Publ. Math. Debrecen
66/1-2 (2005), 41–61

Isotropic Jacobi fields on naturally reductive spaces

By J. C. GONZÁLEZ-DÁVILA (La Laguna) and R. O. SALAZAR (La Laguna)

Abstract. We derive existence results of isotropic Jacobi fields on naturally reductive spaces and we prove that a naturally reductive space (M, g) of dimension ≤ 5 with the property that all Jacobi fields vanishing at two points are $\mathrm{Tr}(M, \tilde{\nabla})$-isotropic, for some adapted canonical connection $\tilde{\nabla}$ and where $\mathrm{Tr}(M, \tilde{\nabla})$ denotes the corresponding transvection group, is locally symmetric. Moreover, for the three-dimensional case (M, g) is locally symmetric if all Jacobi fields vanishing at two points are isotropic.

1. Introduction

As it is well-known, restrictions of Killing vector fields to geodesics on a Riemannian manifold are Jacobi fields. In particular, on a homogeneous Riemannian manifold $(M = G/G_o, g)$ with adapted reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_o$, the restriction of the fundamental vector field X^* of some $X \in \mathfrak{g}$ to a geodesic $\gamma(t)$ starting at the origen o of M, is a Jacobi field induced by the geodesic variation $\phi(t, s) = (\exp sX)\gamma(t)$. When $X \in \mathfrak{g}_o$, the geodesic variation ϕ is called the *isotropic variation* of γ induced by X and the corresponding Jacobi field $V = X^* \circ \gamma$ is said to be *G-isotropic.* Moreover, if G is the identity component $I_o(M, g)$ of the isometry group $I(M, g)$ of (M, g), the set of all isotropic Jacobi fields along γ – we shall

Mathematics Subject Classification: 53C30, 53B20, 53C22.
Key words and phrases: isotropic Jacobi fields, naturally reductive homogeneous spaces, infinitesimal models, invariant unit vectors.
Partially supported by DGI Grant No. BFM2001-3548.

say simply *isotropic* Jacobi fields – coincides with that of the restrictions of all Killing vector fields to γ which vanish at the origin. It is worthwhile to note that in some texts, as for example in [18], the term "isotropic" is used in a broader sense; namely, as Jacobi fields which are restrictions of arbitrary Killing vector fields.

We focus our attention on naturally reductive homogeneous spaces. On these reductive coset spaces, the canonical connection $\tilde{\nabla}$ [9, I, p. 110] has the same geodesics as the Levi Civita connection and thus also the same Jacobi fields. Since its torsion and its curvature are parallel, the Jacobi equation can then be written as a differential equation with constant coefficients (the differential equation (5.6)) [3], [18]. Moreover, it leads to the consideration of Jacobi fields by means of *naturally reductive models* [17].

It is useful to remark here that, in general, the same homogeneous Riemannian manifold (M, g) may have more than one naturally reductive quotient representation G/G_o and once G/G_o is fixed, more than one naturally reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_o$ and thus, more than one *adapted* canonical connection $\tilde{\nabla}$. So, given a naturally reductive space $(M = G/G_o, g)$ with adapted canonical connection $\tilde{\nabla}$, we need to set some determined naturally reductive quotient representation. We consider the quotient representation of (M, g) in the form $M = \mathrm{Tr}(M, \tilde{\nabla})/K$, where $\mathrm{Tr}(M, \tilde{\nabla})$ is the *transvection group* of the affinely connected manifold $(M, \tilde{\nabla})$ [10, Chapter I] with isotropic subgroup K isomorphic to the restricted holonomy group of $(M, \tilde{\nabla})$ at the origin. Then, any $\mathrm{Tr}(M, \tilde{\nabla})$-isotropic Jacobi field is G-isotropic and, obviously, it is isotropic.

I. Chavel proved in [3] (see also [4]) that all simply connected normal Riemannian homogeneous space $(M = G/G_o, g)$ of rank one with the property that all Jacobi fields vanishing at two points are G-isotropic, are homeomorphic to a rank one symmetric space. In fact, he showed that the (non-symmetric) rank one normal homogeneous spaces $Sp(2)/SU(2)$ and $SU(5)/(Sp(2) \times T)$ admit a conjugate point of the origin which is not isotropically conjugate. Afterwards, W. Ziller in [18] proposed the following conjecture:

All naturally reductive spaces with the property that all Jacobi fields vanishing at two points are isotropic are locally symmetric.

The main purpose of this paper is to examine this conjecture and, at the same time, to derive existence results of nonzero isotropic and G-isotropic Jacobi fields on a homogeneous Riemannian manifold $(M = G/G_o, g)$. For naturally reductive spaces, we show that the integral curves of a G-invariant vector field are characterized as those geodesics without nonzero G-isotropic Jacobi fields (Proposition 3.6) and moreover, we determine all conjugate points to the origin on these curves (Theorem 5.3). Then we give a positive answer to the above conjecture when $\dim M \leq 5$ and the Jacobi fields are considered $\mathrm{Tr}(M, \tilde{\nabla})$-isotropic. Moreover, for the three-dimensional case the conjecture is completely resolved (Theorem 5.7).

The paper is organized as follows: in Section 2 we give some preliminaries about the canonical connection adapted to a reductive decomposition and infinitesimal models. In Section 3, we analize some aspects related with the existence of isotropic Jacobi fields on homogeneous Riemannian manifolds and in Section 4, on semi-simple symmetric spaces (Theorem 4.2). Finally, in Section 5 we recall the Jacobi equation for an adapted canonical connection on naturally reductive spaces and we prove the already mentioned results.

The authors wish to thank to L. Vanhecke for his useful suggestions made during the preparation of this paper.

2. Preliminaries

Let (M, g) be an n-dimensional, connected, homogeneous *Riemannian* manifold with $n \geq 2$. Then (M, g) can be expressed as a coset space G/G_o, where G is a Lie group, which is supposed to be connected, acting transitively and effectively on M, G_o is the isotropy subgroup of G at some point $o \in M$ and g is a G-invariant Riemannian metric on G/G_o. Moreover, there is an $Ad(G_o)$-invariant subspace $\mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of G such that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_o$, $\mathfrak{g}_o$ being the Lie algebra of G_o. Hence, G/G_o is a *reductive homogeneous space* (with respect to the given decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_o$). (M, g) is said to be *naturally reductive* if there exists a reductive representation $M = G/G_o$, with respect to a decomposition

$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_o$, satisfying

$$\langle [X,Y]_{\mathfrak{m}}, Z\rangle + \langle [X,Z]_{\mathfrak{m}}, Y\rangle = 0 \tag{2.1}$$

for all $X, Y, Z \in \mathfrak{m}$, where $[X,Y]_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of $[X,Y]$ and $\langle\, ,\rangle$ is the metric induced by g on $\mathfrak{m}$, or equivalently, $[X,\cdot]_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{m}$ is skew-symmetric for all $X \in \mathfrak{m}$. (M,g) is said to be *normal homogeneous* if there exists a bi-invariant metric on $\mathfrak{g}$ whose restriction to $\mathfrak{m} = \mathfrak{g}_o^{\perp}$ is the metric $\langle\, ,\rangle$. In particular, all $[X,\cdot] : \mathfrak{g} \to \mathfrak{g}$ are skew-symmetric and thus, a normal homogeneous metric is also naturally reductive.

