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Mann iterative algorithm for a system of operator inclusions

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Abstract. In this paper, we introduce and study a new system of operator inclusions in Hilbert spaces. We prove the existence and uniqueness of solution for this system of operator inclusions. We also construct a new Mann iterative algorithm for approximating the solution of this system of operator inclusions and discuss the convergence analysis of the algorithm.

1. Introduction and preliminaries

Let H be a Hilbert space and $T : H \to 2^H$ be a multivalued operator, where 2^H denotes the family of all the nonempty subsets of H. The operator inclusion problem formulated by finding $u \in H$ such that $0 \in T(u)$ has been studied extensively because of its role in modelization of unilateral problems, nonlinear dissipative systems, variational inequalities, complementarity problems, convex optimizations, equilibrium problems, etc. For details, we refer to [1]–[3], [8]–[18], [24]–[26], [30] and the references therein.

Recently, some new and interesting problems were considered by some authors. They are systems of variational inequalities, systems of complementarity problems, and systems of equilibrium problems (see [4]–[6], [19],

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[20], [22], [27]–[29]). In 1999, Ansari et al. [6] studied a system of variational inequalities by the fixed point theorem. In the paper [20], Kassay and Kolumbán introduced a system of variational inequalities and proved an existence theorem by Ky Fan lemma. Recently, Kassay, Kolumbán and Páles [22] introduced and studied Minty and Stampacchia variational inequality systems by Kakutani–Fan–Glicksberg fixed point theorem. Very recently, Huang and Fang [19] introduced a system of order complementarity problems and established some existence results by fixed point theory. The study of systems of variational inequalities is interesting and important because of the fact that a Nash equilibrium problem for differentiable functions can be formulated in the form of a variational inequality problem over product of sets (see [7]). In the paper [4], Ansari and Khan further pointed out the equivalence of a system of variational inequalities and a variational inequality over product of sets. On the other hand, up to now, only a few iterative algorithms have been constructed for approximating solution of a system of variational inequalities in Hilbert spaces.

Motivated and inspired by the above works, in this paper, we introduce and study a new system of operator inclusions in Hilbert spaces, which includes the systems of variational inequalities considered in [4], [6], [20], [22] as special cases. We prove the existence and uniqueness of solution for this system of operator inclusions. We also construct a new Mann iterative algorithm for approximating the solution of this system of operator inclusions and discuss the convergence analysis of the algorithm.

In the following, unless otherwise specified, we always suppose that H is a Hilbert space with inner $\langle \cdot\,, \cdot \rangle$ and norm $\| \cdot \|$. For our results, we need some concepts and results.

Definition 1.1. Let $T : H \to H$ be a mapping. T is said to be strongly monotone with constant r if there exists some constant $r > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq r\|x - y\|^2, \quad \forall\, x, y \in H.$$

Definition 1.2. A mapping $T : H \to H$ is said to be Lipschitz continuous with constant s if there exists some constant $s > 0$ such that

$$\|Tx - Ty\| \leq s\|x - y\|, \quad \forall\, x, y \in H.$$

Remark 1.1. If a mapping is both strongly monotone with constant r and Lipschitz continuous with constant s, then $r \leq s$.

Definition 1.3. A multi-valued mapping $M : H \to 2^H$ is said to be

(1) monotone if

$$\langle x - y, u - v \rangle \geq 0, \quad \forall\, u, v \in H, x \in M(u), \quad \text{and } y \in M(v);$$

(2) maximal monotone if M is monotone and $(I + \lambda M)(H) = H$ for every (equivalently, for some)$\lambda > 0$, where I denotes the identity mappings on H.

Definition 1.4 (See [9]). Let $M : H \to 2^H$ be a maximal monotone mapping. The resolvent operator $J_M^\lambda : H \to H$ is defined by

$$J_M^\lambda(x) = (I + \lambda M)^{-1}(x), \quad \forall\, x \in H,$$

where $\lambda > 0$ is a constant.

