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Warped product contact CR-submanifolds of Sasakian space forms

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Abstract. In this paper we study contact CR products and contact CR-warped products (in the sense of B. Y. Chen [19]) in Sasakian manifolds. We show that a contact CR submanifold M of a Sasakian manifold with $\xi \in \mathcal{D}$ and with parallel f-structure P is a CR-product of an integral curve of ξ and a ϕ-anti-invariant submanifold of $\widetilde{M}$. If M is a strictly proper contact CR-product in S^7 with $\|B\| = \sqrt{6}$, then M is the Riemannian product between S^3 and S^1 and up to a rigid transformation of $\mathbf{R}^8$ the embedding is given by $r : S^3 \times S^1 \longrightarrow S^7 \hookrightarrow \mathbf{R}^8$, $r(x_1, y_1, x_2, y_2, u, v) = (x_1u, y_1u, -y_1v, x_1v, x_2u, y_2u, -y_2v, x_2v)$. Then we prove that if $M = N^\perp \times_f N^T$ is a warped product contact CR-submanifold such that $N^\perp$ is ϕ-anti-invariant and N^T is ϕ-invariant, then M is a CR-product. Next, we define a contact CR-warped product and we show that the second fundamental form of a contact CR warped product of a Sasakian space form satisfies a "good" inequality, namely $\|B\|^2 \geq 2p\left[\|\nabla \ln f\|^2 - \Delta \ln f + \frac{c+3}{2}\ s + 1\right]$.

1. Introduction

A submanifold M of a Hermitian manifold $(\widetilde{M}, J, \widetilde{g})$ is a CR *submanifold* if it carries a holomorphic distribution $\mathcal{D}$ i.e. $J_x(\mathcal{D}_x) = \mathcal{D}_x$, for any $x \in M$, such that the orthogonal complement (with respect to $g = j^*\widetilde{g}$)

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$\mathcal{D}^\perp$ of $\mathcal{D}$ in $T(M)$ is anti-invariant, i.e. $J_x(\mathcal{D}_x^\perp) \subseteq T(M)_x^\perp$, for any $x \in M$, where $T(M)^\perp$ is the normal bundle (of the given immersion $j : M \subset \widetilde{M}$). CR-submanifolds were first considered by A. Bejancu, [6], in an effort to unify notions such as complex ($\mathcal{D}^\perp = (0)$), anti-invariant ($\mathcal{D} = (0)$), totally real ($T(M) \cap JT(M) = (0)$), or generic ($J\mathcal{D}^\perp = T(M)^\perp$) submanifolds. Although it had been known for some time (cf. A. Andreotti and C. D. Hill, [3]) that real analytic CR manifolds are at least locally embedable, it appears that the notion of a CR submanifold was introduced independently of the theory of CR manifolds (cf. e.g. S. Greenfield, [29]), and it was not until the result by D. E. Blair and B-Y. Chen, [13], that CR submanifolds $(M, \mathcal{D})$ where recognized to posses an (integrable) CR structure $T_{1,0}(M) = \{X - \sqrt{-1}JX : X \in \mathcal{D}\}$ (provided they are *proper*, i.e. $\mathcal{D} \neq (0)$ and $\mathcal{D}^\perp \neq (0)$). Also, the study of CR submanifolds was confined to Kählerian ambient spaces (cf. also B-Y. Chen, [19]). Subsequently, the theory of CR submanifolds was developed to include ambient spaces such as locally conformal Kähler manifolds (cf. e.g. D. E. Blair and S. Dragomir, [14], S. Dragomir and L. Ornea, [27], N. Papaghiuc, [38], M. H. Shahid, [42]), quasi and nearly Kähler manifolds (cf. e.g. S. H. Kon and S. L. Tan, [33], T. Sasahara, [41]), or quaternionic Kähler manifolds (cf. e.g. B. J. Papantoniou and M. H. Shahid, [40]). Another line of thought, similar to that concerning Sasakian geometry as an odd dimensional version of Kählerian geometry (cf. D. E. Blair, [11]), led to the concept of a *contact CR-submanifold*, that is a submanifold M of an almost contact Riemannian manifold $(\widetilde{M}, (\phi, \xi, \widetilde{\eta}, \widetilde{g}))$ carrying an invariant distribution $\mathcal{D}$, i.e. $\phi_x \mathcal{D}_x \subseteq \mathcal{D}_x$, for any $x \in M$, such that the orthogonal complement $\mathcal{D}^\perp$ of $\mathcal{D}$ in $T(M)$ is anti-invariant, i.e. $\phi_x \mathcal{D}_x^\perp \subseteq T(M)_x^\perp$, for any $x \in M$. This notion was already used by A. Bejancu and N. Papaghiuc in [8] by using the terminology of semi-invariant submanifold. It is customary to require that ξ be tangent to M (cf. K. Yano and M. Kon, [49]), rather than normal which is too restrictive (by Prop. 1.1 in [49], p. 43, M must be anti-invariant, i.e. $\phi_x T_x(M) \subseteq T(M)_x^\perp$, $x \in M$), or oblique (which leads to highly complicated embedding equations). Although a formal analogue to the notion of a CR-submanifold to start with, contact CR-submanifolds turn out to have a precise geometric meaning by combining a result by S. Ianuş, [31] (according to which any *normal* almost contact Riemannian manifold is actually a CR-manifold, with the

CR structure $T_{1,0}(\widetilde{M}) = \{X - \sqrt{-1}\phi X : X \in \mathrm{Ker}(\eta)\}$) and the observation that a contact CR-manifold is a CR-manifold (with the induced CR-structure $T_{1,0}(M) = [T(M)\otimes \mathbf{C}] \cap T_{1,0}(\widetilde{M})$). Any hypersurface M of a Sasakian manifold $\widetilde{M}$ is a contact CR-submanifold and a nondegenerated CR-manifold of CR codimension two. Of course, the inclusion $j : M \to \widetilde{M}$ is a CR *immersion*, i.e. an immersion and a CR map. A theory of CR immersions, related to certain aspects of analysis in complex variables, has been started by S. Webster, [48]. There one is interested in rigidity of CR submanifolds $j : M \subset S^{2n+1}$ (up to a fractional linear transformation of S^{2n+1}), the ambient Levi–Civita connection appearing in the theory of Riemannian immersions (cf. [18]) is replaced by the Tanaka–Webster connection of S^{2n+1} (cf. [47] and S. Dragomir, [26]) thus producing CR, or pseudohermitian, analogs to the Gauss–Weingarten and Gauss–Ricci–Codazzi equations, and the relationship between the resulting theory (of CR and pseudohermitian immersions, cf. also E. Barletta and S. Dragomir, [4], S. Dragomir, [25]) and the geometry of the second fundamental form of j is perhaps not sufficiently clear, at the present state of research. Given a contact CR submanifold M of a Sasakian manifold, $\widetilde{M}$ either $\xi \in \mathcal{D}$, or $\xi \in \mathcal{D}^\perp$. Therefore, the tangent space at each point decomposes orthogonally as

$$T(M) = H(M) \oplus \mathbf{R}\xi \oplus E(M),$$

where $\phi H(M) = H(M)$ and $\phi^2 = -I$ along $H(M)$ ($H(M)$ is the Levi, or maximally complex, distribution of M) and $\phi E(M) \subseteq T(M)^\perp$. While both $\mathcal{D} := H(M)$ and $\mathcal{D} := H(M) \oplus \mathbf{R}\xi$ organize M as a contact CR submanifold, it should be remarked that $H(M)$ is never integrable (cf. e.g. [17], p. 170), i.e. $(M, T_{1,0}(M))$ is never Levi flat. This appears as a basic difference between the complex and contact case (Chen's CR or warped CR products are always Levi flat). Therefore, to formulate a contact analog of the notion of warped CR product one assumes that $M = N^T \times N^\perp$ where i) N^T is a ϕ-invariant submanifold of $\widetilde{M}$ *tangent* to ξ, ii) $N^\perp$ is a ϕ-anti-invariant submanifold of $\widetilde{M}$, and iii) the induced metric $g = j^*\widetilde{g}$, $j : M \subset \widetilde{M}$, is a warped product metric (in the sense of R. L. Bishop and B. O'Neill, [10]). Then, of course, $H(M) \oplus \mathbf{R}\xi$ is integrable and N^T is one of its leaves.

To give a brief description of our findings let us consider CR-products in Sasakian manifolds. We give a tensorial characterization, namely we prove the following result: *Let M be a contact CR-submanifold of a Sasakian manifold $\widetilde{M}$, with $\xi \in \mathcal{D}$. Then M is a contact CR product if and only if P satisfies $(\nabla_U P)V = -g(U_{\mathcal{D}}, V)\xi + \eta(V)U_{\mathcal{D}}$ for all U, V tangent to M.* When the ambient is a Sasakian space form we classified the contact CR products as follows: *Let M be a complete, generic, simply connected contact CR submanifold of a complete, simply connected Sasakian space form $\widetilde{M}^{2m+1}(c)$. If M is a contact CR product then either $c \neq -3$ and M is a ϕ anti-invariant submanifold of $\widetilde{M}$ case in which M is locally a Riemannian product of an integral curve of ξ and a totally real submanifold $N^{\perp}$ of $\widetilde{M}$, or $c = -3$ and M is locally a Riemannian product of $\mathbf{R}^{2s+1}$ and $N^{\perp}$ where $\mathbf{R}^{2s+1}$ is endowed with the usual Sasakian structure and $N^{\perp}$ is a totally real submanifold of $\mathbf{R}^{2m+1}$* (with the usual Sasakian structure). [Here $2s = \dim H(M)$.] Our purpose was to introduce and to study an analog of B.Y. Chen's CR warped products suitable for use in Sasakian geometry. As we have mentioned in the Abstract we may consider only warped products CR-submanifolds of the form $N^T \times_f N^{\perp}$. Our next result characterizes contact CR warped products in Sasakian manifolds: *A strictly proper CR submanifold M of a Sasakian manifold $\widetilde{M}$, and tangent to the structure vector field ξ is locally a contact CR warped product if and only if $A_{\phi Z}X = (\eta(X) - (\phi X)(\mu))\, Z$, $X \in \mathcal{D}$, $Z \in \mathcal{D}^{\perp}$ for some function μ on M satisfying $W\mu = 0$ for all $W \in \mathcal{D}^{\perp}$.* This characterization is similar to that of B.Y. Chen (for warped CR products) and a natural generalization of his.

Among other results, we obtain an analog of B. Y. Chen's inequality (satisfied by the norm of the second fundamental form): *Let $M = N^T \times_f N^{\perp}$ be a contact CR warped product of a Sasakian space form $\widetilde{M}^{2m+1}(c)$ and let $h = 2s + 1 = \dim N^T$ and $p = \dim N^{\perp}$. Then the second fundamental form of M satisfies the following inequality*

$$\|B\|^2 \geq 2p\left[\|\nabla \ln f\|^2 - \Delta \ln f + \frac{c+3}{2}\, s + 1\right]. \tag{a}$$

If the ambient space is but a Sasakian manifold (and not necessarily a Sasakian space form) we obtain a weaker inequality

$$\|B\|^2 \geq 2p\left(\|\nabla \ln f\|^2 + 1\right)\ . \tag{b}$$

Here, if equality holds, then N^T is totally geodesic (in $\widetilde{M}$), while $N^\perp$ is totally umbilical. Moreover, M is minimal in $\widetilde{M}$ and if $\widetilde{M} = \mathbf{R}^{2m+1}$ (endowed with the standard Sasakian structure) then $\ln f$ is a harmonic function. Finally, we were able to give an example of a contact CR warped product in $\mathbf{R}^{2m+1}(-3)$ satisfying the inequality (a), yet not satisfying the inequality (b).

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2. Contact *CR* products

Let $\widetilde{M}$ be a $(2m+1)$-dimensional Sasakian manifold with the contact metric structure $(\phi, \xi, \eta, \widetilde{g})$ i.e. $\phi \in \mathcal{T}^1_1(\widetilde{M})$, $\xi \in \chi(\widetilde{M})$ and $\eta \in \Lambda^1(\widetilde{M})$ with the following properties: $\phi^2 = -I + \eta \otimes \xi$, $\phi\xi = 0$, $\eta \circ \phi = 0$, $\eta(\xi) = 1$, $d\eta(X,Y) = \widetilde{g}(X, \phi Y)$ (the contact condition) and $\widetilde{g}(\phi X, \phi Y) = \widetilde{g}(X,Y) - \eta(X)\eta(Y)$ (the compatibility condition). If $\widetilde{\nabla}$ denotes its Levi–Civita connection the following relation

$$(\widetilde{\nabla}_U \phi)V = -\widetilde{g}(U,V)\xi + \eta(V)U, \quad U, V \in \chi(\widetilde{M}), \tag{1}$$

holds on $\widetilde{M}$ and actually characterizes Sasakian manifolds among almost contact Riemannian manifolds. A plane section $\sigma \subset T_x(\widetilde{M})$ is a *ϕ-section* if σ is spanned by $\{v, \phi_x v\}$, for some $v \in T_x(\widetilde{M})$. The restriction k_ϕ to ϕ-planes of the Riemannian sectional curvature (of $(\widetilde{M}, \widetilde{g})$) is the *$\phi$-sectional curvature*. A *Sasakian space form* is a Sasakian manifold of constant ϕ-sectional curvature and if this is the case the Riemannian curvature tensor field $\widetilde{R}$ is given by

$$\begin{aligned}\widetilde{R}(X,Y)Z = \frac{c+3}{4}\big\{\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y\big\} - \frac{c-1}{4}\big\{\eta(Z)[\eta(Y)X - \eta(X)Y] \\ + [\widetilde{g}(Y,Z)\eta(X)\widetilde{g}(X,Z)\eta(Y)]\xi - \widetilde{g}(\phi Y, Z)\phi X + \widetilde{g}(\phi X, Z)\phi Y \\ + 2\widetilde{g}(\phi X, Y)\phi Z\big\},\end{aligned} \tag{2}$$

for any $X, Y, Z \in \chi(\widetilde{M})$ (actually, by the Schur-like result in [11], p. 97, it suffices that k_ϕ be a point function; then k_ϕ is constant and $\widetilde{R}$ is given by (2)).