For each $X \in \mathfrak{g}$, denote by X^* the corresponding *fundamental vector field* on M, that is,

$$X^*_p = \frac{d}{dt}_{|t=0} (\exp tX)p, \quad p \in M.$$

It is clear that X^* is a Killing vector field. Under the canonical identification of $\mathfrak{m}$ with the tangent space T_oM of the origin o, the *canonical connection* $\tilde{\nabla}$ of (M,g) adapted to the reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_o$ is the unique G-invariant affine connection on M such that for every $u \in \mathfrak{m}$ and for every vector field X on M, one has ([10])

$$(\tilde{\nabla}_{u^*} X)_o = [u^*, X]_o. \tag{2.2}$$

Let $\tilde{T}$ denote its torsion tensor and $\tilde{R}$ the corresponding curvature tensor defined by the sign convention $\tilde{R}(X,Y) = \tilde{\nabla}_{[X,Y]} - [\tilde{\nabla}_X, \tilde{\nabla}_Y]$ and $\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y]$, for all $X, Y \in \mathfrak{X}(M)$, the Lie algebra of smooth vector fields on M. Then, these tensors are given by

$$\tilde{T}_o(X,Y) = -[X,Y]_{\mathfrak{m}} \quad \tilde{R}_o(X,Y) = \mathrm{ad}_{[X,Y]_{\mathfrak{g}_o}} \tag{2.3}$$

for $X, Y \in \mathfrak{m}$, where $[X,Y]_{\mathfrak{g}_o}$ denotes $\mathfrak{g}_o$-component of $[X,Y]$.

Let $\mathfrak{k} \subset \mathfrak{g}_o$ be the subalgebra generated by all projections $[X,Y]_{\mathfrak{g}_o}$, $X, Y \in \mathfrak{m}$. The subalgebra $\mathfrak{tr}(\mathfrak{m}) \subset \mathfrak{g}$ given by $\mathfrak{tr}(\mathfrak{m}) = \mathfrak{m} \oplus \mathfrak{k}$ is called the *transvection algebra* and the corresponding connected Lie subgroup $\mathrm{Tr}(M, \tilde{\nabla})$ of G with Lie algebra $\mathfrak{tr}(\mathfrak{m})$ is said to be the *transvection group* of the reductive space $(M = G/G_o, g)$ with the fixed canonical connection $\tilde{\nabla}$, i.e., of the affinely connected manifold $(M, \tilde{\nabla})$ [10, Chapter I]. Taking into account that the linear isotropy representation of G_o in the tangent space T_oM is faithful, $\mathfrak{k}$ coincides precisely with the algebra generated by all curvature transformations $\tilde{R}(X,Y)$ on T_oM [10, Proposition I.2].

Next, we introduce the notion of *infinitesimal model.* Let $(\mathfrak{m}, \langle\, , \rangle)$ be a Euclidean vector space and $\tilde{T}$ and $\tilde{R}$ two tensors on $\mathfrak{m}$ of type $(1,2)$ and $(1,3)$, respectively. Consider $\tilde{R}$ also as the bilinear map

$$\tilde{R} : \mathfrak{m} \times \mathfrak{m} \to \mathrm{End}(\mathfrak{m}), \quad (X,Y) \mapsto \tilde{R}(X,Y).$$

$\mathfrak{M} = (\mathfrak{m}, \tilde{T}, \tilde{R}, \langle\, , \rangle)$ is called an *infinitesimal model* (of a locally homogeneous Riemannian manifold) if $\tilde{T}$ and $\tilde{R}$ satisfy

(i) $\tilde{T}(X,Y) = -\tilde{T}(Y,X)$, $\tilde{R}(X,Y) = -\tilde{R}(Y,X)$,

(ii) $\langle \tilde{R}(X,Y)Z, W\rangle = -\langle \tilde{R}(X,Y)W, Z\rangle$,

(iii) $\tilde{R}(X,Y) \cdot \tilde{T} = \tilde{R}(X,Y) \cdot \tilde{R} = 0$,

(iv) $\underset{XYZ}{\mathfrak{S}} \left\{\tilde{R}(X,Y)Z + \tilde{T}(\tilde{T}(X,Y),Z)\right\} = 0$,

(v) $\underset{XYZ}{\mathfrak{S}} \left\{\tilde{R}(\tilde{T}(X,Y),Z)\right\} = 0$

for all $X,Y,Z \in \mathfrak{m}$. It follows that if $(M = G/G_o, g)$ is a reductive space with adapted canonical connection $\tilde{\nabla}$, then $\tilde{T}$ and $\tilde{R}$ at the origin o determine an infinitesimal model on $(\mathfrak{m} = T_oM, \langle\, , \rangle = g_o)$. Conversely, following Nomizu [16] (see also [15]), we can reconstruct the *transvection algebra* $\mathfrak{tr}(\mathfrak{m}) = \mathfrak{m} \oplus \mathfrak{k}$ by putting

$$\begin{cases} [X,Y] = -\tilde{T}(X,Y) + \tilde{R}(X,Y), \\ [A,X] = AX, \\ [A,B] = AB - BA, \end{cases} \tag{2.4}$$

for all $X,Y \in \mathfrak{m}$ and $A,B \in \mathfrak{k}$. Let $\tilde{G}$ be the unique connected and simply connected Lie group with Lie algebra $\mathfrak{tr}(\mathfrak{m})$ and $\tilde{K}$ the connected Lie subgroup of $\tilde{G}$ with associated Lie subalgebra $\mathfrak{k}$. If $\tilde{K}$ is *closed*, i.e., the model is *regular*, then $\tilde{M} = \tilde{G}/\tilde{K}$ is a (simply connected) manifold [15], [16] and by identifying the tangent space $T_o\tilde{M}$ at the origin $o \in \tilde{M}$ with $\mathfrak{m}$, $\tilde{T}$ and $\tilde{R}$ can be extended to unique $\tilde{G}$-invariant tensor fields, the torsion and the curvature of the canonical connection of $\tilde{M} = \tilde{G}/\tilde{K}$, respectively. We say that $\tilde{M} = \tilde{G}/\tilde{K}$ is *associated* to the infinitesimal model $\mathfrak{M}$.

Finally, given an infinitesimal model $\mathfrak{M} = (\mathfrak{m}, \tilde{T}, \tilde{R}, \langle\,,\rangle)$, a vector $V \in \mathfrak{m}$ is said to be *invariant* (with respect to $\mathfrak{M}$) if $\tilde{R}(X,Y)V = 0$, for all $X, Y \in \mathfrak{m}$. We denote by $\mathrm{Inv}(\mathfrak{M})$ the subspace of the Euclidean space $(\mathfrak{m}, \langle\,,\rangle)$ of all invariant vectors with respect to $\mathfrak{M}$. If $\mathfrak{M}$ is regular, each $V \in \mathrm{Inv}(\mathfrak{M})$ is invariant with respect to the holonomy group and there exists a unique $\tilde{\nabla}$-parallel extension on the associated (simply connected) homogeneous Riemannian manifold $\tilde{M} = \tilde{G}/\tilde{K}$ and it is $\tilde{G}$-invariant (see [6], [10] for more details).