Lemma 1.1 (See [9]). *Let $M : H \to 2^H$ be a maximal monotone mapping. Then J_M^λ is nonexpensive, i.e.,*

$$\|J_M^\lambda(x) - J_M^\lambda(y)\| \leq \|x - y\|, \quad \forall\, x, y \in H.$$

In what follows, unless otherwise specified, we always suppose that H_1 and H_2 are two real Hilbert spaces, $A \subset H_1$ and $B \subset H_2$ are two nonempty, closed and convex sets. Let $F : H_1 \times H_2 \to H_1$ and $G : H_1 \times H_2 \to H_2$ be two mappings, $M : H_1 \to 2^{H_1}$ and $N : H_2 \to 2^{H_2}$ be two maximal monotone mappings. The system of operator inclusions is formulated by finding $(a, b) \in H_1 \times H_2$ such that

$$\begin{cases} 0 \in F(a, b) + M(a); \\ 0 \in G(a, b) + N(b). \end{cases} \tag{1.1}$$

If $M(x) = \partial\varphi(x)$ and $N(y) = \partial\phi(y)$ for all $x \in H_1$ and $y \in H_2$, where $\varphi : H_1 \to R \cup \{+\infty\}$ and $\phi : H_2 \to R \cup \{+\infty\}$ are two proper, convex and lower semi-continuous functionals, $\partial\varphi$ and $\partial\phi$ denote the subdifferetial operators of φ and ϕ, respectively, then problem (1.1) reduces to the following problem: find $(a, b) \in A \times B$ such that

$$\begin{cases} \langle F(a, b), x - a \rangle + \varphi(x) - \varphi(a) \geq 0, \quad \forall\, x \in H_1, \\ \langle G(a, b), y - b \rangle + \phi(y) - \phi(b) \geq 0, \quad \forall\, y \in H_2, \end{cases} \tag{1.2}$$

which is called a system of nonlinear variational inequalities.

If $M(x) = \partial\delta_A(x)$ and $N(y) = \partial\delta_B(y)$ for all $x \in H_1$ and $y \in H_2$, where δ_A and δ_B denote the indicator functions of A and B, respectively, then problem (1.1) reduces to the following problem: find $(a,b) \in A \times B$ such that

$$\begin{cases} \langle F(a,b), x-a\rangle \geq 0, & \forall\, x \in A, \\ \langle G(a,b), y-b\rangle \geq 0, & \forall\, y \in B, \end{cases} \tag{1.3}$$

which is just the problem in [20] with both F and G being single-valued.

The purpose of this paper is to prove the existence and uniqueness of solution for problem (1.1) and construct a Mann iterative algorithm to approximate the unique solution of problem (1.1).

2. Existence and uniqueness

For the main results, we give a characterization of solution of problem (1.1) as follows:

Lemma 2.1. *For any given $(a,b) \in H_1 \times H_2$, (a,b) is a solution of problem* (1.1) *if and only if (a,b) satisfies*

$$\begin{cases} a = J_M^{\lambda}[a - \lambda F(a,b)], \\ b = J_N^{\beta}[b - \beta G(a,b)], \end{cases}$$

where $\lambda > 0$ and $\beta > 0$ are two constants.

Proof. The conclusion directly follows from Definition 1.4. □

Theorem 2.1. *Let $M : H_1 \to 2^{H_1}$ and $N : H_2 \to 2^{H_2}$ be two maximal monotone mappings. Let $F : H_1 \times H_2 \to H_1$ be a mapping such that for any given $(a,b) \in H_1 \times H_2$, $F(\cdot\,,b)$ is strongly monotone and Lipschitz continuous with constants r_1 and s_1, respectively, and $F(a,\cdot)$ is Lipschitz continuous with constant τ. Let $G : H_1 \times H_2 \to H_2$ be a mapping such that for any given $(x,y) \in H_1 \times H_2$, $G(x,\cdot)$ is strongly monotone and Lipschitz continuous with constant r_2 and s_2, and $G(\cdot\,,y)$ is Lipschitz continuous with constant ξ. If $\xi < r_1$ and $\tau < r_2$, then problem* (1.1) *admits a unique solution.*