Let M be a real m-dimensional submanifold of $\widetilde{M}$, tangent to the contact vector ξ. We shall need the Gauss and Weingarten formulae

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X,Y), \quad \widetilde{\nabla}_X N = -A_N X + \nabla^\perp_X N, \tag{3}$$

for any $X, Y \in \chi(M)$, and $N \in \Gamma^\infty(T(M)^\perp)$. Here $T(M)^\perp$ is the normal bundle of the given immersion. Also, ∇ is the induced connection, $\nabla^\perp$ is the normal connection (a connection in the normal bundle), B is the second fundamental form (of the given immersion), and A_N is the Weingarten operator (corresponding to the normal section N). Cf. e.g. [18]. Then

$$g(A_N X, Y) = \widetilde{g}(N, B(X,Y)). \tag{4}$$

For any $X \in \chi(M)$ we set $PX = \tan(\phi X)$ and $FX = \mathrm{nor}(\phi X)$, where $\tan_x$ and nor_x are the natural projections associated to the direct sum decomposition

$$T_x(\widetilde{M}) = T_x(M) \oplus T(M)^\perp_x, \quad x \in M.$$

Then P is an endomorphism of the tangent bundle $T(M)$ of and F is a normal bundle valued 1-form on M. Since ξ is tangent to M we get

$$P\xi = 0, \quad F\xi = 0, \quad \nabla_X \xi = PX, \quad B(X,\xi) = FX. \tag{5}$$

Similarly, for a normal vector field N, we put $tN = \tan(\phi N)$ and $fN = \mathrm{nor}(\phi N)$ for the tangential and the normal part of ϕN, respectively.

The Riemannian curvature tensor R of M is given by

$$\begin{aligned} R_{XY}Z = {} & \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ & - \frac{c-1}{4}\big\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(Y,Z)\eta(X)\xi \\ & - g(X,Z)\eta(Y)\xi - g(PY,Z)PX + g(PX,Z)PY \\ & + 2g(PX,Y)PZ\big\} + A_{B(Z,Y)}X - A_{B(Z,X)}Y \end{aligned} \tag{6}$$

for all X, Y, Z vector fields on M. We recall the equation of Gauss

$$\begin{aligned} \widetilde{g}(\widetilde{R}_{XY}Z, W) = {} & g(R_{XY}Z, W) - \tilde{g}(B(X,W), B(Y,Z)) \\ & + \tilde{g}(B(Y,W), B(X,Z)) \end{aligned} \tag{7}$$

and the equation of Codazzi

$$\begin{aligned}&(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z)\\ &= \frac{c-1}{4}\{g(PY,Z)FX - g(PX,Z)FY - 2g(PX,Y)FZ\}\end{aligned} \tag{8}$$

where

$$(\nabla_X B)(Y,Z) = \nabla_X^{\perp} B(Y,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z). \tag{9}$$

The second fundamental form B satisfies the *classical Codazzi equation* (according to [7], [32]) if

$$(\nabla_X B)(Y,Z) = (\nabla_Y B)(X,Z). \tag{10}$$

Lemma 2.1. *Let M be a submanifold of Sasakian space form $\widetilde{M}^{2m+1}(c)$ with $c \neq 1$ and tangent to the structure vector field ξ. If the second fundamental form B of M satisfies the classical Codazzi equation then M is ϕ invariant or ϕ anti-invariant.*

Proof. By using (8) and (10) one gets

$$\begin{aligned}g(PY,Z)FX - g(PX,Z)FY - 2g(PX,Y)FZ = 0,\\ \forall X,Y,Z \in T(M).\end{aligned} \tag{11}$$

We will give the proof by contradiction. Suppose that there exists $U_x \in T_x(M)$ such that $PU_x \neq 0$ and $FU_x \neq 0$. From (11) we deduce $3g_x(PU_x, PU_x)\ FU_x = 0$, false. Therefore, for $U_x \in T_x(M)$ we have either $PU_x = 0$ or $FU_x = 0$. It can also be proved that we cannot have $U_x, V_x \in T_x(M)$ such that $PU_x \neq 0$, $FU_x = 0$, $PV_x = 0$ and $FV_x \neq 0$. Consequently $P = 0$ or $F = 0$ which means that M is a ϕ-invariant manifold (if $F = 0$) or M is a ϕ-anti-invariant manifold (if $P = 0$). □

Putting

$$(\nabla_U P)V = \nabla_U(PV) - P\nabla_U V, \quad (\nabla_U F)V = \nabla_U^{\perp}(FV) - F\nabla_U V \tag{12}$$

for $U, V \in \chi(M)$ we have (cf. [49])

$$(\nabla_U P)V = A_{FV}U + tB(U,V) - g(U,V)\xi + \eta(V)U \tag{13}$$

$$(\nabla_U F)V = -B(U, PV) + fB(U, V). \tag{14}$$

We set $\dim M = n+1$, $\dim \mathcal{D} = h$ and $\dim \mathcal{D}^\perp = p$.

It is known the following remarkable result (cf. e.g. [49] p. 55): *In order for a submanifold M, tangent to the structure vector field ξ, of a Sasakian manifold $\widetilde{M}$ to be a contact CR submanifold, it is necessary and sufficient that $FP = 0$.*

Lemma 2.2. *Let M be a contact CR submanifold of a Sasakian manifold $\widetilde{M}$. Then for any $Z, W \in \mathcal{D}^\perp$ we have*

$$A_{FZ}W - A_{FW}Z = \eta(W)Z - \eta(Z)W \tag{15}$$

$$(\nabla_W P)Z = (\nabla_Z P)W. \tag{16}$$

In [49] it is proved that the distribution $\mathcal{D}^\perp$ is always completely integrable. The idea of the proof is to show that $\phi[Z, W] = F[Z, W]$ for all $Z, W \in \mathcal{D}^\perp$.

In the following we will suppose that $\xi \in \mathcal{D}$.

Lemma 2.3. *Let M be a contact CR submanifold of a Sasakian manifold $\widetilde{M}$ with $\xi \in \mathcal{D}$. Then the following three statements are equivalent.*

(i) $B(X, PY) = B(PX, Y)\ \ \forall X, Y \in \mathcal{D}$

(ii) $\widetilde{g}(B(X, PY), \phi Z) = \widetilde{g}(B(PX, Y), \phi Z)\ \ \forall X, Y \in \mathcal{D},\ \forall Z \in \mathcal{D}^\perp$

(iii) *$\mathcal{D}$ is completely integrable.*

Proof. We will sketch out only the implication (ii) $\Rightarrow$ (iii).
Since $B(X, PY) - B(PX, Y) = (\nabla_Y F)X - (\nabla_X F)Y$ it follows that $[X, Y] \in \mathcal{D}$ for all $X, Y \in \mathcal{D}$. (For details see also [9].) □

Let now $N^\perp$ be a leaf of anti-invariant distribution $\mathcal{D}^\perp$. We may state the following

Proposition 2.1. *A necessary and sufficient condition for the submanifold $N^\perp$ to be totally geodesic in M is that*

$$\widetilde{g}\left(B(H(M), \mathcal{D}^\perp), \phi\mathcal{D}^\perp\right) = 0. \tag{17}$$

Proof. Denote by $\overset{(2)}{\nabla}$ the Levi–Civita connection on $N^\perp$. Denote also by σ_2 the second fundamental form of $N^\perp$ in M and let $Z, W \in \mathcal{D}^\perp$

(i.e. tangent to $N^\perp$). The Gauss formula is $\nabla_Z W = \overset{(2)}{\nabla}_Z W + \sigma_2(Z,W)$. With $X \in \mathcal{D}$ we can write

$$g(\sigma_2(Z,W),X) = g(\nabla_Z W - \overset{(2)}{\nabla}_Z W, X) = g(\nabla_Z W, X) = -g(W, \nabla_Z X).$$

$\Rightarrow$: Suppose $N^\perp$ is totally geodesic in M (i.e. $\sigma_2 = 0$) and thus, $g(\nabla_Z X, W) = 0$, for all $X \in \mathcal{D}$ and $Z, W \in \mathcal{D}^\perp$. Since $\mathcal{D}$ is invariant we can replace in the equality above X by ϕX. One obtains

$$0 = g(\nabla_Z(\phi X), W) = \widetilde{g}(\widetilde{\nabla}_Z(\phi X) - B(Z,\phi X), W) = -\widetilde{g}(B(Z,X),\phi W)$$

i.e. (17).
$\Leftarrow$: Conversely, suppose we have $\widetilde{g}(B(X,Z),\phi W) = 0$ for all $X \in H(M)$ and for all $Z, W \in \mathcal{D}^\perp$. Doing the computation in the same manner as above one obtains

$$g(\nabla_Z(\phi X), W) = 0, \quad \forall X \in H(M),\ \forall Z, W \in \mathcal{D}^\perp.$$

Replacing X by ϕX and taking into account that $H(M) \subset \ker \eta$ one has $g(\sigma_2(Z,W),X) = 0$. The component of $\sigma_2(Z,W)$ along ξ vanishes since $\eta(\sigma_2(Z,W)) = -g(W,\nabla_Z \xi) = -g(W, FZ) = 0$. It follows that $\sigma_2(Z,W) = 0$, $\forall Z, W \in \mathcal{D}^\perp$ which means that $N^\perp$ is totally geodesic in M. □

A contact CR submanifold M in a Sasakian manifold $\widetilde{M}$ (with $\xi \in \mathcal{D}$) is called *strictly proper* if $\dim H(M) > 0$ and $\dim \mathcal{D}^\perp > 0$.

Proposition 2.2. *Let M be a strictly proper contact CR submanifold of a Sasakian space form $\widetilde{M}^{2m+1}(c)$. If the second fundamental form of M satisfies the classical Codazzi equation then $c = 1$.*

Proof. The proof follows easily from Lemma 1. □

We give the following definition: A contact CR submanifold M of a Sasakian manifold $\widetilde{M}$ is called *contact CR product* if it is locally a Riemannian product of a ϕ-invariant submanifold N^T tangent to ξ and a totally real submanifold $N^\perp$ of $\widetilde{M}$, i.e. $N^\perp$ is ϕ anti-invariant submanifold of $\widetilde{M}$.

Let us remark that in [36] N.Papaghiuc used the notion of *semi-invariant product*, according to the terminology *semi-invariant submanifolds* in Sasakian manifolds (for further details we refer to [8], [9], [37]).

Let ν be the complementary orthogonal subbundle of $\phi\mathcal{D}^{\perp}$ in the normal bundle $T(M)^{\perp}$. Thus we have the following direct sum decomposition

$$T(M)^{\perp} = \phi\mathcal{D}^{\perp} \oplus \nu. \tag{18}$$

Lemma 2.4. *Let M be a contact CR submanifold in a Sasakian manifold $\widetilde{M}$ with $\xi \in \mathcal{D}$. Then for all $X, Y \in \mathcal{D}$ we have $\phi B(X,Y) \in \mathcal{D}^{\perp} \oplus \nu$.*

Proof. The proof is based on the remark that $\phi\nu = \nu$. Since B is normal to M and $\eta(\mathcal{D}^{\perp}) = 0$ we easily get the statement. □

In view of B. Y. Chen's characterization of CR-products in Kählerian manifolds (cf. [19], I, theorem 4.1: *A CR-submanifold of a Kählerian manifold $\widetilde{M}$ is a CR-product if and only if P is parallel, i.e. $\nabla P = 0$*) it is natural to study the contact CR-submanifolds M in Sasakian manifolds $\widetilde{M}$ (with $\xi \in \mathcal{D}$) satisfying $\nabla P = 0$.

First of all suppose that the distribution $\mathcal{D}$ contains another vector field except ξ, non zero and belonging to $\ker\eta$; call it X_0. Let us take $U, V \in \mathcal{D} \cap \ker\eta$. It follows that

$$0 = (\nabla_U P)V = t\ B(U,V) - g(U,V)\xi.$$

As we have already seen $\phi B(U,V) \in \mathcal{D}^{\perp} \oplus \nu$ and thus $t\ B(U,V)$ belongs to $\mathcal{D}^{\perp}$ while $f\ B(U,V)$ belongs to ν. We get $g(U,V)\xi \in \mathcal{D}^{\perp}$ for all $U, V \in \mathcal{D} \cap \ker\eta$. If we take $U = V = X_0$ we obtain a contradiction (because $g(X_0, X_0) \neq 0$ and $\xi \in \mathcal{D}$). The conclusion is that we cannot have this situation. Consequently, $\mathcal{D} = \text{span}\,[\xi]$ and thus $\mathcal{D}$ is completely integrable. Moreover, $H(M)$ is empty and the condition (17) is automatically fulfilled which yields to the totally geodesy of the orthogonal distribution (more precisely of its integral manifold $N^{\perp}$). Since N^T (the integral curve of ξ) is obvious totally geodesic in M it follows that M is (locally) a Riemannian product between N^T and $N^{\perp}$. We can state now the following theorem.