3. Isotropic Jacobi fields

Let $(M = G/G_o, g)$ be a connected homogeneous Riemannian manifold with an adapted reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_o$. Denote by $\mathrm{Isot}_G(\gamma)$ the vector space of all G-isotropic Jacobi fields along a unit-speed geodesic γ starting at the origin of M and by $i(M)$ the set of all (complete) Killing vector fields on M. Let $F : \mathfrak{g} \to i(M)$ be the mapping given by $F(X) = X^*$, $X \in \mathfrak{g}$. Then F is a one-to-one linear map and, moreover, it is a Lie anti-homomorphism, that is, the linear map F satisfies

$$[X^*, Y^*] = -[X,Y]^* \tag{3.1}$$

for all $X, Y \in \mathfrak{g}$. Furthermore, if $G = I_o(M)$, then F is a Lie anti-isomorphism. Hence, it follows that the *set of all isotropic Jacobi fields along γ coincides with that of the restrictions of all Killing vector fields to γ which vanish at $o \in M$.*

From (2.2) and (3.1), we have

$$\nabla_u X^* = [X, u] \tag{3.2}$$

for all $u \in \mathfrak{m} \cong T_oM$ and $X \in \mathfrak{g}_o$. It implies that a Jacobi field V along γ with $\gamma'(0) = u$, $\|u\| = 1$, is G-isotropic if and only if there exists an $A \in \mathfrak{g}_o$ such that

$$(V(0), V'(0)) = (0, [A, u]). \tag{3.3}$$

Then $V = A^* \circ \gamma$ and

$$\dim \mathrm{Isot}_G(\gamma) = \mathrm{rank}(\mathrm{ad}\, u_{|\mathfrak{g}_o}). \tag{3.4}$$

Proposition 3.1. *All G-isotropic Jacobi field along a geodesic γ is orthogonal to γ' everywhere.*

Proof. Let V be a G-isotropic Jacobi field along γ and let $A \in \mathfrak{g}_o$ such that $V'(0) = [A, u]$. Because the inner product $\langle\, ,\rangle$ on $\mathfrak{m}$ is $Ad(G_o)$-invariant, it follows that ad_A is a skew-symmetric endomorphism of $(\mathfrak{m}, \langle\, ,\rangle)$. Then $V'(0)$ is orthogonal to u. This gives at once the required result. $\square$

Next, we analize the existence of nonzero G-isotropic Jacobi fields.

Proposition 3.2. *A homogeneous Riemannian manifold $(M{=}G/G_o, g)$ admits a geodesic starting at the origin with some nonzero G-isotropic Jacobi field if and only if $\dim G > \dim M$.*

Proof. If $\dim G = \dim M$ then the isotropy subgroup G_o is finite and, consequently, $\mathfrak{g}_o$ is trivial. It implies that (M, g) does not possess any nonzero G-isotropic Jacobi field.

If $\dim G > \dim M$ then there exists $A \neq 0$ in $\mathfrak{g}_o$ and, because $F : \mathfrak{g} \to i(M)$ is one-to-one, it follows that A^* is not identically zero and it is a Killing vector field with $A^*_o = 0$. It implies the existence of a unit tangent vector u at o such that $\nabla_u A^*$ is a nonzero vector. Let γ be the geodesic with $\gamma'(0) = u$. Then $V = A^* \circ \gamma$ is a nonzero G-isotropic Jacobi field along γ. $\square$

O. Kowalski and L. Vanhecke proved in [13, Proposition 2] that the isotropy subgroup of the group of isometries of a simply connected naturally reductive space is at least one-dimensional. This yields the following existence result.

Corollary 3.3. *Any simply connected naturally reductive space admits a geodesic starting at the origin with some nonzero isotropic Jacobi field.*

Remark 3.4. In the class of all simply connected three-dimensional unimodular Lie groups equipped with a left-invariant metric, one can find examples of (non-naturally reductive) homogeneous Riemannian manifolds without isotropic Jacobi fields. First, note that the Lie algebra of these Lie groups (G, g) admit an orthonormal basis $\{e_1, e_2, e_3\}$ such that

$$[e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3$$

where $\lambda_1, \lambda_2, \lambda_3$ are constants. From Proposition 3.2, (G, g) does not admit any isotropic Jacobi field if and only if $\dim I(G, g) = 3$ and it occurs at least in the following cases [12]:

(i) The group $E(1, 1)$ of the rigid motions of the three-dimensional Minkowski space equipped with a one-parameter family of left-invariant metrics such that $\lambda_1 = -\lambda_3 > 0$, $\lambda_2 = 0$.

(ii) The universal covering of $SL(2, \mathbb{R})$ equipped with a two-parameter family of left-invariant metrics such that $\lambda_1 > \lambda_2 > 0$, $\lambda_3 = -(\lambda_1+\lambda_2)$.

On the other hand, one can also find examples of isotropic Jacobi fields which are not G-isotropic, for some Lie group $G \subset I(M, g)$: Consider (M, g) the n-dimensional hyperbolic space. Then, $\dim I(M, g) = n(n+1)/2 > n$ and we shall show in Corollary 4.3 that $\dim \mathrm{Isot}(\gamma) = n - 1$, for every geodesic γ. Moreover, (M, g) can be identified with a solvable Lie group G of isometries (G is a semi-direct product of the multiplicative group $\mathbb{R}_0^+ = \{x \in \mathbb{R} \mid x > 0\}$ and the additive group $\mathbb{R}^{n-1}$, see for example [17]). Then (M, g) *does not admit any G-isotropic Jacobi field.*

Denote by $\mathfrak{M}$ the infinitesimal model determined by the torsion and the curvature of the adapted canonical connection $\tilde{\nabla}$ on $\mathfrak{m} \cong T_oM$. Because $u \in \mathrm{Inv}(\mathfrak{M})$ if and only if $[u, \mathfrak{k}] = 0$, we have

Proposition 3.5. *A geodesic γ on a homogeneous Riemannian manifold $(M = G/G_o, g)$ starting at the origin does not admit any nonzero $\mathrm{Tr}(M, \tilde{\nabla})$-isotropic Jacobi field if and only if $\gamma'(0) \in \mathrm{Inv}(\mathfrak{M})$.*

On naturally reductives spaces $(M = G/G_o, g)$, any G-invariant unit vector field is Killing [6, Lemma 6.1] and so, each one of its integral curves is a geodesic. Put $u = U_o \in \mathfrak{m} = T_oM$. Then u is $Ad(G_o)$-invariant and it implies $[\mathfrak{g}_o, u] = 0$. The converse holds if G_o is connected (see [9]). Hence, we have

Proposition 3.6. *Let $(M = G/G_o, g)$ be a naturally reductive space. The integral curve through the origin of a G-invariant vector field U does not admit any nonzero G-isotropic Jacobi field. Moreover, if G_o is connected then each geodesic through the origin without nonzero G-isotropic Jacobi fields is the integral curve of a G-invariant vector field.*

Note that G_o is always connected if M is simply connected.

