_* - x_k}$$
$$- \sum_{k=j+1}^{n} \frac{(1-x^2)}{\sqrt{1-x_k^2}} \frac{1}{x - x_k} + \sum_{k=1}^{j+1} \frac{(1-x_*^2)}{\sqrt{1-x_k^2}} \frac{1}{x_k - x_*} \tag{9}$$
$$- \sum_{k=1}^{j} \frac{(1-x^2)}{\sqrt{1-x_k^2}} \frac{1}{x_k - x}.$$

Consider the case when $n = 2m$ is even. Since $x_k = -x_{n-k+1}$, (9) can be written as

$$\frac{n}{2}|T_n(x)|^{-1}\left[\lambda_{n+2}(x_*) - \lambda_{n+2}(x)\right] = \sum_{k=j+2}^{m} \frac{x_*}{\sqrt{1-x_k^2}} \frac{1-x_*^2}{x_*^2 - x_k^2}$$
$$+ \sum_{k=1}^{j+1} \frac{x_k}{\sqrt{1-x_k^2}} \frac{1-x_*^2}{x_k^2 - x_*^2} - \sum_{k=j+1}^{m} \frac{x}{\sqrt{1-x_k^2}} \frac{1-x^2}{x^2 - x_k^2}$$
$$- \sum_{k=1}^{j} \frac{x_k}{\sqrt{1-x_k^2}} \frac{1-x^2}{x_k^2 - x^2}.$$

On using the representation

$$\frac{1-a^2}{a^2 - x_k^2} = -1 + \frac{1-x_k^2}{a^2 - x_k^2} \tag{10}$$

in each summand, we obtain

$$f(x) := n|T_n(x)|^{-1}\left[\lambda_{n+2}(x_*) - \lambda_{n+2}(x)\right]$$

$$
\begin{aligned}
&- 2x \csc\theta_{j+1} - 2(x - x_*) \sum_{k=j+2}^{m} \csc\theta_k \\
&= 2\cot\theta_{j+1} + \sum_{k=1}^{j+1} \frac{\sin 2\theta_k}{x_k^2 - x_*^2} - \sum_{k=1}^{j} \frac{\sin 2\theta_k}{x_k^2 - x^2} \\
&+ 2x_* \sum_{k=j+2}^{m} \frac{\sin\theta_k}{x_*^2 - x_k^2} - 2x \sum_{k=j+1}^{m} \frac{\sin\theta_k}{x^2 - x_k^2}.
\end{aligned}
\tag{11}
$$

Since $x > x_*$, the lemma will be established for even n if it can be shown that $f(x) > 0$.

Now, on using the trigonometric identity

$$
\frac{\sin 2A}{\cos^2 A - \cos^2 B} = \cot(B - A) - \cot(B + A),
\tag{12}
$$

and noting $\theta_* - \theta_{k+1} = \theta - \theta_k$ for $1 \le k \le n-1$, we obtain

$$
\begin{aligned}
\sum_{k=2}^{j+1} \frac{\sin 2\theta_k}{x_k^2 - x_*^2} - \sum_{k=1}^{j} \frac{\sin 2\theta_k}{x_k^2 - x^2} &= \sum_{k=1}^{j} \left(\frac{\sin 2\theta_{k+1}}{\cos^2\theta_{k+1} - \cos^2\theta_*} - \frac{\sin 2\theta_k}{\cos^2\theta_k - \cos^2\theta} \right) \\
&= \sum_{k=1}^{j} \left(\cot(\theta + \theta_k) - \cot(\theta_* + \theta_{k+1})\right) \\
&= \cot(\theta + \theta_1) + \cot(\theta + \theta_2) \\
&\quad - \cot(\theta + \theta_{j+1}) - \cot(\theta + \theta_{j+2}).
\end{aligned}
$$

Therefore, because $\theta_* - \theta_1 = \theta + \theta_1$,

$$
\begin{aligned}
\sum_{k=1}^{j+1} \frac{\sin 2\theta_k}{x_k^2 - x_*^2} - \sum_{k=1}^{j} \frac{\sin 2\theta_k}{x_k^2 - x^2} &= 2\cot(\theta + \theta_1) \\
&\quad - \cot(\theta + \theta_{j+1}) - \cot(\theta + \theta_{j+2}).
\end{aligned}
\tag{13}
$$