Proof. Choose $\rho > 0$ such that

$$\rho < \min\left\{\frac{2(r_2-\tau)}{s_2^2-\tau^2}, \frac{2(r_1-\xi)}{s_1^2-\xi^2}\right\}. \tag{2.1}$$

Define $T_\rho : H_1 \times H_2 \to H_1$ and $S_\rho : H_1 \times H_2 \to H_2$ by

$$T_\rho(u,v) = J_M^\rho[u-\rho F(u,v)] \quad \text{and} \quad S_\rho(u,v) = J_N^\rho[v-\rho G(u,v)] \tag{2.2}$$

for all $(u,v) \in H_1 \times H_2$.

For any $(u_1,v_1),(u_2,v_2) \in H_1\times H_2$, it follows from (2.2) and Lemma 1.1 that

$$\begin{aligned}\|T_\rho(u_1,v_1) - T_\rho(u_2,v_2)\| &\\ \le \|u_1-u_2-\rho(F(u_1,v_1)-F(u_2,v_2))\| &\\ \le \|u_1-u_2-\rho(F(u_1,v_1)-F(u_2,v_1))\| &\\ +\rho\|F(u_2,v_1)-F(u_2,v_2)\| &\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\|S_\rho(u_1,v_1) - S_\rho(u_2,v_2)\| &\\ \le \|v_1-v_2-\rho(G(u_1,v_1)-G(u_2,v_2))\| &\\ \le \|v_1-v_2-\rho(G(u_1,v_1)-G(u_1,v_2))\| &\\ +\rho\|G(u_1,v_2)-G(u_2,v_2)\|. &\end{aligned} \tag{2.4}$$

By assumptions, we have

$$\begin{aligned}&\|u_1-u_2-\rho(F(u_1,v_1)-F(u_2,v_1))\|^2\\ &\quad= \|u_1-u_2\|^2 - 2\rho\langle F(u_1,v_1)-F(u_2,v_1), u_1-u_2\rangle\\ &\qquad+\rho^2\|F(u_1,v_1)-F(u_2,v_1)\|^2\\ &\quad\le (1-2\rho r_1+\rho^2 s_1^2)\|u_1-u_2\|^2,\end{aligned} \tag{2.5}$$

$$\begin{aligned}&\|v_1-v_2-\rho(G(u_1,v_1)-G(u_1,v_2))\|^2\\ &\quad= \|v_1-v_2\|^2 - 2\rho\langle G(u_1,v_1)-G(u_1,v_2), v_1-v_2\rangle\\ &\qquad+\rho^2\|G(u_1,v_1)-G(u_1,v_2)\|^2\\ &\quad\le (1-2\rho r_2+\rho^2 s_2^2)\|v_1-v_2\|^2,\end{aligned} \tag{2.6}$$

$$\|F(u_2,v_1)-F(u_2,v_2)\| \le \tau\|v_1-v_2\|, \tag{2.7}$$

and

$$\|G(u_1,v_2)-G(u_2,v_2)\| \le \xi\|u_1-u_2\|. \tag{2.8}$$

It follows from (2.3)–(2.8) that

$$\begin{cases} \|T_\rho(u_1,v_1)-T_\rho(u_2,v_2)\| \\ \qquad \le \sqrt{1-2\rho r_1+\rho^2 s_1^2}\,\|u_1-u_2\|+\rho\tau\|v_1-v_2\|, \\ \|S_\rho(u_1,v_1)-S_\rho(u_2,v_2)\| \\ \qquad \le \sqrt{1-2\rho r_2+\rho^2 s_2^2}\,\|v_1-v_2\|+\rho\xi\|u_1-u_2\|. \end{cases} \tag{2.9}$$