Next, let $\lambda_1, \dots, \lambda_{n-1}$ be the eigenvalues and $\{e_1, \dots, e_{n-1}\}$ the eigenvectors of the self-adjoint operator $\tilde{R}_u = \tilde{R}(u, \cdot)u$ of $E = \{v \in \mathfrak{m} \mid \langle u, v\rangle = 0\}$ such that $\lambda_i \neq 0$ if $i = n-r, \dots, n-1$, and $\lambda_1 = \dots = \lambda_{n-r-1} = 0$, being $r = \operatorname{rank} \tilde{R}_u \leq n-1$. Then $E = E_0 \oplus E_1$, where $E_0 = \operatorname{Ker} \tilde{R}_u$ and E_1 is the r-dimensional subspace of $\mathfrak{m}$ generated by $e_{n-r}, \dots, e_{n-1}$.

Proposition 3.7. *Let $(M = G/G_o, g)$ be a homogeneous Riemannian manifold and $\tilde{\nabla}$ some of its adapted canonical connections. Let V be a Jacobi field along a geodesic γ on (M, g) starting at the origin such that $V(0) = 0$. If $V'(0) \in E_1$, then V is $\mathrm{Tr}(M, \tilde{\nabla})$-isotropic and we have*

$$\dim \mathrm{Isot}_G(\gamma) \geq \dim \mathrm{Isot}_{\mathrm{Tr}(M,\tilde{\nabla})}(\gamma) \geq \operatorname{rank} \tilde{R}_u. \tag{3.5}$$

Moreover, if $(M = G/G_o, g)$ is a normal homogeneous space, then the converse holds, (3.5) becomes into an equality and

$$\mathrm{Isot}_G(\gamma) = \mathrm{Isot}_{\mathrm{Tr}(M,\tilde{\nabla})}(\gamma).$$

Proof. Put $V'(0) = \sum_{i=n-r}^{n-1} x^i e_i$ and consider $A \in \mathfrak{k} \subset \mathfrak{g}_o$ given by

$$A = \sum_{i=n-r}^{n-1} \frac{x^i}{\lambda_i} \tilde{R}(u, e_i).$$

Then, $A(u) = [A, u] = V'(0)$ which proves that $V = V(t)$ is $\mathrm{Tr}(M, \tilde{\nabla})$-isotropic and, consequently, G-isotropic. This also implies (3.5).

If $(M = G/G_o, g)$ is normal, then we get

$$\langle \tilde{R}_u v, v\rangle = \big\langle [[u, v]_{\mathfrak{g}_o}, u], v\big\rangle = \big\langle [u, v]_{\mathfrak{g}_o}, [u, v]_{\mathfrak{g}_o}\big\rangle$$

for all $v \in \mathfrak{m} = \mathfrak{g}_o^{\perp}$. Hence, $\tilde{R}_u v = 0$, for $v \in E$, if and only if $[u, v]_{\mathfrak{g}_o} = 0$. But,

$$[u, v]_{\mathfrak{g}_o} = 0 \Leftrightarrow 0 = \big\langle [u, v]_{\mathfrak{g}_o}, \mathfrak{g}_o\big\rangle = -\big\langle v, [u, \mathfrak{g}_o]\big\rangle \Leftrightarrow v \in [u, \mathfrak{g}_o]^{\perp}.$$

So, we have $E_1 = [u, \mathfrak{g}_o]$. Then the proof is completed using (3.4) and (3.5). □

Hence, a Jacobi field V is G-isotropic on a normal homogeneous space $(M = G/G_o, g)$ if and only if it has initial condition

$$(V(0), V'(0)) = (0, X), \quad X \in E_1.$$

Remark 3.8. To finish this section, we consider a class of examples of naturally reductive homogeneous spaces, the *generalized Heisenberg groups* $H(p,1)$, $p \geq 1$, which admit G-isotropic Jacobi fields, being G the simply connected Lie group obtained by using the Nomizu construction for suitable infinitesimal models. $H(p,1)$ is the group of matrices of the form

$$a = \begin{pmatrix} 1 & A & c \\ 0 & I_p & B^t \\ 0 & 0 & 1 \end{pmatrix}$$

where I_p denotes the identity matrix of type $p \times p$ and where $A = (a_1, \ldots, a_p) \in \mathbb{R}^p$, $B = (b_1, \ldots, b_p) \in \mathbb{R}^p$ and $c \in \mathbb{R}$. It is a connected, simply connected nilpotent Lie group of dimension $2p+1$ and the dimension of its center is one.

The following coordinates (x^i, x^{p+i}, z), $1 \leq i \leq p$, provide a system of global coordinates:

$$x^i(a) = a_i, \quad x^{p+i}(a) = b_i, \quad z(a) = c$$

and a basis of left-invariant one-forms is given by

$$\alpha_i = dx^i, \quad \alpha_{p+i} = dx^{p+i}, \quad \eta = dz - \sum_{j=1}^{p} x^j dx^{p+j}.$$

For the dual left-invariant vector fields, we then have

$$X_i = \frac{\partial}{\partial x^i}, \quad X_{p+i} = \frac{\partial}{\partial x^{p+i}} + x^i \frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z}.$$

On $H(p,1)$ we consider the Riemannian metric g for which these vectors form an orthonormal basis at each point. Denote by $\langle , \rangle$ the inner product on the Lie algebra $\mathfrak{h}(p,1)$ of $H(p,1)$ determined by g. Then $\mathfrak{h}(p,1)$ can be expressed as the orthogonal decomposition $\mathfrak{h}(p,1) = \mathfrak{v} \oplus \mathfrak{z}$ with respect to $\langle , \rangle$, where $\mathfrak{z}$ is the one-dimensional subspace generated by Z and $\mathfrak{v}$ is the $2p$-dimensional subspace generated by $\{X_i, X_{p+i}\}$, $1 \leq i \leq p$. Let J be the endomorphism on $\mathfrak{v}$ given by

$$JX_i = X_{p+i}, \quad JX_{p+i} = -X_i, \quad i = 1, \ldots, p.$$

Then $J^2 = -id_{\mathfrak{v}}$, $\langle JU, JV\rangle = \langle U, V\rangle$ and $[U, V] = \langle JU, V\rangle Z$, for all $U, V \in \mathfrak{v}$. From now on and as before, we suppose that $U, V, W \in \mathfrak{v}$ and $\lambda, \mu, \delta \in \mathbb{R}$. For the Levi Civita connection of $(H(p, 1), g)$ we obtain

$$\nabla_{(V+\mu Z)}(U + \lambda Z) = -\frac{1}{2}(\mu JU + \lambda JV + \langle JU, V\rangle Z)$$

and the corresponding curvature tensor R is given by

$$\begin{aligned} &R((U + \lambda Z), (V + \mu Z))(W + \delta Z) \\ &\quad = \frac{1}{4}\langle JV, W\rangle JU - \frac{1}{4}\langle JU, W\rangle JV - \frac{1}{2}\langle JU, V\rangle JW \\ &\qquad + \frac{1}{4}\langle \mu U - \lambda V, W\rangle Z - \frac{1}{4}\delta(\mu U - \lambda V). \end{aligned} \tag{3.6}$$

Following [5], $S = -\frac{1}{2}\tilde{T}$ is a homogeneous structure of type $\mathcal{S}_3$ on $H(p, 1)$ where $\tilde{T}$ is the $(1, 2)$-tensor field given by

$$\tilde{T}(U + \lambda Z, V + \mu Z) = \mu JU - \lambda JV - \langle JU, V\rangle Z.$$