Theorem 2.1. *Let M be a contact CR-submanifold of a Sasakian manifold $\widetilde{M}$ with $\xi \in \mathcal{D}$ and $\nabla P = 0$. Then M is a contact CR-product* *between an integral curve of ξ and a ϕ-anti-invariant submanifold $N^{\perp}$ of $\widetilde{M}$.*

In the sequel we give a tensorial characterization for a contact CR submanifold to be a contact CR product. Thus, we prove the following

Theorem 2.2. *Let M be a contact CR submanifold of a Sasakian manifold $\widetilde{M}$ and set $\xi \in \mathcal{D}$. Then M is a contact CR product if and only if P satisfies*

$$(\nabla_U P)V = -g(U_{\mathcal{D}}, V)\xi + \eta(V)U_{\mathcal{D}} \tag{19}$$

for all U, V tangent to M where $U_{\mathcal{D}}$ is the $\mathcal{D}$-component of U.

Proof. $\Rightarrow$: Since $\phi \equiv P$ on N^T the Gauss formula becomes $(\widetilde{\nabla}_X \phi)Y = (\nabla_X P)Y + B(X, PY) - \phi B(X, Y)$ with $X, Y \in N^T$. Due to the Sasakian structure of $\widetilde{M}$ we obtain $(\nabla_X P)Y = -g(X, Y)\xi + \eta(Y)X + \phi B(X, Y) - B(X, PY)$. Taking the component in $\mathcal{D}$ one gets

$$(\nabla_X P)Y = -g(X, Y)\xi + \eta(Y)X. \tag{20}$$

Consider now $Z \in N^{\perp}$ and $Y \in N^T$. Making similar computations as above we can prove

$$(\nabla_Z P)Y = 0. \tag{21}$$

As consequence

$$B(Z, PY) = \phi B(Z, Y) + \eta(Y)Z, \quad \forall\, Y \in N^T,\ Z \in N^{\perp}. \tag{22}$$

Now it is easy to show that $(\nabla_U P)Z = 0$ for all $U \in \chi(M)$, $Z \in \mathcal{D}^{\perp}$ and hence the conclusion.

$\Leftarrow$: Let us prove the converse, i.e. suppose we have satisfied (19) and prove that M is a contact CR product. Consider $U = X$, $V = Z$ with $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$. The relation (19) becomes $(\nabla_X P)Z = 0$ and by using (13) we obtain $tB(X, Z) = -A_{FZ}X$. Considering $U = Z$, $V = X$ (with X, Z as above) we obtain $(\nabla_Z P)X = 0$. Using again (13) we obtain $tB(Z, X) = -\eta(X)Z$. Thus one gets

$$A_{FZ}X = \eta(X)Z \tag{23}$$

for all $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$. After the computations we obtain $\widetilde{g}(B(X, PY) - B(PX, Y), \phi Z) = 0$. From Lemma 2.3 it follows that the distribution

$\mathcal{D}$ is completely integrable. Denote by N^T and $N^\perp$ the leaves of two distributions $\mathcal{D}$ and $\mathcal{D}^\perp$, respectively (the orthogonal distribution $\mathcal{D}^\perp$ is always completely integrable). Let $X \in H(M)$, $Z, W \in \mathcal{D}^\perp$. Due to (23) we have

$$\widetilde{g}(B(X,Z), \phi W) = \widetilde{g}(A_{\phi W}X, Z) = \widetilde{g}(\eta(X)W, Z) = \eta(X)g(W,Z) = 0.$$

Thus, by virtue of the Proposition 2.1, $N^\perp$ is totally geodesic in M. Let now $X, Y \in \mathcal{D}$ (i.e. tangent to N^T). From (13) and (19) we obtain $tB(X,Y) = 0$. If $Z \in \mathcal{D}^\perp$ we have

$$\begin{aligned} 0 &= \widetilde{g}(tB(X,Y), Z) = -\widetilde{g}(\widetilde{\nabla}_X Y, \phi Z) \\ &= \widetilde{g}(Y, (\widetilde{\nabla}_X \phi)Z) + \widetilde{g}(Y, \phi\widetilde{\nabla}_X Z) = -g(\phi Y, \nabla_X Z). \end{aligned}$$

Replacing Y by ϕY (since $\mathcal{D}$ is invariant by ϕ) one obtains $0 = g(Y, \nabla_X Z) - \eta(Y)g(\xi, \nabla_X Z)$. But $g(\xi, \nabla_X Z) = 0$, so $g(Y, \nabla_X Z) = 0$ for all $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$. It follows that $g(\nabla_X Y, Z) = 0$ which means that N^T is also totally geodesic in M. We may conclude that M is a contact CR product in $\widetilde{M}$. □

Remark 2.1. A similar calculus as in Lemma 2.4 leads to $B(X,Y) \in \nu$ and $\phi B(X,Y) = B(X, PY)$ for all X, Y tangent to N^T. On N^T we have an induced Sasakian structure.

It can be proved, independently of the previous theorem the following

Proposition 2.3. *Let M be a contact CR-submanifold in a Sasakian manifold $\widetilde{M}^{2m+1}$ with $\xi \in \mathcal{D}$. Then M is a contact CR product if and only if*

$$A_{\phi Z}X = \eta(X)Z \tag{24}$$

for all $X \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$.

Proof. First we shall prove the converse. Suppose that (24) holds. We have

$$\widetilde{g}(B(X,Z), \phi W) = g(A_{\phi W}X, Z) = \eta(X)g(Z,W) = 0,$$

$$\forall\, X \in H(M),\ \forall\, Z, W \in \mathcal{D}^\perp.$$

From Proposition 2.1 we get that $N^\perp$ (the integral manifold of $\mathcal{D}^\perp$) is totally geodesic in M.

Consider now $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$. We have $\widetilde{g}(B(X, \phi Y), \phi Z) = \widetilde{g}(A_{\phi Z} X, \phi Y) = \widetilde{g}(\eta(X) Z, \phi Y) = 0$. Similarly $\widetilde{g}(B(Y, \phi X), \phi Z) = 0$ and by Lemma 2.3 it follows that $\mathcal{D}$ is completely integrable. To prove that N^T (the integral manifold of $\mathcal{D}$) is totally geodesic in M we will prove that $\nabla_X Y$ belongs to N^T for all X, Y tangent to N^T. We have $g(\nabla_X Y, Z) = -g(Y, \nabla_X Z)$. On the other hand, from the hypothesis $\widetilde{g}(B(X, Y), \phi Z) = 0$. Then

$$\widetilde{g}(B(X,Y), \phi Z) = -\widetilde{g}(Y, \widetilde{\nabla}_X(\phi Z)) = \widetilde{g}(\phi Y, \widetilde{\nabla}_X Z) = \widetilde{g}(\phi Y, \nabla_X Z).$$

So, we obtain $g(\phi Y, \nabla_X Z) = 0$, $\forall X, Y \in \mathcal{D}$, $\forall Z \in \mathcal{D}^{\perp}$. But $g(\xi, \nabla_X Z) = 0$ and hence $g(Y, \nabla_X Z) = 0$. We may conclude now that $\nabla_X Y \in N^T$ for all $X, Y \in N^T$. Therefore the two integral manifolds N^T and $N^{\perp}$ are both totally geodesic in M. Consequently, M is locally a Riemannian product of N^T and $N^{\perp}$.

To prove the direct implication we have to take into account the totally geodesy of N^T and $N^{\perp}$. Using the Gauss formula we get $\widetilde{g}(\widetilde{\nabla}_X Y, \phi Z) = \widetilde{g}(B(X, Y), \phi Z)$ with $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$. The right side is exactly $g(A_{\phi Z} X, Y)$ while the left side equals to

$$X(\widetilde{g}(Y, \phi Z)) - \widetilde{g}(Y, \widetilde{\nabla}_X(\phi Z)) = \widetilde{g}(\phi Y, \widetilde{\nabla}_X Z) = -g(\nabla_X(\phi Y), Z) = 0.$$

It follows that $A_{\phi Z} X \in \mathcal{D}^{\perp}$. Again by using the Gauss formula we obtain after the computations

$$\eta(X) g(Z, W) = \widetilde{g}(A_{\phi Z} X, W).$$

Taking into account that $A_{\phi Z} X \in \mathcal{D}^{\perp}$ it follows $A_{\phi Z} X = \eta(X) Z$. This completes the proof. $\square$

The next result is a geometric description of contact CR products in Sasakian space forms.

Theorem 2.3. *Let M be a complete, generic, simply connected contact CR submanifold of a complete, simply connected Sasakian space form $\widetilde{M}^{2m+1}(c)$. If M is a contact CR product then either $c \neq -3$ and M is a ϕ anti-invariant submanifold of $\widetilde{M}$ case in which M is locally a Riemannian product of an integral curve of ξ and a totally real submanifold $N^{\perp}$ of $\widetilde{M}$, or $c = -3$ and M is locally a Riemannian product of $\mathbf{R}^{2s+1}$ and*

$N^\perp$ where $\mathbf{R}^{2s+1}$ is endowed with the usual Sasakian structure and $N^\perp$ is a totally real submanifold of $\mathbf{R}^{2m+1}$ (with the usual Sasakian structure). [Here $2s = \dim H(M)$.]

Proof. Since M is generic it follows that $\xi \in \mathcal{D}$. By Remark 2.1 we have $B(X,Y) = 0$ (for all $X,Y \in \mathcal{D}$) and $A_{FZ}X = \eta(X)Z$ (for all $X \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$). Since $T(M)^\perp = \phi\mathcal{D}^\perp$ and $B \in T(M)^\perp$ by using the Weingarten formula we immediately see that $g(B(X,Z),\phi W) = g(A_{\phi W}X,Z) = \eta(X)g(W,Z)$. Consequently $B(X,Z) = \eta(X)\phi Z$ for all $X \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$. By using similar arguments we can show that $B(U,PV) = 0$ for all $U,V \in T(M)$.

By making use of (9) we obtain for $X,U,V \in T(M)$:

$$\begin{aligned}(\nabla_X B)(U,PV) &= -B(U,(\nabla_X P)V + P\nabla_X V)\\ &= g(X_{\mathcal{D}},V)B(U,\xi) - \eta(V)B(U,X_{\mathcal{D}})\\ &= [g(X_{\mathcal{D}},V_{\mathcal{D}}) - \eta(V_{\mathcal{D}})\eta(X_{\mathcal{D}})]\ \ FU.\end{aligned}$$

Thus

$$(\nabla_X B)(U,PV) = g(PX,PV)\ FU \tag{25}$$

for all $X,U,V \in T(M)$. Substitute in (8) Z by PZ (with $Z \in T(M)$ arbitrary) the following identity holds:

$$(\nabla_X B)(Y,PZ) - (\nabla_Y B)(X,PZ) = \frac{c-1}{4}\{g(PY,PZ)FX - g(PX,PZ)\ FY\}.$$

Combining with (25) the relation above yields to

$$g(PX,PZ)FY - g(PY,PZ)FX = \frac{c-1}{4}\{g(PY,PZ)FX - g(PX,PZ)FY\}$$

which is equivalent to

$$\frac{c+3}{4}\{g(PY,PZ)FX - g(PX,PZ)FY\} = 0, \quad \forall X,Y,Z \in T(M). \tag{26}$$

Now we have to discuss two situations: $c \neq -3$ and $c = -3$.

Case 1. From the equation (26) we obtain
$g(PY,PZ)FX - g(PX,PZ)FY = 0,\ \ \forall\ X,Y,Z \in T(M)$. Since M is generic we have $F \neq 0$ and it is not difficult to prove that $P = 0$. Thus

M is ϕ-anti-invariant. Moreover, by Theorem 2.2 we can say that M is a contact CR product between an integral curve of ξ and a totally real submanifold $N^{\perp}$ of $\widetilde{M}$.