Similarly, from

$$
\frac{2\cos A \sin B}{\cos^2 A - \cos^2 B} = \csc(B - A) + \csc(B + A),
$$

and $\theta_m + \theta_* = \pi - (\theta_m - \theta)$, it follows that

$$2x_* \sum_{k=j+2}^{m} \frac{\sin\theta_k}{x_*^2 - x_k^2} - 2x \sum_{k=j+1}^{m} \frac{\sin\theta_k}{x^2 - x_k^2} = -\csc(\theta+\theta_{j+1}) - \csc(\theta+\theta_{j+2}). \tag{14}$$

Substituting (13) and (14) into (11), and using $\cot A + \csc A = \cot A/2$, gives

$$f(x) = 2\left[\cot\theta_{j+1} + \cot(\theta+\theta_1)\right] - \left[\cot\frac{\theta+\theta_{j+1}}{2} + \cot\frac{\theta+\theta_{j+2}}{2}\right]. \tag{15}$$

To show this is positive, we employ

$$\cot A + \cot B = \frac{\sin(A+B)}{\sin A \sin B} = \frac{2\sin(A+B)}{\cos(A-B) - \cos(A+B)}$$

and $\theta_{j+2} = \theta_{j+1} + 2\theta_1$, so that (15) can be written as

$$f(x) = \frac{\sin(\theta_{j+1}+\theta+\theta_1)\left[2\cos\theta_1 - \cos(\theta_{j+1}+\theta+\theta_1) - \cos(\theta_{j+1}-\theta-\theta_1)\right]}{2\sin\theta_{j+1}\sin(\theta+\theta_1)\sin\frac{\theta+\theta_{j+1}}{2}\sin\frac{\theta+\theta_{j+2}}{2}}.$$

All the sine terms in this expression for $f(x)$ are positive, and

$$2\cos\theta_1 - \cos(\theta_{j+1}+\theta+\theta_1) - \cos(\theta_{j+1}-\theta-\theta_1)$$

$$= 2\left(\cos\theta_1 - \cos\theta_{j+1}\cos(\theta+\theta_1)\right) \geq 2\left(\cos\theta_1 - \cos\theta_1\cos(\theta+\theta_1)\right) > 0.$$

Thus the lemma is established if n is even.

We next consider the case when $n = 2m+1$ is odd. Similar calculations to those for the even case can be made (note that $x_{m+1} = 0$), with the equivalent expression to (11) being

$$\begin{aligned} g(x) &:= n|T_n(x)|^{-1}\left[\lambda_{n+2}(x_*) - \lambda_{n+2}(x)\right] - 2x\csc\theta_{j+1} \\ &\quad - 2(x-x_*)\sum_{k=j+2}^{m}\csc\theta_k = 2\cot\theta_{j+1} + \sum_{k=1}^{j+1}\frac{\sin 2\theta_k}{x_k^2 - x_*^2} \\ &\quad - \sum_{k=1}^{j}\frac{\sin 2\theta_k}{x_k^2 - x^2} + 2x_*\sum_{k=j+2}^{m}\frac{\sin\theta_k}{x_*^2 - x_k^2} - 2x\sum_{k=j+1}^{m}\frac{\sin\theta_k}{x^2 - x_k^2} \end{aligned}$$

$$+\frac{1-x_*^2}{x_*}-\frac{1-x^2}{x}.$$

Observe that the lemma will be established for odd n if it can be shown that $g(x)>0$.