$\tilde{T}$ is the torsion of the connection determined by $\tilde{\nabla} = \nabla - S$ and, from (3.6), the curvature tensor $\tilde{R}$ of $\tilde{\nabla}$ is given by

$$\tilde{R}((U + \lambda Z), (V + \mu Z))(W + \delta Z) = -\langle JU, V\rangle JW.$$

Then, $\mathfrak{M} = (\mathfrak{h}(p, 1), \tilde{T}, \tilde{R}, \langle, \rangle)$ is a naturally reductive model for $H(p, 1)$ and the holonomy algebra $\mathfrak{k}$ of $\tilde{\nabla}$ is generated by the endomorphism φ given by $\varphi(U + \lambda Z) = JU$. Hence and by using (2.4), the transvection algebra $\mathfrak{tr}(\mathfrak{h}(p, 1)) = \mathfrak{h}(p, 1) \oplus \mathfrak{k}$ is isomorphic to a semidirect sum of $\mathfrak{h}(p, 1)$ and $\mathfrak{k}$ and $(H(p, 1), g)$ admits a naturally reductive quotient representation G/K where G is a semidirect product $H(p, 1) \times_\phi SO(2)$ [5].

Next, using [1, Section 3.7] we can directly determine all G-isotropic Jacobi fields:

Let $U + \lambda Z$ be a unit vector of $\mathfrak{h}(p, 1)$ and $\gamma : \mathbb{R} \to H(p, 1)$ the geodesic in $H(p, 1)$ with $\gamma(0) = e$, where e denotes the identity element, and $\gamma'(0) = U + \lambda Z$. We have

(i) If $\lambda = 0$, then $\mathrm{Isot}_G(\gamma) = \mathrm{span}\{tJU - \frac{1}{2}t^2 Z\}$.

(ii) If $U = 0$, then $\mathrm{Isot}_G(\gamma) = \{0\}$.

(iii) If $U \neq 0$, $\lambda \neq 0$, then

$$\mathrm{Isot}_G(\gamma) = \mathrm{span}\Big\{(\cos(|\lambda|t) - 1))U + \mathrm{sig}(\lambda)\sin(|\lambda|t)JU + \frac{1-\lambda^2}{\lambda}(\cos(|\lambda|t) - 1)Z\Big\}.$$

Moreover, it follows from [1, Section 3.8] that γ *admits* G*-isotropic conjugate points if and only if* $U \neq 0$, $\lambda \neq 0$, *and the* G*-isotropic conjugate points are all* $\gamma(t)$ *satisfying* $|\lambda|t \in 2\pi\mathbb{Z}^*$.

4. Isotropic Jacobi fields on symmetric spaces

Let $(M = G/G_o, g)$ be a Riemannian globally symmetric space. Consider $G = I_o(M, g)$ and σ the involutive automorphism of G given by $\sigma : g \to s_o g s_o$, where s_o is the geodesic symmetry at the origin $o \in M$. Then $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_o$ with $\mathfrak{g}_o = \{X \in \mathfrak{g} \mid \sigma_* X = X\}$ and $\mathfrak{m} = \{X \in \mathfrak{g} \mid \sigma_* X = -X\}$ and $\mathfrak{M} = (\mathfrak{m}, \tilde{T} = 0, \tilde{R} = R, \langle,\rangle)$ is a (symmetric) infinitesimal model of M, where the Levi Civita connection of (M, g) is precisely the corresponding canonical connection [8, Theorem 3.3, Ch. IV]. Moreover, the transvection group $\mathrm{Tr}(M)$ of this affine reductive space coincides with the group generated by all *elementary transvections* $s_p \circ s_q$, for all $p, q \in M$ (see [10]).

When G is semi-simple, $G = \mathrm{Tr}(M) = I_o(M)$ and then $\mathfrak{g}_o = \mathrm{span}\{R(X,Y) \mid X, Y \in T_oM\}$ [8, Theorem 4.1, Ch. V]. The decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ of $\mathfrak{g}$ into its simple ideals $\mathfrak{g}_i$, $1 \leq i \leq r$, allows to express the metric g at $o \in M$ as restriction of an inner product $\langle,\rangle = \beta_1 B_1 \perp \ldots \perp \beta_r B_r$ of $\mathfrak{g}$ to $\mathfrak{m}$, where B_i is the Killing form of $\mathfrak{g}_i$ and $\beta_i \in \mathbb{R} - \{0\}$. In particular, semi-simple symmetric spaces are normal homogeneous spaces. For the corresponding infinitesimal models, called *semi-simple symmetric models*, the following result has been proved in [6].

Lemma 4.1. *Let* $\mathfrak{M}$ *be a* n*-dimensional symmetric infinitesimal model,* $n \geq 2$. *If it is irreducible or semi-simple, then it does not admit invariant vectors. Moreover, if* $\dim \mathrm{Inv}(\mathfrak{M}) = k \leq n$, *then* $\mathfrak{M} = \mathfrak{M}^{n-k} \oplus \mathbb{R}^k$ *where* $\mathfrak{M}^{n-k}$ *is a semi-simple symmetric model and* $\mathbb{R}^k$ *is the* k*-dimensional flat model.*

Hence, we have

Theorem 4.2. *On an n-dimensional semi-simple symmetric space (M, g), every unit-speed geodesic γ starting at the origin admits nonzero isotropic Jacobi fields and*

$$\dim \operatorname{Isot}(\gamma) = \operatorname{rank} R_{\gamma'(0)}.$$

Moreover, Riemannian symmetric spaces admitting geodesics without isotropic Jacobi fields are locally isometric to a Riemannian product $M = M_1 \times \mathbb{R}^k$, where $1 \leq k \leq n$ and M_1 is an $(n-k)$-dimensional semi-simple symmetric space.

Proof. From Proposition 3.5 and Lemma 4.1, there exist nonzero isotropic Jacobi fields along γ and from Proposition 3.7, we get $\dim \operatorname{Isot}(\gamma) = \operatorname{rank} R_{\gamma'(0)}$. For the last part of the theorem, we use again Lemma 4.1. □

The rank one symmetric spaces have been classified completely. They are the Euclidean spheres S^m, the projective spaces $\mathbb{K}P^m$, where $\mathbb{K}$ means either the field $\mathbb{R}$ of real numbers, or the field $\mathbb{C}$ of complex numbers, or the non-commutative field $\mathbb{H}$ of quaternions and the Cayley plane $\mathbb{C}aP^2$ and their non-compact duals. On an n-dimensional rank one symmetric space, we have $\operatorname{rank} R_u = n - 1$, for all unit vector u. Moreover, it is well-known that all geodesics are periodic with the same length on the compact case. From this and taking into account that each $V \in \operatorname{Isot}(\gamma)$ is a normal Jacobi field, we then obtain

Corollary 4.3. *On rank one symmetric spaces, all normal Jacobi fields vanishing at a point are isotropic. In particular, for the compact case, they are all periodic.*

All Jacobi fields on Riemannian symmetric spaces which vanish at two points are isotropic [2]. Next, we give a simple proof of the same result but considering only the class of the $\operatorname{Tr}(M)$-isotropic Jacobi fields:

Theorem 4.4. *On Riemannian symmetric spaces, every Jacobi field vanishing at two points is* $\operatorname{Tr}(M)$*-isotropic.*