Case 2. From [11] we know that $\widetilde{M}^{2m+1}$ is equivalent to $\mathbf{R}^{2m+1}$ with the usual Sasakian structure (see for details [35]). M is a contact CR product of the invariant submanifold N^T and the anti-invariant submanifold $N^{\perp}$. Since N^T is totally geodesic in M and $B(X,Y)=0$ for all $X,Y\in\mathcal{D}$ then N^T is totally geodesic in $\widetilde{M}$. Thus, from [49], Theorem 1.3, p. 49 it follows that M has constant ϕ sectional curvature $c=-3$. Since M is simply connected and since M is the Riemannian product of N^T and $N^{\perp}$ it follows that N^T is simply connected. It is also known that the completeness of the product manifold inherits the completeness of the two factors. Thus, from [11] it follows that N^T is equivalent to $\mathbf{R}^h$ where $h=2s+1$, with $2s=\dim H(M)$. So, M is locally a Riemannian product of $\mathbf{R}^{2s+1}$ and $N^{\perp}$, where $N^{\perp}$ is a ϕ-anti-invariant submanifold of $\mathbf{R}^{2m+1}$. □

The notion of holomorphic bisectional curvature on Kählerian manifolds (see [28]) was extended to ϕ holomorphic bisectional curvature in Sasakian manifolds. Let $\widetilde{H}_B(U,V)$ be the ϕ-holomorphic bisectional curvature of the plane $U\wedge V$, i.e.

$$\widetilde{H}_B(U,V)=\widetilde{R}(\phi U,U,\phi V,V)\quad\text{for } U,V\in T(\widetilde{M}). \tag{27}$$

For later use we give the following

Lemma 2.5. *Let M be a contact CR product of a Sasakian manifold $\widetilde{M}^{2m+1}$. Then, for any unit vector fields $X\in\mathcal{D}$ and $Z\in\mathcal{D}^{\perp}$ we have*

$$\widetilde{H}_B(X,Z)=2\,\left(\|B(X,Z)\|^2-1\right). \tag{28}$$

Proof. We have $\widetilde{H}_B(X,Z)=\widetilde{g}(\phi Z,(\widetilde{R}_{\phi X,X}Z)^{\perp})$. Using the equation of Codazzi and the definition of ∇B we get:

$$\begin{aligned}\widetilde{H}_B(X,Z)&=\widetilde{g}(\phi Z,\nabla^{\perp}_{\phi X}B(X,Z)-B(\nabla_{\phi X}X,Z)-B(X,\nabla_{\phi X}Z))\\&\quad-\widetilde{g}(\phi Z,\nabla^{\perp}_{X}B(\phi X,Z)-B(\nabla_X(\phi X),Z)-B(\phi X,\nabla_X Z)).\end{aligned}$$

Since N^T is parallel and by using the relation $\widetilde{g}(\phi W,B(X,Z))=\eta(X)g(W,Z),\ \forall\ X\in\mathcal{D},\ \forall\ Z,W\in\mathcal{D}^{\perp}$ (obtained in the proof of Theorem 2.2) we deduce

$$\widetilde{H}_B(X,Z)=\widetilde{g}(\phi Z,\nabla^{\perp}_{\phi X}B(X,Z))-\widetilde{g}(\phi Z,\nabla^{\perp}_{X}B(\phi X,Z))+\eta([X,\phi X]).$$

After some computations one obtains

$$\widetilde{H}_B(X,Z) = (\phi X)(\eta(X)) + \eta([X,\phi X]) - 2\widetilde{g}(\phi B(\phi X,Z), B(X,Z)).$$

Due to the Sasakian structure of $\widetilde{M}$ we have

$$\begin{aligned} 1 = \widetilde{g}(X,X) &= d\eta(X,\phi X) + \eta(X)^2 \\ &= -\frac{1}{2}\left\{(\phi X)(\eta(X)) + \eta([X,\phi X])\right\} + \eta(X)^2 \end{aligned}$$

and hence

$$(\phi X)(\eta(X)) + \eta([X,\phi X]) = -2\left(1-\eta(X)^2\right). \tag{29}$$

In order to compute $\widetilde{g}(\phi B(\phi X,Z), B(X,Z))$ we find firstly that the normal component of $\phi B(\phi X,Z)$ is $\eta(X)\phi Z - B(Z,X)$. Consequently $\widetilde{g}(\phi B(\phi X,Z), B(X,Z)) = \eta(X)^2 - \|B(X,Z)\|^2$ which ends the proof. □

Notice that $\widetilde{H}_B(U,\xi) = 0$ and $B(U,\xi) = FU$. So, when we will refer to the ϕ-holomorphic bisectional curvature of the plane $U \wedge V$ we intend that this plane is orthogonal to ξ. Thus for X in the above lemma we can suppose that it belongs to $H(M)$. Moreover, since the ϕ-holomorphic planes $(X,\phi X)$ and $(Z,\phi Z)$ in $T_x(\widetilde{M})$, $x \in M$, are orthogonal, then $H_B(X,Z)$ is called *ϕ-holomorphic special bisectional curvature* (cf. e.g. [34], [45]).

Theorem 2.4. *Let $\widetilde{M}$ be a Sasakian manifold with the ϕ holomorphic bisectional curvature less strictly than -2. Then every contact CR product M in $\widetilde{M}$ is either an invariant submanifold or an anti-invariant submanifold, case in which M is (locally) a Riemannian product of an integral curve of ξ and a ϕ-anti-invariant submanifold of $\widetilde{M}$.*

Proof. If dim $H(M) > 0$ then, by taking $X \in H(M)$ and $Z \in \mathcal{D}^\perp$, from the previous lemma we get a contradiction. So, either $\dim H(M) = 0$ or $\dim \mathcal{D}^\perp = 0$. The second part of the theorem follows from Theorem 2.2. □

Proposition 2.4. a) *Let $\widetilde{M}^{2m+1}(c)$ be a Sasakian space form and let X,Z be two unit vector fields orthogonal to ξ. Then the ϕ-holomorphic bisectional curvature of the plane $X \wedge Z$ is given by*

$$\widetilde{H}_B(X,Z) = \frac{c-1}{2} + \frac{c+1}{2}\widetilde{g}(\phi X,Z)^2. \tag{30}$$

b) *Let M be a contact CR submanifold of a Sasakian space form $\widetilde{M}^{2m+1}(c)$. Then the ϕ-holomorphic bisectional curvature of the plane $X \wedge Z$, where $X \in H(M)$ and $Z \in \mathcal{D}^{\perp}$ are unit vector fields, is given by the formula*

$$\widetilde{H}_B(X,Z) = \frac{c-1}{2}. \tag{31}$$

Consequently, if $c = -3$ then $\widetilde{H}_B(X,Z) = -2$. In this case it follows that $B(X,Z) = 0$ for all $X \in H(M)$ and $Z \in \mathcal{D}^{\perp}$.

PROOF. Direct calculations. □

Corollary 2.1. *Let $\widetilde{M}^{2m+1}(c)$, $c < -3$ be a Sasakian space form. Then there exists no strictly proper contact CR product in $\widetilde{M}$.*

Corollary 2.2. *Let $\widetilde{M}^{2m+1}$ be a Sasakian manifold with $\widetilde{H}_B > -2$ and let M be a strictly proper contact CR product in $\widetilde{M}$. Then $B(\mathcal{D},\mathcal{D}^{\perp}) \neq 0$ and hence M is never totally geodesic in $\widetilde{M}$.*

We prove now a inequality satisfied by the norm of the second fundamental form of a contact CR product in Sasakian space form. So we give the following theorem.

Theorem 2.5. *Let $\widetilde{M}^{2m+1}(c)$ be a Sasakian space form and let $M = N^T \times N^{\perp}$ be a contact CR product in $\widetilde{M}$. Then the norm of the second fundamental form of M satisfies the inequality*

$$\|B\|^2 \geq p\left((c+3)s + 2\right). \tag{32}$$

The equality sign holds if and only if both N^T and $N^{\perp}$ are totally geodesic in $\widetilde{M}$.

PROOF. For $X \in H(M)$ and $Z \in \mathcal{D}^{\perp}$ we have $\|B(X,Z)\|^2 = \frac{c+3}{4}$. Thus

$$\begin{aligned}\|B\|^2 &= \|B(\mathcal{D},\mathcal{D})\|^2 + \|B(\mathcal{D}^{\perp},\mathcal{D}^{\perp})\|^2 + 2\|B(\mathcal{D},\mathcal{D}^{\perp})\|^2 \geq 2\|B(\mathcal{D},\mathcal{D}^{\perp})\|^2 \\ &= 2\left(\sum_{i=1}^{2s}\sum_{\alpha=1}^{p}\|B(X_i,Z_\alpha)\|^2 + \sum_{\alpha=1}^{p}\|B(\xi,Z_\alpha)\|^2\right) = 2p\left(\frac{c+3}{2}\,s+1\right)\end{aligned}$$

where $\{X_i\}$ and $\{Z_\alpha\}$ are orthonormal basis in $H(M)$ and $\mathcal{D}^{\perp}$ respectively. The equality sign holds if and only if $B(\mathcal{D},\mathcal{D}) = 0$ and $B(\mathcal{D}^{\perp},\mathcal{D}^{\perp}) = 0$, which is equivalent to the totally geodesy of N^T and $N^{\perp}$. □

In the following we will give an example in which the equality sign holds.

Consider the odd spheres S^{2s+1} and S^{2p+1} naturally embedded in Euclidian spaces $\mathbf{R}^{2(s+1)}$ and $\mathbf{R}^{2(p+1)}$ respectively. Take the Riemannian product $S^{2s+1} \times S^{2p+1}$ and the application $r : S^{2s+1} \times S^{2p+1} \longrightarrow S^{2m+1}$ given as follows

$$(x_0, y_0, \ldots, x_s, y_s; u_0, v_0, \ldots, u_p, v_p) \overset{r}{\longmapsto} (\ldots, x_j u_\alpha - y_j v_\alpha, x_j v_\alpha + y_j u_\alpha, \ldots)$$

where $m = sp + s + p$. Here the sphere S^{2m+1} is also embedded in the Euclidian space $\mathbf{R}^{2(m+1)}$. On these spheres we have the usual Sasakian structures and the map r has the property that it is an isometric immersion and maps the Sasakian structure of each sphere component into the Sasakian structure of S^{2m+1}. It is known the fact that the natural almost complex structure on the product manifold is integrable since the contact structures on spheres are normal. Moreover the Hermitian structure is not Kählerian (cf. e.g. [11]).

Let now L be a linear subspace of dimension $p+1$ in $\mathbf{R}^{2(p+1)}$ and passing by the origin such that JL is orthogonal to L (here J is the natural complex structure of $\mathbf{R}^{2(p+1)}$). We also know that the structure vector field is obtained by multiplication with J of the position vector field. So we obtain (as intersection of L with the sphere S^{2p+1}) a p dimensional sphere which is normal to the structure vector field (see [11]). Now, applying Proposition 1.1, p. 43 from [49] we get that S^p is ϕ-anti-invariant submanifold.

Consider now $M = S^{2s+1} \times S^p \longrightarrow S^{2s+1} \times S^{2p+1} \overset{r}{\longrightarrow} S^{2m+1}$. We obtain a contact CR product in S^{2m+1} and we have that S^{2s+1} and S^p are totally geodesic in S^{2m+1}. Consequently the equality holds.

In the end of this section we obtain the smallest dimension for a Sasakian space form $\widetilde{M}^{2m+1}(c)$ which admits a contact CR product. The above example will show us that this estimation for the dimension is the best possible. First we prove the following proposition.

Proposition 2.5. *Let $M = N^T \times N^\perp$ be a contact CR submanifold in a Sasakian space form $\widetilde{M}^{2m+1}(c)$.*

a) *If X, Y are unitary and orthogonal belonging to $H(M)$ and Z is unitary in $\mathcal{D}^\perp$ then*

$$\langle B(X,Z), B(Y,Z)\rangle = 0.$$

b) *If* $\{X, Y\}$ *and* $\{Z, W\}$ *are two pairs of unitary and orthogonal vector fields belonging to* $H(M)$ *and* $\mathcal{D}^\perp$ *respectively, then*

$$\langle B(X, Z), B(Y, W)\rangle = 0.$$

PROOF. Easy calculation. □

Let M be a strictly proper contact CR product. From Proposition 2.1 we know that $B(H(M), \mathcal{D}^\perp) \in \nu$. If $\dim \mathcal{D}^\perp = p = 1$ then let $\{X_j\}_{j=1,\dots,2s}$ be an orthonormal basis in $H(M)$ and let Z be a unitary vector field in $\mathcal{D}^\perp$. From the statement a) in the proposition above we have that $\langle B(X_j, Z), B(X_k, Z)\rangle = 0$ which show that, if $c \neq -3$, $\{B(X_j, Z)\}$ is an orthogonal system. Thus $\dim \nu \geq 2s$.

If $\dim \mathcal{D}^\perp = p \geq 2$, let $\{X_j\}_{j=1,\dots,2s}$ and $\{Z_\alpha\}_{\alpha=1,\dots,p}$ be orthonormal basis in $H(M)$ and $\mathcal{D}^\perp$ respectively. From statement b), with similar arguments, $\{B(X_j, Z_\alpha)\}$ is an orthogonal system in ν. We deduce that $\dim \nu \geq 2sp$. But this is still available even in the first case.

We establish:

Theorem 2.6. *Let* M *be a strictly proper contact* CR *product in a Sasakian space form* $\widetilde{M}^{2m+1}(c)$, *with* $c \neq -3$. *Then*

$$m \geq sp + s + p. \tag{33}$$

PROOF. We know: $\{B(X_j, Z_\alpha)\}$, $i = 1, \dots, 2s$, $\alpha = 1, \dots, p$ is a linearly independent system in ν and $B(\xi, Z_\alpha) = \phi Z_\alpha \in \phi\mathcal{D}^\perp$. Counting the dimensions we obtain (33). □

Let us remark that since the example $S^{2s+1} \times S^p \longrightarrow S^{2m+1}$ with $m = sp + s + p$ satisfies the equality case it follows that the estimation in (33) is the best possible.

A particular example is the product of spheres S^3 and S^1. The last one is obtained by $S^3 \subset \mathbf{R}^4$ intersected with the 2-plane L spanned by $\{(1, 0, 0, 0), (0, 0, 0, 1)\}$ which has the property that JL is orthogonal to L. So we obtain the sphere $S^1 = \{(u, 0, 0, v) \in \mathbf{R}^4 : u^2 + v^2 = 1\}$. The vector field $Z = (-v, 0, 0, u)$ is a generator for the tangent space of S^1 at an arbitrary point. The isometric immersion is given by

$$r : S^3 \times S^1 \longrightarrow S^7$$
$$r(x_1, y_1, x_2, y_2, u, v) = (x_1u, y_1u, -y_1v, x_1v, x_2u, y_2u, -y_2v, x_2v).$$

It is easy to check that $r_*\xi_1 = \xi$ (where ξ_1 and ξ are the structure vector fields on S^3 and S^7 respectively, as Sasakian manifolds). We also can verify that $r_*H(M)$ is orthogonal to r_*Z and ϕr_*Z is normal to $r(S^3 \times S^1)$. So we have a contact CR product in S^7.