Now, identity (13) remains valid for odd n, while from $\theta_m=\pi/2-\pi/n$, it follows that the equivalent expression to (14) is

$$\begin{aligned}2x_*\sum_{k=j+2}^{m}\frac{\sin\theta_k}{x_*^2-x_k^2}-2x\sum_{k=j+1}^{m}\frac{\sin\theta_k}{x^2-x_k^2}\\ =-\csc(\theta+\theta_{j+1})-\csc(\theta+\theta_{j+2})+\frac{1}{x}-\frac{1}{x_*}.\end{aligned}$$

Therefore,

$$\begin{aligned}g(x)&=2\left[\cot\theta_{j+1}+\cot(\theta+\theta_1)\right]\\ &\quad-\left[\cot\frac{\theta+\theta_{j+1}}{2}+\cot\frac{\theta+\theta_{j+2}}{2}\right]+(x-x_*),\end{aligned}$$

which is positive by the same considerations as for the even case, and $x>x_*$. Thus the lemma is proved. □

3. Proof of Theorem 2

If n is even, then by Theorem 1 and symmetry considerations, $\Lambda_{n+2}(T_a) = \lambda_{n+2}(T_a,0)$, and so, by (8),

$$\begin{aligned}\Lambda_{n+2}(T_a)&=1+\frac{2}{n}\sum_{k=1}^{n/2}(\cot\theta_k+\tan\theta_k)\\ &=1+\frac{4}{n}\sum_{k=1}^{n/2}\cot\frac{(2k-1)\pi}{2n}=1+2\Lambda_{n/2}(T),\end{aligned}$$

where the final equality is a consequence of (3). The result (7) then follows from (4).

If n=$2m$+1 is odd, then by Theorem 1, $\Lambda_{n+2}(T_a) = \max_{0<x<x_m} \lambda_{n+2}(T_a, x)$. Now, if $0 < x < x_m$ and $x = \cos\theta$, then by (8), (10) and (12),

$$\begin{aligned}\lambda_{n+2}(T_a, x) = (-1)^m T_n(x) \Bigg[1 + \frac{2}{n} \sum_{k=1}^{m} \cot\theta_k \\ + \frac{1}{n} \sum_{k=1}^{m} [\cot(\theta - \theta_k) - \cot(\theta + \theta_k)] + \frac{1 - x^2}{nx} \Bigg].\end{aligned} \tag{16}$$

Define $\phi_m = m\pi/(2m+1)$. Then, on choosing $x = \cos\phi_m = \sin(\pi/(2n)) = \sin\theta_1$ in (16), we obtain

$$\begin{aligned}\lambda_{n+2}(T_a, \cos\phi_m) &= 1 + \frac{4}{n} \sum_{k=1}^{m} \cot\theta_k - \frac{1}{n} \cot\theta_1 + \frac{\cos^2\theta_1}{n \sin\theta_1} \\ &= 1 + \frac{4}{n} \sum_{k=1}^{m} \cot\theta_k + O\left(\frac{1}{n^2}\right).\end{aligned} \tag{17}$$

To obtain an asymptotic expression for $n^{-1} \sum_{k=1}^{m} \cot\theta_k$, we adapt the methods of Shivakumar and Wong [8] that were used to find a complete asymptotic expansion of (3). From the expansion

$$\cot z = \frac{1}{z} - \sum_{r=1}^{\infty} 2^{2r} |B_{2r}| \frac{z^{2r-1}}{(2r)!} \qquad (0 < |z| < \pi), \tag{18}$$

where the B_{2r} are Bernoulli numbers, it follows that

$$\begin{aligned}&\frac{1}{2m+1} \sum_{k=1}^{m} \cot\left(\frac{(2k-1)\pi}{2(2m+1)}\right) \\ &\qquad = \frac{2}{\pi} \sum_{k=1}^{m} \frac{1}{2k-1} - \frac{2}{\pi} \sum_{r=1}^{\infty} \frac{|B_{2r}|}{(2r)!} \frac{\pi^{2r}}{(2m+1)^{2r}} \sum_{k=1}^{m} (2k-1)^{2r-1}.\end{aligned}$$

Now, it is well-known that

$$\sum_{k=1}^{m} \frac{1}{2k-1} = \frac{1}{2} \log(4m) + \frac{\gamma}{2} + O\left(\frac{1}{m^2}\right).$$

Also, as derived by Shivakumar and Wong using the Euler–Maclaurin formula,

$$\sum_{k=1}^{m}(2k-1)^{2r-1} = \frac{(2m)^{2r}}{4r} + O\left(r(2m)^{2r-2}\right).$$