(2.9) implies that

$$\begin{aligned} &\|T_\rho(u_1,v_1)-T_\rho(u_2,v_2)\|+\|S_\rho(u_1,v_1)-S_\rho(u_2,v_2)\| \\ &\quad\le \left(\sqrt{1-2\rho r_1+\rho^2 s_1^2}+\rho\xi\right)\|u_1-u_2\| \\ &\qquad+\left(\sqrt{1-2\rho r_2+\rho^2 s_2^2}+\rho\tau\right)\|v_1-v_2\| \\ &\quad\le \max\left\{\sqrt{1-2\rho r_1+\rho^2 s_1^2}+\rho\xi,\sqrt{1-2\rho r_2+\rho^2 s_2^2}+\rho\tau\right\} \\ &\qquad\times(\|u_1-u_2\|+\|v_1-v_2\|). \end{aligned} \tag{2.10}$$

Now define $\|\cdot\|_1$ on $H_1\times H_2$ by

$$\|(u,v)\|_1=\|u\|+\|v\|,\quad \forall\,(u,v)\in H_1\times H_2.$$

It is easy to see that $(H_1\times H_2,\|\cdot\|_1)$ is a Banach space. Define $Q_\rho : H_1\times H_2\to H_1\times H_2$ by

$$Q_\rho(u,v)=(T_\rho(u,v),S_\rho(u,v)),\quad \forall\,(u,v)\in H_1\times H_2. \tag{2.11}$$

Let

$$k=\max\left\{\sqrt{1-2\rho r_1+\rho^2 s_1^2}+\rho\xi,\sqrt{1-2\rho r_2+\rho^2 s_2^2}+\rho\tau\right\}.$$

Since $\xi<r_1$ and $\tau<r_2$, it is easy to see that $0\le k<1$ from (2.1) and Remark 1.1. It follows from (2.10) and (2.11) that

$$\|Q_\rho(u_1,v_1)-Q_\rho(u_2,v_2)\|_1\le k\|(u_1,v_1)-(u_2,v_2)\|_1.$$

This proves that $Q_\rho : H_1\times H_2\to H_1\times H_2$ is a contractive mapping. Hence there exists a unique $(a,b)\in H_1\times H_2$ such that

$$Q_\rho(a,b)=(a,b),$$

i.e.,

$$\begin{cases} a = J_M^\rho[a - \rho F(a,b)], \\ b = J_N^\rho[b - \rho G(a,b)]. \end{cases}$$

By Lemma 2.1, (a,b) is the unique solution of problem (1.1). □

The following existence results are immediate consequences of Theorem 2.1.

Theorem 2.2. *Let $\varphi : H_1 \to R \cup \{+\infty\}$ and $\phi : H_2 \to R \cup \{+\infty\}$ be two proper convex lower semicontinuous functionals. Let $F : H_1 \times H_2 \to H_1$ be a mapping such that for any given $(a,b) \in H_1 \times H_2$, $F(\cdot\,,b)$ is strongly monotone and Lipschitz continuous with constants r_1 and s_1, respectively, and $F(a,\cdot)$ is Lipschitz continuous with constant τ. Let $G : H_1 \times H_2 \to H_2$ be a mapping such that for any given $(x,y) \in H_1 \times H_2$, $G(x,\cdot)$ is strongly monotone and Lipschitz continuous with constant r_2 and s_2, and $G(\cdot\,,y)$ is Lipschitz continuous with constant ξ. If $\xi < r_1$ and $\tau < r_2$, then problem* (1.2) *admits a unique solution.*

Theorem 2.3. *Let $F : A \times B \to H_1$ be a mapping such that for any given $(a,b) \in A \times B$, $F(\cdot\,,b)$ is strongly monotone and Lipschitz continuous with constants r_1 and s_1, respectively, and $F(a,\cdot)$ is Lipschitz continuous with constant τ. Let $G : A \times B \to H_2$ be a mapping such that for any given $(x,y) \in A \times B$, $G(x,\cdot)$ is strongly monotone and Lipschitz continuous with constant r_2 and s_2, and $G(\cdot\,,y)$ is Lipschitz continuous with constant ξ. If $\xi < r_1$ and $\tau < r_2$, then problem (1.3) admits a unique solution.*

3. Iterative algorithms and convergence

In this section, we construct a Mann iterative algorithm to approximate the unique solution of problem (1.1) and discuss the convergence analysis of the algorithm. As consequence, Mann iterative algorithms for problems (1.2) and (1.3) are defined and the convergence of the iterative sequences are proved, too.