We will see that this example is quite important. First we will give a kind of converse of Theorem 2.5. Hence we give the following

Theorem 2.7. *Let $M = N^T \times N^\perp$ be a contact CR product in a Sasakian space form $\widetilde{M}^{2m+1}(c)$, $c \neq -3$. Let $\dim N^T = 2s+1$, $\dim N^\perp = p$ and suppose that $m = sp+s+p$. Then N^T is a totally geodesic submanifold in $\widetilde{M}$.*

Proof. The idea of the proof is to apply again the equation of Gauss. A basic calculus leads us to

$$\langle B(X,W), B(Y,U)\rangle = \langle B(Y,W), B(X,U)\rangle \qquad (*)$$

for any X, Y, U tangent to N^T and W tangent to $N^\perp$. In the sequel we consider $X, U \in H(M)$. We have

$$\langle B(\phi X,W), B(X,U)\rangle = \langle B(X,W), B(\phi X,U)\rangle = -\langle \phi B(X,W), B(X,U)\rangle.$$

For any X tangent to N^T (including ξ) and W tangent to $N^\perp$ we easily observe that

$$B(\phi X,W) = \eta(X)W + \phi B(X,W).$$

So $\langle \phi B(X,U), B(X,W)\rangle$ vanishes for all $X, U \in \mathcal{D}$, $W \in \mathcal{D}^\perp$. Consequently $\langle B(X,\phi U), B(X,W)\rangle$ vanishes too, and hence, according to $(*)$ one gets

$$\langle B(X,U), B(Y,W)\rangle = 0, \quad \forall\, X, Y, U \in H(M),\ \forall\, W \in \mathcal{D}^\perp. \qquad (**)$$

Consider orthonormal basis $\{X_j\}$ and $\{Z_\alpha\}$ in $H(M)$ and $\mathcal{D}^\perp$ respectively, and since $m = sp+s+p$ then $\{B(X_j, Z_\alpha)\}$ form a basis in ν. The relation $(**)$ yields to $B(X,U) = 0$ for all $X, U \in H(M)$. But $B(\xi,U) = FU = 0$

for $U \in \mathcal{D}$ and since N^T is totally geodesic in M it follows that N^T is also totally geodesic in $\widetilde{M}$. □

Corollary 2.3. *Let $M = N^T \times N^\perp$ be a strictly proper contact CR product in S^7. Then M is a Riemannian product between the sphere S^3 and a curve. Moreover, if the norm of the second fundamental form of M satisfies the equality case in the inequality we have that M is the Riemannian product between S^3 and S^1.*

Proof. We have $sp + s + p \leq 3$ so $s = p = 1$ and we are in the case of the 'minimum dimension'. Thus N^T is totally geodesic in S^7 and having dimension 3 is the sphere S^3. The ϕ anti-invariant manifold $N^\perp$ has dimension 1, so it is a curve. If the equality case is satisfied then both N^T and $N^\perp$ are totally geodesic in S^7 and thus M is a Riemannian product between S^3 and S^1. In this situation $\|B\| = \sqrt{6}$. □

We end this section with the following

Theorem 2.8. *Let $M = N^T \times N^\perp$ be a strictly proper contact CR product in S^7 whose second fundamental form has the norm $\sqrt{6}$. Then M is the Riemannian product between S^3 and S^1 and, up to a rigid transformation of $\mathbf{R}^8$ the embedding is given by*

$$r : S^3 \times S^1 \longrightarrow S^7 \tag{34}$$

$$r(x_1, y_1, x_2, y_2, u, v) = (x_1u, y_1u, -y_1v, x_1v, x_2u, y_2u, -y_2v, x_2v).$$

Proof. We are interested to find the equations of the isometrical immersion

$$S^3 \times S^1 \xrightarrow{r} S^7 \qquad (x, y, z; t) \mapsto (\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_8)$$

where S^7 is thought to be embedded in $\mathbf{R}^8$ and thus we have $\sum_{I=1}^{8} \mathcal{X}_I^2 = 1$.

The Levi–Civita connection on the sphere S^7 is $\widetilde{\nabla} = \tan(\overset{0}{\nabla})$ where $\overset{0}{\nabla}$ is the flat connection on the Euclidian space $\mathbf{R}^8$ and tan denotes the projection operator on the tangent bundle of the sphere. Notice that (x, y, z) and t are the spherical coordinates on the two spheres S^3 and S^1 respectively. If $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ are the Levi–Civita connections on the two spheres, we have

$$\begin{cases} \overset{1}{\nabla}_{\frac{\partial}{\partial x}} \dfrac{\partial}{\partial x} = 0, \quad \overset{1}{\nabla}_{\frac{\partial}{\partial x}} \dfrac{\partial}{\partial y} = -\operatorname{tg} x \dfrac{\partial}{\partial y}, \quad \overset{1}{\nabla}_{\frac{\partial}{\partial x}} \dfrac{\partial}{\partial z} = -\operatorname{tg} x \dfrac{\partial}{\partial z} \\ \overset{1}{\nabla}_{\frac{\partial}{\partial y}} \dfrac{\partial}{\partial y} = \sin x \cos x \dfrac{\partial}{\partial x}, \quad \overset{1}{\nabla}_{\frac{\partial}{\partial y}} \dfrac{\partial}{\partial z} = -\operatorname{tg} y \dfrac{\partial}{\partial z} \\ \overset{1}{\nabla}_{\frac{\partial}{\partial z}} \dfrac{\partial}{\partial z} = \sin x \cos x \cos^2 y \dfrac{\partial}{\partial x} + \sin y \cos y \dfrac{\partial}{\partial y} \end{cases} \tag{35}$$

$$\overset{2}{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = 0. \tag{36}$$

Then $\widetilde{\nabla}_{r_* \frac{\partial}{\partial t}} r_* \frac{\partial}{\partial t} = \left(\frac{\partial^2 \mathcal{X}_I}{\partial t^2} - \mathcal{X}_I \sum_J \mathcal{X}_J \frac{\partial^2 \mathcal{X}_J}{\partial t^2} \right) \frac{\partial}{\partial \mathcal{X}_I}$ and from the totally geodesy of S^1 in S^7 we get

$$\frac{\partial^2 \mathcal{X}_I}{\partial t^2} - \mathcal{X}_I \sum_J \mathcal{X}_J \frac{\partial^2 \mathcal{X}_J}{\partial t^2} = 0. \tag{37}$$

The isometry condition yields to

$$\sum_J \left(\frac{\partial \mathcal{X}_J}{\partial t} \right)^2 = 1 \tag{38}$$

and consequently one gets $\sum_J \mathcal{X}_J \frac{\partial^2 \mathcal{X}_J}{\partial t^2} = -1$. Thus we obtain the following PDE equation system:

$$\frac{\partial^2 \mathcal{X}_I}{\partial t^2} + \mathcal{X}_I = 0 \tag{39}$$

with the solution

$$\mathcal{X}_I = \alpha_I \cos t + \beta_I \sin t, \tag{40}$$

where α_I, β_I are smooth functions on S^3. From (38) we have

$$\sum_I \left(\alpha_I^2 \sin^2 t + \beta_I^2 \cos^2 t - 2\alpha_I \beta_I \sin t \cos t \right) = 1 \quad \text{for all } t.$$

Consequently we obtain

$$\sum \alpha_I^2 = 1, \quad \sum \beta_I^2 = 1, \quad \sum \alpha_I \beta_I = 0. \tag{41}$$

By using the totally geodesy of the sphere S^3 in S^7 Gauss equation $\widetilde{\nabla}_{r_*\frac{\partial}{\partial x}} r_*\frac{\partial}{\partial x} = r_* \overset{1}{\nabla}_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = 0$ yields to $\frac{\partial^2 \mathcal{X}_I}{\partial x^2} + \mathcal{X}_I = 0$. From (40) one gets

$$\alpha_I = a_I \cos x + b_I \sin x, \quad \beta_I = c_I \cos x + d_I \sin x \tag{42}$$

where a_I, b_I, c_I, d_I are smooth functions on S^3 depending on y and z. Hence

$$\mathcal{X}_I = (a_I \cos x + b_I \sin x)\cos t + (c_I \cos x + d_I \sin x)\sin t. \tag{43}$$

By (41) the following relations hold:

$$\left.\begin{array}{l} \sum a_I^2 = \sum b_I^2 = 1, \quad \sum c_I^2 = \sum d_I^2 = 1, \quad \sum a_I b_I = \sum c_I d_I = 0 \\ \sum a_I c_I = \sum b_I d_I = 0\ , \quad \sum (a_I d_I + b_I c_I) = 0. \end{array}\right] \tag{44}$$

From the isometry condition $\langle r_*\frac{\partial}{\partial y}, r_*\frac{\partial}{\partial y}\rangle = \langle \frac{\partial}{\partial y}, \frac{\partial}{\partial y}\rangle\ = \cos^2 x$ we obtain

$$\sum_I \left(\frac{\partial \mathcal{X}_I}{\partial y}\right)^2 = \cos^2 x. \tag{45}$$

In the same way, Gauss equation $\widetilde{\nabla}_{r_*\frac{\partial}{\partial y}} r_*\frac{\partial}{\partial y}\ = r_* \overset{1}{\nabla}_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y}$ yields to the following PDE system

$$\frac{\partial^2 \mathcal{X}_I}{\partial y^2} + \cos^2 x\ \mathcal{X}_I = \sin x \cos x\ \frac{\partial \mathcal{X}_I}{\partial x}\ . \tag{46}$$

Replacing (43) in (46) and taking into account that sin and cos are independently functions we obtain

$$\begin{array}{ll} \dfrac{\partial^2 a_I}{\partial y^2} + a_I = 0, & \dfrac{\partial b_I}{\partial y^2} = 0, \\[2ex] \dfrac{\partial^2 c_I}{\partial y^2} + c_I = 0, & \dfrac{\partial d_I}{\partial y^2} = 0 \end{array} \tag{47}$$

with the solution

$$\begin{array}{ll} a_I = A_I \cos y + B_I \sin y, & b_I = C_I y + D_I, \\ c_I = E_I \cos y + F_I \sin y, & d_I = G_I y + H_I \end{array} \tag{48}$$

where A_I, B_I, C_I, D_I, E_I, F_I, G_I, H_I are C^∞ functions depending on z and satisfying

$$\left.\begin{aligned}
&C_I = G_I = 0,\\
&\sum D_I^2 = \sum H_I^2 = \sum A_I^2 = \sum B_I^2 = \sum E_I^2 = \sum F_I^2 = 1\\
&\sum A_I B_I = \sum E_I F_I = \sum A_I D_I = \sum B_I D_I = \sum E_I H_I\\
&\quad = \sum F_I H_I = \sum A_I E_I = 0\\
&\sum B_I F_I = \sum D_I H_I = 0,\\
&\sum (A_I F_I + B_I E_I) = \sum (A_I H_I + E_I D_I)\\
&\quad = \sum (B_I H_I + F_I D_I) = 0.
\end{aligned}\right] \tag{49}$$

Thus we have

$$\begin{aligned}
\mathcal{X}_I &= [(A_I \cos y + B_I \sin y)\cos x + D_I \sin x]\cos t\\
&\quad + [(E_I \cos y + F_I \sin y)\cos x + H_I \sin x]\sin t.
\end{aligned} \tag{50}$$

Finally, using the isometry condition $\langle r_*\frac{\partial}{\partial z}, r_*\frac{\partial}{\partial z}\rangle = \langle \frac{\partial}{\partial z}, \frac{\partial}{\partial z}\rangle = \cos^2 x \cos^2 y$ we get

$$\sum \left(\frac{\partial \mathcal{X}_I}{\partial z}\right)^2 = \cos^2 x \cos^2 y.$$

Similarly as above we use the Gauss equation $\widetilde{\nabla}_{r_*\frac{\partial}{\partial z}} r_*\frac{\partial}{\partial z} = r_* \overset{1}{\nabla}_{\frac{\partial}{\partial z}} \frac{\partial}{\partial z}$ and obtain the equation

$$\frac{\partial^2 \mathcal{X}_I}{\partial z^2} + \cos^2 x \cos^2 y\ \mathcal{X}_I = \sin x \cos x \cos^2 y\ \frac{\partial \mathcal{X}_I}{\partial x} + \sin y \cos y\ \frac{\partial \mathcal{X}_I}{\partial y}. \tag{51}$$