Further, from the Maclaurin series expansion of $(1+x)^{-2r}$ for $x>0$, it follows that

$$\left(1+\frac{1}{2m}\right)^{-2r} = 1 - \frac{r}{m} + O\left(\frac{r^2}{m^2}\right).$$

Thus

$$\frac{1}{2m+1}\sum_{k=1}^{m}\cot\left(\frac{(2k-1)\pi}{2(2m+1)}\right)$$

$$= \frac{1}{\pi}\left(\log(4m)+\gamma\right) - \frac{1}{2\pi}\sum_{r=1}^{\infty}\frac{|B_{2r}|}{r(2r)!}\,\pi^{2r} + \frac{1}{2\pi m}\sum_{r=1}^{\infty}\frac{|B_{2r}|}{(2r)!}\,\pi^{2r} + O\left(\frac{1}{m^2}\right).$$

The two series on the right of this equation can be evaluated by substituting $z=\pi/2$ in (18) and in the series

$$\log\sin z = \log z - \sum_{r=1}^{\infty}\frac{2^{2r-1}|B_{2r}|}{r(2r)!}\,z^{2r} \qquad (0<|z|<\pi),$$

which is obtained by integrating (18), so that

$$\frac{1}{2m+1}\sum_{k=1}^{m}\cot\left(\frac{(2k-1)\pi}{2(2m+1)}\right)$$
$$= \frac{1}{\pi}\left(\log\left(8m/\pi\right)+\gamma\right) + \frac{1}{2\pi m} + O\left(\frac{1}{m^2}\right).$$

On recalling that $n=2m+1$, we obtain from (17),

$$\begin{aligned}\Lambda_{n+2}(T_a) &\geq \lambda_{n+2}(T_a,\cos\phi_m)\\ &= \frac{4}{\pi}\log n + \frac{4}{\pi}\left(\gamma+\log\frac{4}{\pi}\right) + 1 + O\left(\frac{1}{n^2}\right).\end{aligned} \tag{19}$$

To show that $\Lambda_{n+2}(T_a)$ is equal to $\lambda_{n+2}(T_a,\cos\phi_m)$ to within $O(1/n^2)$ terms, we use a method that was employed by Brutman [2] to establish

(5). Now, it follows from (16) that

$$
\begin{aligned}
\lambda'_{n+2}&(T_a, \cos\phi_m) \\
&= \frac{1}{n}\left[\sec\theta_1 \sum_{k=1}^{m}\left[\csc^2(\phi_m-\theta_k)-\csc^2(\phi_m+\theta_k)\right]-\csc^2\theta_1-1\right] \qquad (20)\\
&= \frac{1}{n}\left[\sec\theta_1\left(\csc^2\theta_1-1\right)-\csc^2\theta_1-1\right] = -\frac{2+\cos\theta_1}{n(1+\cos\theta_1)}.
\end{aligned}
$$

Thus $\lambda'_{n+2}(T_a,\cos\phi_m)<0$, and so $\Lambda_{n+2}(T_a)=\max_{0<x<\cos\phi_m}\lambda_{n+2}(T_a,x)$.

On noting that $T''_n(\cos\phi_m) = (-1)^{m+1}n^2/\cos^2\theta_1$, we obtain, also from (16),

$$
\begin{aligned}
\lambda''_{n+2}(T_a,\cos\phi_m) &= \frac{-n^2}{\cos^2\theta_1}\left[1+\frac{4}{n}\sum_{k=1}^{m}\cot\theta_k-\frac{\cos\theta_1(1-\cos\theta_1)}{n\sin\theta_1}\right]\\
&+\frac{2}{n\cos^2\theta_1}\sum_{k=1}^{m}\left[\cot(\phi_m-\theta_k)\csc^2(\phi_m-\theta_k)-\cot(\phi_m+\theta_k)\csc^2(\phi_m+\theta_k)\right]\\
&+\frac{\sin\theta_1}{n\cos^3\theta_1}\sum_{k=1}^{m}\left[\csc^2(\phi_m-\theta_k)-\csc^2(\phi_m+\theta_k)\right]+\frac{2}{n\sin^3\theta_1}\\
&= -\frac{4}{n\cos^2\theta_1}\sum_{k=1}^{m}\left[n^2-\csc^2\theta_k\right]\cot\theta_k-\frac{n}{\cos\theta_1}\left[\frac{n}{\cos\theta_1}-\frac{1-\cos\theta_1}{\sin\theta_1}\right]\\
&-\frac{1}{n}\left[\frac{2}{\cos\theta_1\sin^3\theta_1}-\frac{1}{\cos\theta_1\sin\theta_1}-\frac{2}{\sin^3\theta_1}\right].
\end{aligned}
$$