Lemma 3.1. *Let $\{c_n\}$ and $\{k_n\}$ be two real sequences of nonnegative numbers satisfying the following conditions:*

(i) $0 \leq k_n < 1$, $n = 0, 1, 2 \dots$ *and* $\limsup_n k_n < 1$;

(ii) $c_{n+1} \leq k_n c_n, \quad n = 0, 1, 2 \dots$.

Then c_n *converges to* 0 *as* $n \to \infty$.

Proof. Condition (ii) implies that $\{c_n\}$ is decreasing and so c_n has a limit c. Suppose by contradiction that $c \neq 0$. Choose a subsequence $\{k_{n_j}\} \subset \{k_n\}$ such that k_{n_j} converges to $\limsup_n k_n$ as $j \to \infty$. By condition (ii), $c_{n_j} \leq k_{n_j} c_{n_j}$ and so $c \leq (\limsup_n k_n)c$, which contradicts condition (i). Hence c_n converges to 0 as $n \to \infty$. □

Theorem 3.1. *Let* $M : H_1 \to 2^{H_1}$ *and* $N : H_2 \to 2^{H_2}$ *be two maximal monotone mappings,* $F : H_1 \times H_2 \to H_1$ *and* $G : H_1 \times H_2 \to H_2$ *be two mappings. Assume that all the conditions of Theorem 2.1 hold. For any given* $(a_0, b_0) \in H_1 \times H_2$*, define Mann iterative sequences* $\{(a_n, b_n)\}$ *by*

$$\begin{cases} a_{n+1} = \alpha_n a_n + (1 - \alpha_n) J_M^\rho [a_n - \rho F(a_n, b_n)], \\ b_{n+1} = \alpha_n b_n + (1 - \alpha_n)) J_N^\rho [b_n - \rho G(a_n, b_n)], \end{cases} \tag{3.1}$$

where

$$0 \leq \alpha_n < 1 \quad \text{and} \quad \limsup_n \alpha_n < 1. \tag{3.2}$$

Then (a_n, b_n) *converges strongly to the unique solution* (a, b) *of problem* (1.1).

Proof. By Theorem 2.1, problem (1.1) admits a unique solution (a, b). It follows from Lemma 2.1 that

$$\begin{cases} a = \alpha_n a + (1 - \alpha_n) J_M^\rho [a - \rho F(a, b)], \\ b = \alpha_n b + (1 - \alpha_n) J_N^\rho [b - \rho G(a, b)]. \end{cases} \tag{3.3}$$

By (3.1) and (3.3),

$$\begin{aligned} &\|a_{n+1} - a\| \\ &\quad \leq \alpha_n \|a_n - a\| + (1 - \alpha_n) \|J_M^\rho [a_n - \rho F(a_n, b_n)] - J_M^\rho [a - \rho F(a, b)]\| \\ &\quad \leq \alpha_n \|a_n - a\| + (1 - \alpha_n) \|a_n - a - \rho(F(a_n, b_n) - F(a, b))\| \\ &\quad \leq \alpha_n \|a_n - a\| + (1 - \alpha_n) \|a_n - a - \rho(F(a_n, b_n) - F(a, b_n))\| \\ &\qquad + (1 - \alpha_n)\rho \|F(a, b_n) - F(a, b)\| \end{aligned}$$

$$\leq \alpha_n\|a_n - a\| + (1-\alpha_n)\sqrt{1-2\rho r_1+\rho^2 s_1^2}\,\|a_n - a\| \\ + (1-\alpha_n)\rho\tau\|b_n - b\| \tag{3.4}$$