Replacing (50) in (51), after straightforward computations we obtain the following PDE system

$$A_I'' + A_I = 0,\ B_I'' = 0,\ D_I'' = 0,\ E_I'' + E_I = 0,\ F_I'' = 0,\ H_I'' = 0. \tag{52}$$

The solution of this system is given by

$$\begin{cases}
A_I = \lambda_I \cos z + \mu_I \sin z,\ B_I = \psi_I z + \tau_I,\ D_I = \varepsilon_I z + \rho_I\\
E_I = \widetilde{\lambda}_I \cos z + \widetilde{\mu}_I \sin z,\ F_I = \widetilde{\psi}_I z + \widetilde{\tau}_I,\ H_I = \widetilde{\varepsilon}_I z + \widetilde{\rho}_I
\end{cases} \tag{53}$$

where λ_I, μ_I, ψ_I, τ_I, ε_I, ρ_I, $\widetilde{\lambda}_I$, $\widetilde{\mu}_I$, $\widetilde{\psi}_I$, $\widetilde{\tau}_I$, $\widetilde{\varepsilon}_I$, $\widetilde{\rho}_I$ are some real constants. Moreover, we have

$$\left.\begin{aligned}
&\psi_I = \varepsilon_I = \widetilde{\psi}_I = \widetilde{\varepsilon}_I = 0,\\
&\sum \tau_I^2 = \sum \rho_I^2 = \sum \widetilde{\tau}_I^2 = \sum \widetilde{\rho}_I^2 = \sum \lambda_I^2 = \sum \mu_I^2\\
&\quad = \sum \widetilde{\lambda}_I^2 = \sum \widetilde{\mu}_I^2 = 1\\
&\sum \lambda_I\mu_I = \sum \widetilde{\lambda}_I\widetilde{\mu}_I = \sum \lambda_I\tau_I = \sum \mu_I\tau_I = \sum \widetilde{\lambda}_I\widetilde{\tau}_I = \sum \widetilde{\mu}_I\widetilde{\tau}_I\\
&\quad = \sum \lambda_I\rho_I = \sum \mu_I\rho_I = 0\\
&\sum \tau_I\rho_I = \sum \widetilde{\lambda}_I\widetilde{\rho}_I = \sum \widetilde{\mu}_I\widetilde{\rho}_I = \sum \widetilde{\tau}_I\widetilde{\rho}_I = \sum \lambda_I\widetilde{\lambda}_I = \sum \mu_I\widetilde{\mu}_I\\
&\quad = \sum \tau_I\widetilde{\tau}_I = \sum \rho_I\widetilde{\rho}_I = 0\\
&\sum(\lambda_I\widetilde{\mu}_I + \widetilde{\lambda}_I\mu_I) = \sum(\lambda_I\widetilde{\tau}_I + \widetilde{\lambda}_I\tau_I) = \sum(\mu_I\widetilde{\tau}_I + \widetilde{\mu}_I\tau_I)\\
&\quad = \sum(\lambda_I\widetilde{\rho}_I + \widetilde{\lambda}_I\rho_I) = 0\\
&\sum(\mu_I\widetilde{\rho}_I + \widetilde{\mu}_I\rho_I) = \sum(\tau_I\widetilde{\rho}_I + \widetilde{\tau}_I\rho_I) = 0.
\end{aligned}\right] \tag{54}$$

We can write at this moment the expression of the immersion r

$$\begin{aligned}
\mathcal{X}_I &= \{[(\lambda_I \cos z + \mu_I \sin z)\cos y + \tau_I \sin y]\cos x + \rho_I \sin x\}\cos t\\
&\quad + \{[(\widetilde{\lambda}_I \cos z + \widetilde{\mu}_I \sin z)\cos y + \widetilde{\tau}_I \sin y]\cos x + \widetilde{\rho}_I \sin x\}\sin t.
\end{aligned} \tag{55}$$

Now we use the coordinates of the Euclidian spaces in which the two spheres are embedded, namely $x_1 = \cos x\cos y\cos z$, $y_1 = \cos x\cos y\sin z$, $x_2 = \cos x\sin y$, $y_2 = \sin x$; $u = \cos t$, $v = \sin t$. Consequently, the immersion r can be written as

$$\mathcal{X}_I = (\lambda_I x_1 + \mu_I y_1 + \tau_I x_2 + \rho_I y_2)u + (\widetilde{\lambda}_I x_1 + \widetilde{\mu}_I y_1 + \widetilde{\tau}_I x_2 + \widetilde{\rho}_I y_2)v. \tag{56}$$

We ask $r_*\xi_1 = \xi$ (in any point of S^3), where ξ_1 and ξ are the structure vector fields of the Sasakian structures on S^3 and S^7, respectively. We have

$$r_*\xi_1 = \Big[y_1(\lambda_I u+\widetilde{\lambda}_I v) - x_1(\mu_I u+\widetilde{\mu}_I v) + y_2(\tau_I u+\widetilde{\tau}_I v) - x_2(\rho_I u+\widetilde{\rho}_I v)\Big]\frac{\partial}{\partial \mathcal{X}_I}.$$

Identifying with the components of ξ one obtains

$$\begin{array}{llll} \lambda_{2k-1} = \mu_{2k}, & \widetilde{\lambda}_{2k-1} = \widetilde{\mu}_{2k}, & \mu_{2k-1} = -\lambda_{2k}, & \widetilde{\mu}_{2k-1} = -\widetilde{\lambda}_{2k} \\ \tau_{2k-1} = \rho_{2k}, & \widetilde{\tau}_{2k-1} = \widetilde{\rho}_{2k}, & \rho_{2k-1} = -\tau_{2k}, & \widetilde{\rho}_{2k-1} = -\widetilde{\tau}_{2k} \end{array} \tag{57}$$

for $k = 1, 2, 3, 4$. Now we impose the initial conditions.

1. Let $p_0 = (1, 0, 0, 0; 1, 0) \in S^3 \times S^1$ and let $q_0 = (1, 0, \ldots, 0) \in S^7$. In order to have $r(p_0) = q_0$ we use (56) and we obtain

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = \ldots = \lambda_8 = 0. \tag{58}$$

By virtue of (57) one gets

$$\mu_2 = 1 \quad \text{and} \quad \mu_1 = \mu_3 = \ldots = \mu_8 = 0. \tag{59}$$

2. Let $X_1 = (-x_2,\ y_2, x_1,\ -y_1) \in \chi(S^3)$.

We ask $r_{*,p_0} X_{1,p_0} = (0, 0, 0, 0, 1, 0, 0, 0)$. (Remark that this vector is tangent to the sphere S^7 in q_0.) We obtain

$$\tau_5 = 0 \quad \text{and} \quad \tau_1 = \ldots = \tau_4 = \tau_6 = \tau_7 = \tau_8 = 0. \tag{60}$$

and consequently,

$$\rho_6 = 1 \quad \text{and} \quad \rho_1 = \ldots = \rho_5 = \rho_7 = \rho_8 = 0. \tag{61}$$

3. Consider now $Z = (-v,\ u) \in \chi(S^1)$.

In p_0 we set $r_{*,p_0} Z_{p_0} = (0, 0, 0, 1, 0, 0, 0, 0) \in T_{q_0} S^7$. Hence, we get

$$\widetilde{\lambda}_4 = 1 \quad \text{and} \quad \widetilde{\lambda}_1 = \widetilde{\lambda}_2 = \widetilde{\lambda}_3 = \widetilde{\lambda}_5 = \ldots = \widetilde{\lambda}_8 = 0 \tag{62}$$

and using (57) we have also

$$\widetilde{\mu}_3 = -1 \quad \text{and} \quad \widetilde{\mu}_1 = \widetilde{\mu}_2 = \widetilde{\mu}_4 = \ldots = \widetilde{\mu}_8 = 0. \tag{63}$$

We shall use the relations (54). First, we get $\widetilde{\tau}_4 = 0$ and $\widetilde{\tau}_3 = 0$ and as consequence $\widetilde{\rho}_4 = 0$ and $\widetilde{\rho}_3 = 0$. Then we can prove that $\widetilde{\tau}_5 = 0$, $\widetilde{\rho}_6 = 0$, $\widetilde{\tau}_1 = 0$, $\widetilde{\tau}_2 = 0$, $\widetilde{\rho}_1 = 0$ and $\widetilde{\rho}_2 = 0$. The orthogonality condition $r_{*,p} Z_p \perp \xi_{r(p)}$ (for all $p \in S^3 \times S^1$) yields to $\widetilde{\tau}_6 = 0$ and $\widetilde{\rho}_5 = 0$. Denote

$\widetilde{\tau}_7 = \widetilde{\rho}_8 = A$ and $\widetilde{\tau}_8 = -\widetilde{\rho}_7 = B$, where A, B are real constants which verify $A^2 + B^2 = 1$. Finally, the immersion r is given by

$$r(x_1, y_1, x_2, y_2, u, v) = (x_1 u, y_1 u, -y_1 v, x_1 v, x_2 u, y_2 u, Ax_2 v - By_2 v, Bx_2 v + Ay_2 v). \tag{64}$$

Thus, after a rigid transformation (the rotation $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ applied to the last two components in $\mathbf{R}^8$) we get the conclusion. □

3. *CR* warped product submanifolds in Sasakian manifolds

The main purpose of this section is devoted to the presentation of some properties of warped product contact CR submanifolds in Sasakian manifolds. The notion of warped product (or, more generally warped bundle) was introduced by Bishop and O'Neill in [10] in order to construct a large variety of manifolds of negative curvature. For example, negative space forms can easily be constructed in this way from flat space forms. Along the years the interest was to find an analogous of classical de Rham theorem to warped products. A result was proved by Hiepko and we used it in order to give a characterization of warped product contact CR submanifolds in Sasakian manifolds.

Let B, F be two Riemannian manifolds with Riemannian metrics g_B and g_F respectively. Let $f > 0$ be a smooth positive function on B and consider $B \times F$ the product manifold. Let $\pi_1 : B \times F \longrightarrow B$ and $\pi_2 : B \times F \longrightarrow F$ be the canonical projections. We give the following definition: the manifold $M = B \times_f F$ is called *warped product* if it is equipped with the Riemannian structure such that

$$\|X\|^2 = \|\pi_{1,*}(X)\|^2 + f^2(\pi_1(x))\|\pi_{2,*}(X)\|^2 \tag{65}$$

for all $X \in T_x(M)$, $x \in M$, or, equivalently,

$$g = g_B + f^2\, g_F \tag{66}$$

with the usual meaning. In this case, f is called *the warped function* on the warped product.

By following an idea of B. Y. Chen we give

Theorem 3.1. *Let $\widetilde{M}$ be a Sasakian manifold and let $M = N^\perp \times_f N^T$ be a warped product CR submanifold such that $N^\perp$ is a totally real submanifold and N^T is ϕ holomorphic (invariant) of $\widetilde{M}$. Then M is a CR product.*

Proof. Let X be tangent to N^T and let Z be a vector field tangent to $N^\perp$. From the Levi–Civita formula we find that $\nabla_X Z = (Z \ln f)\, X$. Now we distinguish two cases:

Case 1: ξ is tangent to $N^\perp$. Take $Z = \xi$. Since $\nabla_X \xi = PX = \phi X$ it follows $\phi X = (\xi \ln f)\, X$. But this is impossible if dim $N^T \neq 0$.

Case 2: ξ is tangent to N^T. Take $X = \xi$. Since $\nabla_Z \xi = PZ = 0$ and $\nabla_Z \xi = \nabla_\xi Z$ (ξ is tangent to N^T while Z is tangent to $N^\perp$) one gets $0 = Z(\ln f)\xi$ and hence $Z(\ln f) = 0$ for all Z tangent to $N^\perp$. Consequently f is constant and thus the warped product above is nothing but a product $N^\perp \times N^T_f$ where N^T_f is the manifold N^T with the metric $f^2 g_{N^T}$ which is homothetic with the original metric. □

The previous theorem shows that do not exist warped product contact CR submanifolds in the form $N^\perp \times_f N^T$ other than contact CR products such that N^T is a ϕ-invariant submanifold and $N^\perp$ is a totally real submanifold of $\widetilde{M}$. This is the reason that from now on we will consider warped product contact CR submanifolds in the form $N^T \times_f N^\perp$. We give the following definition: A contact CR submanifold M of a Sasakian manifold $\widetilde{M}$, tangent to the structure vector field ξ is called *a contact CR warped product* if it is the warped product $N^T \times_f N^\perp$ of an invariant submanifold N^T, tangent to ξ and a totally real submanifold $N^\perp$ of $\widetilde{M}$ (where f is the warped function).

Sometimes we will use $\langle\ ,\ \rangle$ for all three metrics g, g_{N^T}, $g_{N^\perp}$ (when there is no confusion).

Lemma 3.1. *Let M be a contact CR submanifold in Sasakian manifold $\widetilde{M}^{2m+1}$ such that $\xi \in \mathcal{D}$. Then we have*

$$g(\nabla_U Z, X) = \widetilde{g}(\phi A_{\phi Z} U, X), \quad \forall X \in \mathcal{D},\ \forall Z \in \mathcal{D}^\perp,\ \forall U \in T(M); \quad (67)$$

$$A_{\phi Z} W = A_{\phi W} Z, \quad \forall Z, W \in \mathcal{D}^\perp; \quad (68)$$

$$A_{\phi\mu}X + A_{\mu}(\phi X) = 0, \quad \forall X \in \mathcal{D},\ \forall \mu \in \nu. \tag{69}$$

Proof. Let us prove the first formula. We have

$$\begin{aligned}
\widetilde{g}(\phi A_{\phi Z}U, X) &= \widetilde{g}(\widetilde{\nabla}_U(\phi Z) - \nabla^{\perp}_U(\phi Z), \phi X)\\
&= \widetilde{g}(-\widetilde{g}(U,Z)\xi + \eta(Z)U, \phi X) + \widetilde{g}(\phi\widetilde{\nabla}_U Z, \phi X)\\
&= \widetilde{g}(\widetilde{\nabla}_U Z, X) - \eta(\widetilde{\nabla}_U Z)\eta(X)\\
&= \widetilde{g}(\nabla_U Z, X) - \eta(X)(U\widetilde{g}(Z,\xi) - \widetilde{g}(Z, \widetilde{\nabla}_U \xi))\\
&= g(\nabla_U Z, X) + \eta(X)\widetilde{g}(Z, \phi U) = g(\nabla_U Z, X).
\end{aligned}$$

In order to prove the formula (68) let us take $U \in T(M)$. We have

$$g(A_{\phi Z}W, U) = g(W,Z)\eta(U) + \widetilde{g}(\widetilde{\nabla}_W Z, \phi U).$$

Hence $g(A_{\phi Z}W - A_{\phi W}Z, U) = \widetilde{g}([W,Z], \phi U)$. Due to the integrability of $\mathcal{D}^{\perp}$, $[Z,W] \in \mathcal{D}^{\perp}$ while $\phi U \in \mathcal{D} \oplus \phi\mathcal{D}^{\perp}$. It follows that $g(A_{\phi Z}W - A_{\phi W}Z, U) = 0$ for all U tangent to M. From here we have the formula.