Proof. With respect to a (parallel) frame field $\{E_1, \ldots, E_n\}$ of eigenvectors of the Jacobi operator R_γ at each point $\gamma(t)$, the Jacobi differential equation

$$\frac{\nabla^2 V}{dt^2} + R_\gamma V = 0$$

gives rise to the following system of differential equations:

$${V^i}'' + \lambda_i V^i = 0, \quad i = 1, \ldots, n,$$

where $\lambda_1, \ldots, \lambda_n$ are the (constant) eigenvalues of R_γ. If $\operatorname{rank} R_\gamma = r$, then the solutions are given by

$$V(t) = \sum_{j=1}^{n-r}(A_j t + B_j)E_j(t) + \sum_{k=n-r+1}^{n}(A_k\alpha_k(t) + B_k\beta_k(t))E_k(t),$$

where $\alpha_k(t) = \cos\sqrt{\lambda_k}t, \beta_k(t) = \sin\sqrt{\lambda_k}t$, if $\lambda_k > 0$ and $\alpha_k(t) = \cosh\sqrt{-\lambda_k}t$, $\beta_k(t) = \sinh\sqrt{-\lambda_k}t$, if $\lambda_k < 0$, $k = n-r+1, \ldots, n$. Because $V(0) = V(t_o) = 0$, for some $t_o \neq 0$, we have $A_j = B_j = 0$ for $j = 1, \ldots, n-r$ and $A_k = 0$ for $k = n-r+1, \ldots, n$. Hence, $V'(0) \in E_1$ and from Proposition 3.7, V is $\operatorname{Tr}(M)$-isotropic. □

Remark 4.5. Note that Riemannian symmetric spaces of noncompact type are Hadamard manifolds and so, they contain no pairs of conjugate points.

5. Isotropic Jacobi fields on naturally reductive spaces

Let $(M = G/G_o, g)$ be a naturally reductive space and $\tilde\nabla$ some of its adapted canonical connections. Let $\mathfrak{M} = (\mathfrak{m}, \tilde T, \tilde R, \langle,\rangle)$ be the infinitesimal model determined by $(M, \tilde\nabla)$. Then, from (2.3), we can also write (2.1) in the form

$$\langle \tilde T(X,Y), Z\rangle + \langle \tilde T(X,Z), Y\rangle = 0. \tag{5.1}$$

Moreover, the *homogeneous structure* [17] $S = \nabla - \tilde\nabla$ is given by ([9, p. 201, Vol. II])

$$S_X Y = -\frac{1}{2}\tilde T_X Y. \tag{5.2}$$

This implies that ∇ and $\tilde{\nabla}$ have the same geodesics and, consequently, the geodesics of (M, g) are orbits of one-parameter subgroups of G of type $\exp tu$ where $u \in \mathfrak{m}$ (see [17]). Moreover, we get

$$\begin{aligned} R(X,Y)Z = \tilde{R}(X,Y)Z &+ \frac{1}{2}\tilde{T}(\tilde{T}(X,Y),Z) + \frac{1}{4}\tilde{T}(\tilde{T}(Z,X),Y) \\ &+ \frac{1}{4}\tilde{T}(\tilde{T}(Y,Z),X). \end{aligned} \tag{5.3}$$

Given a geodesic $\gamma(t) = (\exp tu)o$, $u \in \mathfrak{m}$, $\|u\| = 1$, through the origin o at M, the Jacobi equation for $\tilde{\nabla}$ is given by

$$\frac{\tilde{\nabla}^2 V}{dt^2} - \tilde{T}_\gamma \frac{\tilde{\nabla} V}{dt} + \tilde{R}_\gamma V = 0, \tag{5.4}$$

where $\tilde{R}_\gamma = \tilde{R}(\gamma', \cdot)\gamma'$ and $\tilde{T}_\gamma = \tilde{T}(\gamma', \cdot)$. Note that, from (5.2) and (5.3), we see that the corresponding equations for ∇ and for $\tilde{\nabla}$ are identical. Moreover, (5.3) implies the relation

$$R_\gamma = \tilde{R}_\gamma - \frac{1}{4}\tilde{T}_\gamma^2. \tag{5.5}$$

Hence, $\tilde{R}_\gamma$ is a self-adjoint operator and from (5.1), $\tilde{T}_\gamma$ is skew-symmetric with respect to g.

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $(\mathfrak{m}, \langle , \rangle) \cong (T_oM, g_o)$. Since the parallel translation with respect to $\tilde{\nabla}$ of tangent vectors at o along γ coincides with the differential of $\exp tu \in G$ acting on M, it follows that the

$$E_i(t) = (\exp tu)_{*o} e_i, \quad i = 1, \dots, n,$$

are the vector fields along γ obtained by parallel translation of e_i with respect to $\tilde{\nabla}$. Hence, any vector field $V(t)$ along γ can be expressed as

$$V(t) = (\exp tu)_{*o} X(t)$$

with $X(t) = \sum_{i=1}^n X^i(t) e_i$, where $V(t) = \sum_{i=1}^n X^i(t) E_i(t)$. Since $\tilde{\nabla}\tilde{T} = \tilde{\nabla}\tilde{R} = 0$, the Jacobi equation (5.4) can be expressed as the differential equation

$$X'' - \tilde{T}_u X' + \tilde{R}_u X = 0 \tag{5.6}$$

in the vector space $\mathfrak{m}$, where $\tilde{T}_u X = \tilde{T}(u,X) = -[u,X]_{\mathfrak{m}}$ and $\tilde{R}_u X = \tilde{R}(u,X)u = [[u,X]_{\mathfrak{g}_o}, u]$ or, equivalently, as the system of second order linear differential equations

$$X^{j''}(t) + \sum_{i=1}^{n} \Big(-(\tilde{T}_u)_i^j X'^i(t) + (\tilde{R}_u)_i^j X^i(t) \Big) = 0, \tag{5.7}$$

$1 \leq j \leq n$, with constant coefficients $(\tilde{T}_u)_i^j$ and $(\tilde{R}_u)_i^j$ given by

$$(\tilde{T}_u)_i^j = \langle \tilde{T}_u e_i, e_j \rangle = g(\tilde{T}_\gamma E_i, E_j), \quad (\tilde{R}_u)_i^j = \langle \tilde{R}_u e_i, e_j \rangle = g(\tilde{R}_\gamma E_i, E_j).$$

Definition 5.1. Given a naturally reductive model $\mathfrak{M} = (\mathfrak{m}, \tilde{T}, \tilde{R}, \langle,\rangle)$, a solution $X = X(t)$ of (5.6) is called a *Jacobi solution* of $\mathfrak{M}$ with respect to u.

Each Jacobi solution X is then a curve in $\mathfrak{m}$ obtained as $X(t) = P(t) \cdot e^{mt}$, where m is a complex number and $P(t)$ is a complex vector-valued polynomial (see [18] for more information).