Now, from $\sin\theta > 2\theta/\pi$ $(0<\theta<\pi/2)$, it follows that $\sin\theta_1 > 1/n$, and so $n^2-\csc^2\theta_k \ge n^2-\csc^2\theta_1 > 0$. Also from $\sin\theta_1 > 1/n$, we obtain

$$
\begin{aligned}
\frac{n}{\cos\theta_1}-\frac{1-\cos\theta_1}{\sin\theta_1} &> \frac{n}{\cos\theta_1}-n(1-\cos\theta_1)\\
&= \frac{n}{\cos\theta_1}\left(1-\cos\theta_1+\cos^2\theta_1\right) > 0.
\end{aligned}
$$

Finally,

$$
\frac{2}{\cos\theta_1\sin^3\theta_1}-\frac{1}{\cos\theta_1\sin\theta_1}-\frac{2}{\sin^3\theta_1} = \frac{(1-\cos\theta_1)^2}{\cos\theta_1\sin^3\theta_1} > 0.
$$

On combining these results it follows that $\lambda''_{n+2}(T_a, \cos\phi_m) < 0$.

Now suppose that $x^* \in (0, \cos\phi_m)$ is such that $\Lambda_{n+2}(T_a) = \lambda_{n+2}(T_a, x^*)$. We claim that $\lambda_{n+2}(T_a, x)$ is concave on the interval $(x^*, \cos\phi_m)$. To see this, consider the polynomial

$$p(x) = (-1)^m T_n(x)\left[1 - \sum_{k=1}^{m} \frac{(1-x^2)}{n\sqrt{1-x_k^2}}\,\frac{1}{x-x_k} + \sum_{k=m+1}^{n} \frac{(1-x^2)}{n\sqrt{1-x_k^2}}\,\frac{1}{x-x_k}\right],$$

which, by (8), agrees with $\lambda_{n+2}(T_a, x)$ on (x_{m+1}, x_m). Note that the leading term in $p(x)$ is $(-1)^{m+1}2^{2m}x^{2m+2}/(2m+1)$, and $p(x_j) = (-1)^{m-j}$ for $0 \le j \le m$, $p(x_j) = (-1)^{j-m-1}$ for $m+1 \le j \le 2m+2$. Thus p has (at least) $m+1$ changes of sign in each of the intervals (x_m, ∞) and (x_{2m+2}, x_{m+1}), and so p' has (at least) m zeros in each of these intervals, as well as a zero at $x^* \in (x_{m+1}, x_m)$. Since p' has degree $2m+1$, we have, in fact, identified the location of all its zeros. Hence p'' has at most one zero in (x^*, x_m), and so, because $p''(x^*) < 0$ and $p''(\cos\phi_m) < 0$, it follows that $p''(x) < 0$ on $(x^*, \cos\phi_m)$, as claimed.

To complete the proof, note that from the concavity of $\lambda_{n+2}(T_a, x)$ on $(x^*, \cos\phi_m)$ and (20), it follows that

$$\begin{aligned}\lambda_{n+2}(T_a, x^*) - \lambda_{n+2}(T_a, \cos\phi_m) &\le -\lambda'_{n+2}(T_a, \cos\phi_m) \times (\cos\phi_m - x^*)\\ &\le \frac{2+\cos\theta_1}{n(1+\cos\theta_1)}\cos\phi_m \;=\; O\left(\frac{1}{n^2}\right).\end{aligned}$$

The result (7) for odd values of n then follows from (19).

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(Received February 26, 2003; revised November 17, 2003)