For the proof of (69) we have $g(A_{\phi\mu}X, U) = -\widetilde{g}(\mu, \phi\widetilde{\nabla}_X U)$ and $g(A_{\mu}(\phi X), U) = \widetilde{g}(\mu, \phi\widetilde{\nabla}_U X)$, with $U \in T(M)$. It follows that $g(A_{\phi\mu}X + A_{\mu}(\phi X), U) = 0$ so, $A_{\phi\mu}X + A_{\mu}(\phi X) = 0$, $\forall X \in \mathcal{D}$, $\forall \mu \in \nu$. □

Lemma 3.2. *If $M = N^T \times_f N^{\perp}$ is a contact CR warped product in a Sasakian manifold $\widetilde{M}$ then*

$$\langle B(\mathcal{D},\mathcal{D}), \phi\mathcal{D}^{\perp}\rangle = 0 \tag{70}$$

$$\nabla_X Z = \nabla_Z X = X(\ln f)Z \tag{71}$$

for X tangent to N^T and Z tangent to $N^{\perp}$;

$$\xi(f) = 0 \tag{72}$$

$$\langle B(\phi X, Z), \phi W\rangle = (X \ln f)\langle Z, W\rangle \tag{73}$$

for X tangent to N^T and Z, W tangent to $N^{\perp}$.

Proof. Consider $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$. Then

$$\langle B(X,Y), \phi Z\rangle = \langle \widetilde{\nabla}_X Y, \phi Z\rangle = \langle \phi Y, \widetilde{\nabla}_X Z\rangle = -\langle \nabla_X(\phi Y), Z)\rangle = 0.$$

Take now X tangent to N^T and Z tangent to $N^\perp$. We have that $\langle\nabla_X Z, Y\rangle = 0$ for all Y tangent to N^T so, $\nabla_X Z$ is tangent to $N^\perp$. By using Levi–Civita formula and the orthogonality of the two distributions one gets $2g(\nabla_X Z, W) = X(f^2 g_{N^\perp}(Z, W))$. But $g_{N^\perp}$ depends only of the points of $N^\perp$ so we obtain

$$2g(\nabla_X Z, W) = 2f\ X(f) g_{N^\perp}(Z, W) = 2X(\ln f)g(Z, W).$$

Recall that $\nabla_U \xi = PU$. It follows that $\nabla_Z \xi = 0$ for all Z tangent to $N^\perp$. Combining with (71) one gets $\xi(f) = 0$.

To prove the last statement we will use (67):

$$\langle B(\phi X, Z), \phi W\rangle = \langle A_{\phi W} Z, \phi X\rangle = -\langle\nabla_Z W, X\rangle = X(\ln f)\langle Z, W\rangle.$$

This ends the proof of this lemma. □

In the following we give a characterization of the contact CR warped product in Sasakian manifold, an analogue of Proposition 2.3. We have the following result of S. Hiepko (cf. e.g. [30]): *Let $\mathcal{F}$ be a vector subbundle in the tangent bundle of a Riemannian manifold M and let $\mathcal{F}^\perp$ be its normal bundle. Assume that the two distributions are both involutive and the integral manifold of $\mathcal{F}$ (resp. $\mathcal{F}^\perp$) are extrinsic spheres (resp. totally geodesic). Then M is locally isometric to a warped product $N_1 \times_f N_2$. Moreover, if M is simply connected and complete there exists a global isometry of M with a warped product.*

Theorem 3.2 (of characterization)**.** *A strictly proper CR submanifold M of a Sasakian manifold $\widetilde{M}$, and tangent to the structure vector field ξ is locally a contact CR warped product if and only if*

$$A_{\phi Z} X = (\eta(X) - (\phi X)(\mu))\ Z, \quad X \in \mathcal{D},\ Z \in \mathcal{D}^\perp \tag{74}$$

for some function μ on M satisfying $W\mu = 0$ for all $W \in \mathcal{D}^\perp$.

Proof. "$\Longrightarrow$:" Let $M = N^T \times_f N^\perp$ be a (locally) contact CR warped product and let $X \in \mathcal{D}$, $Z \in \mathcal{D}^\perp$. It can be easily proved that $g(A_{\phi Z} X, Y) = 0$ for all $Y \in \mathcal{D}$ which shows that $A_{\phi Z} X$ belongs to $\mathcal{D}^\perp$. Take $W \in \mathcal{D}^\perp$. We get

$$g(A_{\phi Z} X, W) = [\eta(X) - (\phi X)(\ln f)]\ g(W, Z)$$

from which we obtain the conclusion where $\mu = \ln f$.

"$\Longleftarrow$:" Let us prove now the converse. Suppose that $A_{\phi Z}X = (\eta(X) - (\phi X)(\mu))Z$. We get easily that

$$\widetilde{g}\,(B(X,Y),\phi Z) = 0 \text{ and } \widetilde{g}\,(B(X,W),\phi Z) = (\eta(X) - (\phi X)(\mu))\, g(Z,W)$$

where $X, Y \in \mathcal{D}$ and $Z, W \in \mathcal{D}^{\perp}$. In the second equality replacing X by ϕX (since $\mathcal{D}$ is ϕ invariant) we obtain

$$\widetilde{g}(B(\phi X, W), \phi Z) = (X(\mu) - \eta(X)\xi(\mu))\, g(Z,W).$$

So if $X \in H(M)$ we get $\widetilde{g}(B(\phi X, W), \phi Z) = X(\mu)g(Z,W)$ $(*)$ and if $X = \xi$ we obtain a trivial identity. From now on we will consider $X \in H(M)$.

From the proof of the Proposition 2.3 we have that the distribution $\mathcal{D}$ is integrable and the integral manifold N^T is totally geodesic in M. On the other hand by Lemma 2.6 and $(*)$ we obtain

$$g(\nabla_Z X, W) = -\widetilde{g}(\phi A_{\phi W} Z, X) = \widetilde{g}(B(\phi X, Z), \phi W) = X(\mu)g(Z,W). \quad (**)$$

Let $N^{\perp}$ be the integral manifold of $\mathcal{D}^{\perp}$. Let σ_2 be the second fundamental form of $N^{\perp}$ in M. Computing $g(\nabla_Z W, X)$ in two ways one gets

$$\sigma_2(Z,W) = -(\text{grad } \mu) g_{N^{\perp}}(Z,W)$$

(since the action of a vector from $\mathcal{D}^{\perp}$ to μ vanishes). Thus $\mathcal{D}^{\perp}$ is totally umbilical in M. The spherical condition (see e.g. [24]) is fulfilled

$$g(\nabla_Z(\text{grad } \mu), X) = 0, \quad \forall Z \in \mathcal{D}^{\perp},\ X \in \mathcal{D}.$$

So, we conclude that $\mathcal{D}^{\perp}$ is an extrinsic sphere. Now we apply the result of S. Hiepko and obtain that M is locally isometric to a warped product $N^T \times_f N^{\perp}$. $\square$

If M is simply connected and complete then the result of previous theorem is globally.

3.1. A good geometric inequality for contact CR-warped product in Sasakian space form.

For M a Riemannian manifold of dimension k and a a smooth function on M we recall

1. ∇a, the gradient of a is defined by

$$\langle \nabla a, X \rangle = X(a), \quad \forall X \in \chi(M) \tag{75}$$

2. Δa, the laplacian of a is defined by

$$\Delta a = \sum_{j=1}^{k} \left\{ (\nabla_{e_j} e_j)a - e_j e_j(a) \right\} = -\mathrm{div}\, \nabla a \tag{76}$$

where ∇ is the Levi–Civita connection on M and $\{e_1, \ldots, e_k\}$ is an orthonormal frame on M.

As consequence, we have

$$\|\nabla a\|^2 = \sum_{j=1}^{k} (e_j(a))^2 \,. \tag{77}$$

Theorem 3.3. *Let $M = N^T \times_f N^\perp$ be a contact CR warped product of a Sasakian space form $\widetilde{M}^{2m+1}(c)$ and let $h = 2s+1 = \dim N^T$ and $p = \dim N^\perp$. Then the second fundamental form of M satisfies the following inequality*

$$\|B\|^2 \geq 2p \left[\|\nabla \ln f\|^2 - \Delta \ln f + \frac{c+3}{2}\, s + 1 \right]. \tag{78}$$

Proof. We have

$$\|B(\mathcal{D}, \mathcal{D}^\perp)\|^2 = \sum_{j=1}^{2s+1} \sum_{\alpha=1}^{p} \|B(X_j, Z_\alpha)\|^2 \tag{79}$$

where $\{X_j\}_{j=\overline{1,2s+1}}$ and $\{Z_\alpha\}_{\alpha=\overline{1,p}}$ are (local) orthonormal frames on N^T and $N^\perp$, respectively. On N^T we will consider a ϕ-adapted orthonormal frame, namely $\{e_j, \phi e_j, \xi\}_{j=\overline{1,s}}$.

We have to evaluate $\|B(X,Z)\|^2$ with $X \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$. The second fundamental form $B(X,Y)$ is normal to M so, it splits into two orthogonal components

$$B(X,Z) = B_{\phi\mathcal{D}^\perp}(X,Z) + B_\nu(X,Z) \tag{80}$$

where $B_{\phi\mathcal{D}^\perp}(X,Z) \in \phi\mathcal{D}^\perp$ and $B_\nu(X,Z) \in \nu$. So

$$\|B(X,Z)\|^2 = \|B_{\phi\mathcal{D}^\perp}(X,Z)\|^2 + \|B_\nu(X,Z)\|^2. \tag{81}$$

If $X = \xi$ we have $B(\xi, Z) = FZ = \phi Z$. Hence

$$B_{\phi\mathcal{D}^\perp}(\xi, Z) = \phi Z, \quad B_\nu(\xi, Z) = 0. \tag{82}$$

Consider now $X \in H(M)$ and let us compute the norm of the $\phi\mathcal{D}^\perp$-component of $B(X, Z)$. We have

$$\|B_{\phi\mathcal{D}^\perp}(X, Z)\|^2 = \langle B_{\phi\mathcal{D}^\perp}(X, Z), B(X, Z)\rangle.$$

By using relation (73), after the computations, we obtain

$$\|B_{\phi\mathcal{D}^\perp}(X, Z)\|^2 = -\left[(\phi X)(\ln f)\right]\langle \phi Z, B(X, Z)\rangle = \left[(\phi X)(\ln f)\right]^2\langle Z, Z\rangle.$$

So

$$\begin{aligned}\|B_{\phi\mathcal{D}^\perp}(e_j, Z_\alpha)\|^2 &= \left((\phi e_j)(\ln f)\right)^2, \\ \|B_{\phi\mathcal{D}^\perp}(\phi e_j, Z_\alpha)\|^2 &= (e_j(\ln f))^2.\end{aligned} \tag{83}$$

On the other hand, from (77) we have

$$\|\nabla \ln f\|^2 = \sum_{j=1}^{s}(e_j \ln f)^2 + \sum_{j=1}^{s}\left[(\phi e_j)(\ln f)\right]^2 \tag{84}$$

since $\xi(\ln f) = 0$. Finally we can compute the norm $\|B_{\phi\mathcal{D}^\perp}(\mathcal{D}, \mathcal{D}^\perp)\|^2$. Thus

$$\begin{aligned}\|B_{\phi\mathcal{D}^\perp}(\mathcal{D}, \mathcal{D}^\perp)\|^2 = &\sum_{\substack{j=1,s \\ \alpha=1,p}} \left\{\|B_{\phi\mathcal{D}^\perp}(e_j, Z_\alpha)\|^2 + \|B_{\phi\mathcal{D}^\perp}(\phi e_j, Z_\alpha)\|^2\right\} \\ &+ \sum_{\alpha=1}^{p}\|B_{\phi\mathcal{D}^\perp}(\xi, Z_\alpha)\|^2 = \sum_{\alpha=1}^{p}\|\nabla \ln f\|^2 + \sum_{\alpha=1}^{p}\|\phi Z_\alpha\|^2.\end{aligned}$$

Since $\|\phi Z_\alpha\|^2 = 1$ we can conclude that

$$\|B_{\phi\mathcal{D}^\perp}(\mathcal{D}, \mathcal{D}^\perp)\|^2 = p\ \left\{\|\nabla \ln f\|^2 + 1\right\} \tag{85}$$

Let us compute now the norm of the ν-component of $B(X, Z)$. We have

$$\|B_\nu(X, Z)\|^2 = \langle B_\nu(X, Z), B(X, Z)\rangle = \langle A_{B_\nu(X,Z)}X, Z\rangle.$$

By using formula (69) we can write $A_{B_\nu(X,Z)}X = A_{\phi B_\nu(X,Z)}(\phi X)$ so,

$$\|B_\nu(X, Z)\|^2 = \langle \phi B(X, Z) - \phi B_{\phi\mathcal{D}^\perp}(X, Z), B(\phi X, Z)\rangle.$$

Since $\phi B_{\phi\mathcal{D}^\perp}(X,Z)$ belongs to $\mathcal{D}^\perp$ we obtain

$$\|B_\nu(X,Z)\|^2 = \widetilde{g}(\phi B(X,Z), B(\phi X,Z)), \quad X\in H(M),\ Z\in\mathcal{D}^\perp. \quad (86)$$

Consider the tensor field $\widetilde{H}_B$. As we already have seen

$$\widetilde{H}_B(X,Z) = \langle(\nabla_{\phi X}B)(X,Z) - (\nabla_X B)(\phi X,Z), \phi Z\rangle, \quad X\in H(M),\ Z\in\mathcal{D}^\perp.$$

Using the definition of ∇B, developing the expression above we obtain six terms:

$$T_1 := \langle\nabla^\perp_{\phi X}B(X,Z),\phi Z\rangle \qquad T_2 := -\langle B(\nabla_{\phi X}X,Z),\phi Z\rangle$$

$$T_3 := -\langle B(X,\nabla_{\phi X}Z),\phi Z\rangle \quad T_4 := -\langle\nabla^\perp_X B(\phi X,Z),\phi Z\rangle$$

$$T_5 := \langle B(\nabla_X(\phi X),Z),\phi Z\rangle \quad T_6 := \langle B(\phi X,\nabla_X Z),\phi Z\rangle.$$

We will write the expressions of all these terms.