Next, suppose that the naturally reductive space $(M = G/G_o, g)$ has a G-invariant unit vector field U and put, as in Section 3, $u = U_o \in \mathfrak{m} = T_oM$. Then, using (2.3), $\tilde{R}_u v = 0$, for all $v \in \mathfrak{m}$ and, from (5.5), the Jacobi operator R_u is given by

$$R_u = -\frac{1}{4}\tilde{T}_u^2.$$

Since $\tilde{T}_u : \mathfrak{m} \to \mathfrak{m}$ is a skew-symmetric endomorphism, the rank of $\tilde{T}_u$ is an even number $2k \leq n$ and there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathfrak{m}$ and real positive numbers $c_1, \ldots, c_k$ such that

$$\begin{cases} \tilde{T}_u e_\alpha = c_\alpha e_{k+\alpha}, \quad \tilde{T}_u e_{k+\alpha} = -c_\alpha e_\alpha, & \alpha = 1, \ldots, k \\ \tilde{T}_u e_{2k+\beta} = 0, & \beta = 1, \ldots, n-2k. \end{cases} \tag{5.8}$$

Hence and taking into account that any G-invariant tensor field is parallel with respect to the canonical connection, we have the following result.

Lemma 5.2. *The rank of the Jacobi operator along the integral curves of U is constant and is equal to an even number $2k \leq n$; its eigenvalues are constants and its eigenspaces are $\tilde{\nabla}$-parallel.*

In what follows we denote by $-c_i^2$, $i = 1, \dots, l$, the $l \leq k$ different non-vanishing eigenvalues of $\tilde{T}_u^2$. Then the non-vanishing eigenvalues of R_u are $c_i^2/4$, $i = 1, \dots, l$. The corresponding eigenspaces are denoted by $\mathcal{V}_{c_i^2}$.

Theorem 5.3. *Let $(M = G/G_o, g)$ be a naturally reductive homogeneous space and U a G-invariant unit vector field on it. Then we have:*

(i) *if* $\operatorname{rank} R_u = 0$ *or, equivalently, U is parallel, M is locally a product of an $(n-1)$-dimensional naturally reductive space and the integral curves of U. Then $\gamma(t) = (\exp tu)o$ does not admit any conjugate point to the origin;*

(ii) *if* $\operatorname{rank} R_u = 2k$*, $k \geq 1$, then the conjugate points to the origin are all $\gamma(t)$ satisfying $c_i t \in 2\pi\mathbb{Z}^*$, for each $i = 1, \dots, l$, their multiplicity is* $2\dim \mathcal{V}_{c_i^2}$ *and they are not G-isotropically conjugate.*

Proof. If $\operatorname{rank} R_u = 0$, (5.2) implies that $\tilde{\nabla} U = \nabla U = 0$ and $\mathfrak{m} = \mathfrak{m}^n$ decomposes into $\mathfrak{m}^n = \mathfrak{m}^{n-1} \oplus \mathbb{R}U$, where $\mathfrak{m}^{n-1}$ is the $(n-1)$-dimensional naturally reductive model orthogonal to U (see [6]). From here and taking into account that (5.6) reduces to $X'' = 0$, we get (i).

Next, we show (ii). Suppose that $\operatorname{rank} R_u = \operatorname{rank} \tilde{T}_u = 2k$, $k \geq 1$. Because $\tilde{R}_u \cong 0$, the system (5.7) may be expressed as

$$Y^{j\prime}(t) - \sum_{i=1}^{n} (\tilde{T}_u)_i^j Y^i(t) = 0 \quad 1 \leq j \leq n,$$

where $Y^i(t) = X^{i\prime}(t)$, $i = 1, \dots, n$. From (5.8), it reduces to

$$\begin{cases} Y^{\alpha\prime}(t) + c_\alpha Y^{k+\alpha}(t) = 0, \\ Y^{k+\alpha\prime}(t) - c_\alpha Y^\alpha(t) = 0, \\ Y^{2k+\beta\prime} = 0. \end{cases} \tag{5.9}$$

Then the Jacobi solutions X such that $X(0) = 0$ are given by

$$\begin{aligned} X(t) = \sum_{\alpha=1}^{k} \Big(&(A_\alpha \sin c_\alpha t - B_\alpha (1 - \cos c_\alpha t)) e_\alpha \\ &+ (A_\alpha (1 - \cos c_\alpha t) + B_\alpha \sin c_\alpha t) e_{k+\alpha} \Big) + \sum_{\beta=1}^{n-2k} C_{2k+\beta} t e_{2k+\beta}, \end{aligned}$$

where A_α, B_α, $C_{2k+\beta}$ are constant. Hence, the values of t such that $X(t)$=0, $t \neq 0$, are the solutions of the following system

$$\begin{cases} A_\alpha \sin c_\alpha t - B_\alpha(1-\cos c_\alpha t) = 0, \\ A_\alpha(1-\cos c_\alpha t) + B_\alpha \sin c_\alpha t = 0, \qquad \alpha = 1,\dots,k, \end{cases}$$

which has non-trivial solutions if and only if

$$\prod_{\alpha=1}^{k}(1-\cos c_\alpha t) = 0.$$

So, the conjugate points $\gamma(t)$ are given by the condition $c_i t \in 2\pi\mathbb{Z}^*$, for all $i \in \{1,\dots,l\}$ and, using Proposition 3.6, they are not G-isotropically conjugate. Moreover, because the vector fields

$$\big(A_\alpha \sin c_\alpha t - B_\alpha(1-\cos c_\alpha t)\big)E_\alpha + \big(A_\alpha(1-\cos c_\alpha t) + B_\alpha \sin c_\alpha t\big)E_{k+\alpha}$$

along γ, for each $\alpha = 1,\dots,k$, generate the vector space of Jacobi vector fields V_α such that $V_\alpha(0) = V_\alpha(2p\pi/c_\alpha) = 0$, for all $p \in \mathbb{Z}^*$, it follows that the multiplicity of the conjugate point $\gamma(2p\pi/c_i)$ is $2\dim \mathcal{V}_{c_i^2}$. □

Remark 5.4. In [7], the simply connected Killing-transversally symmetric spaces are introduced as simply connected Riemannian manifolds equipped with a complete unit Killing vector field ξ such that all reflections with respect to the flow lines of ξ are global isometries. These manifolds are naturally reductive spaces and ξ is G-invariant with respect to a naturally reductive representation $(G/G_o, g)$. Moreover, the *canonical connection of the flow* $\mathfrak{F}_\xi$ is an adapted canonical connection $\tilde{\nabla}$ of (M,g) and, if the dual one-form of ξ with respect to g is a contact form, or equivalently, M is irreducible [7, Theorem 5.1], it follows from Theorem 5.3 that *the flow lines on* (M,g) *admit pairs of conjugate points which are not* G*-isotropic.*

Lemma 5.5. *Any non-locally symmetric naturally reductive space* (M,g) *of dimension* $n \le 5$ *admits a non-parallel* $\mathrm{Tr}(M,\tilde{\nabla})$*-invariant unit vector field.*