and

$$\begin{aligned}
&\|b_{n+1} - b\| \\
&\quad\leq \alpha_n\|b_n - b\| + (1-\alpha_n)\|J_N^\rho[b_n - \rho G(a_n, b_n)] - J_N^\rho[b - \rho G(a,b)]\| \\
&\quad\leq \alpha_n\|b_n - b\| + (1-\alpha_n)\|b_n - b - \rho(G(a_n, b_n) - G(a,b))\| \\
&\quad\leq \alpha_n\|b_n - b\| + (1-\alpha_n)\|b_n - b - \rho(G(a_n, b_n) - G(a_n,b))\| \\
&\qquad + (1-\alpha_n)\rho\|G(a_n, b) - G(a,b)\| \\
&\quad\leq \alpha_n\|b_n - b\| + (1-\alpha_n)\sqrt{1-2\rho r_2+\rho^2 s_2^2}\,\|b_n - b\| \\
&\qquad + (1-\alpha_n)\rho\xi\|a_n - a\|.
\end{aligned} \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned}
&\|a_{n+1} - a\| + \|b_{n+1} - b\| \\
&\quad\leq \alpha_n\big(\|a_n - a\| + \|b_n - b\|\big) + (1-\alpha_n)k\big(\|a_n - a\| + \|b_n - b\|\big) \\
&\quad= (k + (1-k)\alpha_n)\big(\|a_n - a\| + \|b_n - b\|\big),
\end{aligned} \tag{3.6}$$

where $0 \leq k < 1$ is defined by

$$k = \max\Big\{\sqrt{1-2\rho r_1+\rho^2 s_1^2} + \rho\xi, \sqrt{1-2\rho r_2+\rho^2 s_2^2} + \rho\tau\Big\}.$$

Let

$$c_n = \|a_n - a\| + \|b_n - b\| \quad \text{and} \quad k_n = k + (1-k)\alpha_n.$$

Then (3.6) can be rewritten as

$$c_{n+1} \leq k_n c_n, \quad n = 0, 1, 2\ldots,$$

By (3.2), we know that $\limsup_n k_n < 1$. It follows from Lemma 3.1 that

$$\|a_n - a\| + \|b_n - b\| \to 0, \quad \text{as } n \to \infty.$$

Therefore, (a_n, b_n) converges strongly to the unique solution (a, b) of problem (1.1). □

By Theorem 3.1, we have the following results:

Theorem 3.2. *Let* $F : H_1 \times H_2 \to H_1$ *and* $G : H_1 \times H_2 \to H_2$ *be two mappings,* $\varphi : H_1 \to R \cup \{+\infty\}$ *and* $\phi : H_2 \to R \cup \{+\infty\}$ *be two proper convex lower semicontinuous functionals. Assume that all the conditions of Theorem 2.2 hold. For any given* $(a_0, b_0) \in H_1 \times H_2$*, define Mann iterative sequences* $\{(a_n, b_n)\}$ *by*

$$\begin{cases} a_{n+1} = \alpha_n a_n + (1 - \alpha_n) J_{\varphi}^{\rho}[a_n - \rho F(a_n, b_n)], \\ b_{n+1} = \alpha_n b_n + (1 - \alpha_n)) J_{\phi}^{\rho}[b_n - \rho G(a_n, b_n)], \end{cases}$$

where

$$0 \leq \alpha_n < 1 \quad \text{and} \quad \limsup_n \alpha_n < 1. \tag{3.7}$$

Then (a_n, b_n) *converges strongly to the unique solution* (a, b) *of problem* (1.2)*.*

Theorem 3.3. *Let* $F : A \times B \to H_1$ *and* $G : A \times B \to H_2$ *be two mappings. Assume that all the conditions of Theorem 2.3 hold. For any given* $(a_0, b_0) \in A \times B$*, define the Mann iterative sequence* $\{(a_n, b_n)\}$ *by*

$$\begin{cases} a_{n+1} = \alpha_n a_n + (1 - \alpha_n) P_A[a_n - \rho F(a_n, b_n)], \\ b_{n+1} = \alpha_n b_n + (1 - \alpha_n)) P_B[b_n - \rho G(a_n, b_n)], \end{cases}$$

where

$$0 \leq \alpha_n < 1 \quad \text{and} \quad \limsup_n \alpha_n < 1$$

Then (a_n, b_n) *converges strongly to the unique solution* (a, b) *of problem* (1.3)*.*

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