Proof. Since for dimension two any Riemannian homogeneous space obviously has constant curvature and hence is a locally symmetric space, we shall consider the cases $n = 3, 4, 5$. For $n = 3$, M is locally isometric to a unimodular Lie group ($SU(2)$, the universal covering of $SL(2,\mathbb{R})$ or the Heisenberg group) equipped with a suitable left-invariant metric [17, Theorem 6.5] and the unit eigenvector corresponding to the single root of the Ricci operator gives a $\mathrm{Tr}(M,\tilde{\nabla})$-invariant vector field U with $\operatorname{rank}\tilde{T}_u = 2$ [6, Theorem 6.12 (ii)]. If $n = 4$, M is locally isometric to a Riemannian product of the form $M^3 \times \mathbb{R}$, where M^3 is a three-dimensional unimodular Lie group as before [13, Theorem 1] and the single root of the Ricci operator on M^3 determines a $\mathrm{Tr}(M,\tilde{\nabla})$-invariant vector field U with $\operatorname{rank}\tilde{T}_u = 2$ [6, Theorem 6.15]. Finally, we study the case $n = 5$. Let $\mathfrak{M} = (\mathfrak{m}, \tilde{T}, \tilde{R}, \langle,\rangle)$ be the infinitesimal model determined by $(M,\tilde{\nabla})$. Then $\tilde{R} \neq 0$ and $\tilde{T} \neq 0$ [14, Lemma 1.1]. If $\mathfrak{M}$ is non-decomposable, then there exists an invariant vector $u \in \mathfrak{m}$ such that $\operatorname{rank}\tilde{T}_u = 4$ [14, Propositions 2.2 and 2.4] (see also [6, Theorem 6.16]). If $\mathfrak{M}$ is decomposable, then $\mathfrak{M} = \mathfrak{M}^2 \oplus \mathfrak{M}^3$, where $\mathfrak{M}^2$ is a two-dimensional symmetric model and $\mathfrak{M}^3$ is a three-dimensional indecomposable naturally reductive model. Then M is locally isometric to a Riemannian product $M^2 \times M^3$, and again, the unit eigenvector corresponding to the single root of the Ricci operator on M^3 gives a $\mathrm{Tr}(M,\tilde{\nabla})$-invariant vector field U with $\operatorname{rank}\tilde{T}_u = 2$ [6, Theorem 6.17]. □

Now, Theorem 5.3 together with this lemma yield the following result:

Theorem 5.6. *Let $(M = G/G_o, g)$ be a naturally reductive space of dimension ≤ 5 and $\tilde{\nabla}$ some of its adapted canonical connections. If all Jacobi fields vanishing at two points are $\mathrm{Tr}(M,\tilde{\nabla})$-isotropic, then (M,g) is locally symmetric.*

For the three-dimensional case, we prove

Lemma 5.7. *Let $(M = G/G_o, g)$ be a non-locally symmetric naturally reductive space of dimension three and let $\tilde{\nabla}$ be an adapted canonical connection. Then, we have*

$$G = \mathrm{Tr}(M,\tilde{\nabla}) = I_o(M,g).$$

Proof. It follows again from [17, Theorem 6.5] and from [11] that $\dim I(M,g) = 4$. On the other hand, in [17, p. 64] it is proved that the holonomy algebra of $\tilde{\nabla}$ is one-dimensional and consequently, $\dim \mathrm{Tr}(M,\tilde{\nabla}) = 4$. Since $\mathrm{Tr}(M,\tilde{\nabla}) \subseteq G \subseteq I_o(M,g)$, we obtain the equality. □

Then, we have

Theorem 5.8. *All three-dimensional naturally reductive spaces with the property that all Jacobi fields vanishing at two points are isotropic are locally symmetric.*

References

[1] J. Berndt, F. Tricerri and L. Vanhecke, Generalized Heisenberg groups and Damek–Ricci harmonic spaces, Lecture Notes in Math. 1598, *Springer-Verlag, Berlin, Heidelberg, New York*, 1995.

[2] R. Bott and H. Samelson, Applications of Morse theory to symmetric spaces, *Amer. J. of Math.* **80** (1958), 964–1029.

[3] I. Chavel, Isotropic Jacobi fields, and Jacobi's equation on Riemannian homogeneous spaces, *Comm. Math. Helv.* **42** (1967), 237–248.

[4] I. Chavel, On normal Riemannian homogeneous spaces of rank 1, *Bull. Amer. Math. Soc.* **73** (1967), 477–481.

[5] D. Chinea, J. C. González-Dávila, Quasi-Sasakian homogeneous structures on the generalized Heisenberg group $H(p, 1)$, *Proc. Amer. Math. Soc.* **105** (1989), 173–184.

[6] O. Gil-Medrano, J. C. González-Dávila and L. Vanhecke, Harmonic and minimal invariant unit vector fields on homogeneous Riemannian manifolds, *Houston J. Math.* **27** (2001), 377–409.

[7] J. C. González-Dávila, M. C. González-Dávila and L. Vanhecke, Classification of Killing-transversally symmetric spaces, *Tsukuba J. Math.* **20** (1996), 321–347.

[8] S. Helgason, Differential geometry, Lie groups and symmetric spaces, *Academic Press, New York*, 1978.

[9] S. Kobayashi and K. Nomizu, Foundations of differential geometry, I, II, *Interscience Publishers, New York*, 1963, 1969.

[10] O. Kowalski, Generalized symmetric spaces, *Lecture Notes in Math. 805, Springer-Verlag, Berlin, Heidelberg, New York* (1980).

[11] O. Kowalski, Spaces with volume-preserving symmetries and related classes of Riemannian manifolds, Rend. Sem. Mat. Univ. Politec. Torino, Fascicolo Speciale Settembre 1983, 131–158.

[12] O. Kowalski and F. Tricerri, Riemannian manifolds of dimension $n \leq 4$ admitting a homogeneous structure of class T_2, jour Conferenze del Seminario di Matematica Univ. di Bari, 222, *Laterza, Bari*, 1987.

[13] O. Kowalski and L. Vanhecke, Four-dimensional naturally reductive homogeneous spaces, Rend. Sem. Mat. Univ. Politec. Torino, *Fascicolo Speciale Settembre 1983*, 223–232.

[14] O. Kowalski and L. Vanhecke, Classification of five-dimensional naturally reductive spaces, *Math. Proc. Cambridge Philos. Soc.* **97** (1985), 445–463.

[15] F. G. Lastaria and F. Tricerri, Curvature-orbits and locally homogeneous Riemannian manifolds, *Ann. Mat. Pura Appl.* **165** (1993), 121–131.

[16] K. Nomizu, Invariant affine connections on homogeneous spaces, *Amer. J. Math.* **76** (1954), 33–65.

[17] F. Tricerri and L. Vanhecke, Homogeneous structures on Riemannian manifolds, *London Math. Soc. Lect. Note Series* 83, *Cambridge Univ. Press, Cambridge*, 1983.

[18] W. Ziller, The Jacobi equation on naturally reductive compact Riemannian homogeneous spaces, *Comm. Math. Helv.* **52** (1977), 573–590.

J. C. GONZÁLEZ-DÁVILA
DEPARTAMENTO DE MATEMÁTICA FUNDAMENTAL
SECCIÓN DE GEOMETRÍA Y TOPOLOGÍA
UNIVERSIDAD DE LA LAGUNA
LA LAGUNA
SPAIN

E-mail: jcgonza@ull.es

R. O. SALAZAR
DEPARTAMENTO DE MATEMÁTICA FUNDAMENTAL
SECCIÓN DE GEOMETRÍA Y TOPOLOGÍA
UNIVERSIDAD DE LA LAGUNA
LA LAGUNA
SPAIN

E-mail: omsalaza@ull.es

(Received March 18, 2003; revised May 12, 2003)