In order to compute T_2 we remark first that $\eta(\nabla_{\phi X}X) = \|X\|^2$ and after the computations we get

$$T_2 = \|Z\|^2\left\{(\phi\nabla_{\phi X}X)(\ln f) - \|X\|^2\right\}. \quad (87)$$

Then, it is not difficult to show that we have

$$T_3 = [(\phi X)(\ln f)]^2\|Z\|^2 \quad \text{and} \quad T_6 = (X\ln f)^2\|Z\|^2. \quad (88)$$

As above, we write down firstly $\eta(\nabla_X(\phi X)) = -\|X\|^2$. It follows

$$T_5 = -\|Z\|^2\left\{(\phi\nabla_X(\phi X))(\ln f) + \|X\|^2\right\}. \quad (89)$$

We direct our attention to the first and the fourth terms:

$$\begin{aligned} T_1 &= \widetilde{g}(\widetilde{\nabla}_{\phi X}B(X,Z),\phi Z)\\ &= -(\phi X)\left((\phi X)(\ln f)\|Z\|^2\right) - \widetilde{g}\left(B(X,Z),\widetilde{\nabla}_{\phi X}(\phi Z)\right)\end{aligned}$$

$$\begin{aligned} T_4 &= \widetilde{g}(-\widetilde{\nabla}_{\phi X}B(X,Z),\phi Z)\\ &= -X\left((X\ln f)\|Z\|^2\right) + \widetilde{g}(B(\phi X,Z),\widetilde{\nabla}_X(\phi Z)).\end{aligned}$$

We also have

$$\begin{cases} (\phi X)((\phi X)(\ln f)\|Z\|^2) = \|Z\|^2\left\{(\phi X)^2(\ln f) + 2[(\phi X)(\ln f)]^2\right\}\\ X((X\ln f)\|Z\|^2) = \|Z\|^2\left\{X^2(\ln f) + 2(X\ln f)^2\right\}\end{cases}$$

and

$$\begin{cases} \widetilde{g}(B(X,Z),\widetilde{\nabla}_{\phi X}(\phi Z)) = -[(\phi X)(\ln f)]^2\|Z\|^2 - \langle \phi B(X,Z), B(\phi X,Z)\rangle \\ \widetilde{g}(B(\phi X,Z),\widetilde{\nabla}_X(\phi Z)) = (X\ln f)^2\|Z\|^2 + \langle B(\phi X,Z),\phi B(X,Z)\rangle. \end{cases}$$

Let us sum now T_1 and T_4; we obtain

$$\begin{aligned} T_1 + T_4 = -\|Z\|^2\{&(\phi X)^2(\ln f) + X^2(\ln f) \\ &+ [(\phi X)(\ln f)]^2 + (X\ln f)^2\} + 2\langle B(\phi X,Z),\phi B(X,Z)\rangle. \end{aligned}$$

If we sum the third and the sixth terms we get

$$T_3 + T_6 = \|Z\|^2\left\{[(\phi X)(\ln f)]^2 + (X\ln f)^2\right\}.$$

In the same way we have

$$T_2 + T_5 = \|Z\|^2\left\{(\phi\nabla_{\phi X}X)(\ln f) - (\phi\nabla_X(\phi X))(\ln f) - 2\|X\|^2\right\}.$$

Consequently

$$\begin{aligned} \widetilde{H}_B(X,Z) = \|Z\|^2\{&(\phi\nabla_{\phi X}X)(\ln f) - (\phi\nabla_X(\phi X))(\ln f) \\ &- (\phi X)^2(\ln f) - X^2(\ln f) - 2\|X\|^2\} \\ &+ 2\langle B(\phi X,Z),\phi B(X,Z)\rangle. \end{aligned} \tag{90}$$

It is not difficult to prove

$$\begin{aligned} (\phi\nabla_{\phi X}X)(\ln f) &= (\nabla_{\phi X}(\phi X))(\ln f), \\ \phi\nabla_X(\phi X))(\ln f) &= -(\nabla_X X)(\ln f). \end{aligned} \tag{91}$$

Using (86) and (91) the expression of $\widetilde{H}_B(X,Z)$ becomes

$$\begin{aligned} \widetilde{H}_B(X,Z) = \|Z\|^2\{&(\nabla_{\phi X}(\phi X))(\ln f) + (\nabla_X X)(\ln f) - (\phi X)^2(\ln f) \\ &- X^2(\ln f) - 2\|X\|^2\} + 2\|B_\nu(X,Z)\|^2. \end{aligned} \tag{92}$$

It is time to work with orthonormal frames. Thus

$$\left[\begin{aligned} \widetilde{H}_B(e_j,Z_\alpha) &= (\nabla_{\phi e_j}(\phi e_j))(\ln f) + (\nabla_{e_j}e_j)(\ln f) - (\phi e_j)^2(\ln f) \\ &\quad - e_j^2(\ln f) - 2 + 2\|B_\nu(e_j,Z_\alpha)\|^2\widetilde{H}_B(\phi e_j,Z_\alpha) \\ &= (\nabla_{e_j}e_j)(\ln f) + (\nabla_{\phi e_j}(\phi e_j))(\ln f) - e_j^2(\ln f) \\ &\quad - (\phi e_j)^2(\ln f) - 2 + 2\|B_\nu(\phi e_j,Z_\alpha)\|^2. \end{aligned}\right. \tag{93}$$

On the other hand we have

$$\Delta(\ln f) = \sum_{j=1}^{s} \big((\nabla_{e_j} e_j)(\ln f) - e_j^2(\ln f)\big) + \sum_{j=1}^{s} \big((\nabla_{\phi e_j}(\phi e_j))(\ln f) - (\phi e_j)^2(\ln f)\big)$$

since $\xi(\ln f) = 0$. Taking the sum in the two relations of (93) one gets

$$\begin{cases} 2\sum_{j=1}^{s}\sum_{\alpha=1}^{p} \|B_\nu(e_j, Z_\alpha)\|^2 = \sum_{j=1}^{s}\sum_{\alpha=1}^{p} \widetilde{H}_B(e_j, Z_\alpha) - p\Delta(\ln f) + 2sp \\ 2\sum_{j=1}^{s}\sum_{\alpha=1}^{p} \|B_\nu(\phi e_j, Z_\alpha)\|^2 = \sum_{j=1}^{s}\sum_{\alpha=1}^{p} \widetilde{H}_B(\phi e_j, Z_\alpha) - p\Delta(\ln f) + 2sp. \end{cases} \tag{94}$$

Using (31) we can write that

$$2\sum_{j=1}^{s}\sum_{\alpha=1}^{p} \big\{\|B_\nu(e_j, Z_\alpha)\|^2 + \|B_\nu(\phi e_j, Z_\alpha)\|^2\big\} = (c+3)sp - 2p\Delta(\ln f).$$

Finally we conclude that B satisfies the inequality. □

Corollary 3.1. *Let $M = N^T \times_f N^\perp$ be a contact CR warped product in a Sasakian space form $\widetilde{M}^{2m+1}(c)$ and suppose N^T to be compact. Denote by dv_T and vol (N^T) the volume element and the volume on N^T. Let λ_1 be the first non zero eigenvalue of the Laplacian on N^T. Then*

$$\int_{N^T} \|B\|^2 dv_T \geq (\ 2p + (c+3)sp\ )\,\mathrm{vol}\,(N^T) + 2p\lambda_1 \int_{N^T} (\ln f)^2 dv_T. \tag{95}$$

Proof. From the minimum principle we have

$$\int_{N^T} \|\nabla \ln f\|^2 dv_T \geq \lambda_1 \int_{N^T} (\ln f)^2 dv_T. \tag{96}$$

Now we have to integrate on N^T the inequality satisfied by the norm of B and obtain immediately the formula (95). □

Corollary 3.2. *Suppose that $\widetilde{M}(c)$ is a Sasakian space form of type 3, i.e. it is a product between* $\mathbf{R}$ *and a simply connected bounded domain B^m in $\mathbf{C}^m$ endowed with a Kähler structure with constant holomorphic sectional curvature $k < 0$. Then the function $\ln f$ is subharmonic, i.e. $\Delta \ln f \leq 0$.*

Proof. From the proof of Theorem 3.3 we have the following relation

$$2\sum_{j=1}^{s}\sum_{\alpha=1}^{p}\left\{\|B_\nu(e_j, Z_\alpha)\|^2 + \|B_\nu(\phi e_j, Z_\alpha)\|^2\right\} = (c+3)sp - 2p\ \Delta(\ln f).$$

Since the left side of the equality is greater than zero and $c = k-3$ one gets $ksp - 2p\ \Delta \ln f \geq 0$. Hence $\Delta \ln f \leq \frac{ks}{2} \leq 0$ which completes the proof. □

Corollary 3.3. *Suppose that $\widetilde{M}(c)$ is a Sasakian space form of type 2, i.e. $\widetilde{M} = \mathbf{R}^{2m+1}$ with the usual Sasakian structure with constant ϕ-sectional curvature $c = -3$. Then we have*

(a) *The function $\ln f$ is a subharmonic function, i.e. $\Delta \ln f \leq 0$*

(b) *The function $\ln f$ is harmonic if and only if $B(\mathcal{D}, \mathcal{D}^\perp) \subset \phi\mathcal{D}^\perp$.*

Proof. We use the same relation as in Corollary 2.2 and the statement (a) follows immediately. The harmonicity of the function $\ln f$ is equivalent with $B_\nu(e_j, Z_\alpha) = 0$, $B_\nu(\phi e_j, Z_\alpha) = 0$ for all $j = 1, \ldots, s$ and $\alpha = 1, \ldots, p$. This means that $B_\nu(\mathcal{D}, \mathcal{D}^\perp) = 0$, i.e. $B(\mathcal{D}, \mathcal{D}^\perp) \subset \phi\mathcal{D}^\perp$. □

Suppose that in previous two corollaries the manifold N^T is compact. It follows easily that f is a constant function and M becomes a contact CR product.

In the following we will prove a general inequality satisfied by the norm of the second fundamental form B of a contact CR warped product in Sasakian manifolds (which are not necessary Sasakian space forms).

Theorem 3.4. *Let $M = N^T \times_f N^\perp$ be a contact CR warped product in a Sasakian manifold $\widetilde{M}$. We have*

(1) *The norm of the second fundamental form of M satisfies*

$$\|B\|^2 \geq 2p\left(\|\nabla \ln f\|^2 + 1\right) \tag{97}$$

where $\nabla \ln f$ is the gradient of $\ln f$ and $p = \dim N^\perp$.

(2) *If the equality sign in* (97) *holds identically, then* N^T *is a totally geodesic submanifold and* $N^\perp$ *is a totally umbilical submanifold of* $\widetilde{M}$. *The product manifold* M *is a minimal submanifold in* $\widetilde{M}$. *Moreover if* $\widetilde{M} = \mathbf{R}^{2m+1}$ *with the usual Sasakian structure then* $\ln f$ *is a superharmonic function, i.e.* $\Delta \ln f \geq 0$.

(3) The case $TM^\perp = \phi\mathcal{D}^\perp$. *If* $p > 1$ *then the equality sign in* (97) *holds identically if and only if* $N^\perp$ *is a totally umbilical submanifold of* $\widetilde{M}$.

(4) *If* $p = 1$ *then the equality sign in* (97) *holds identically if and only if the characteristic vector field* $\phi\mu$ *of* M *satisfies* $A_\mu\phi\mu = -\phi\nabla \ln f - \xi$. (Notice that in this case, M is a hypersurface in $\widetilde{M}$ with the unitary normal vector field denoted by μ.)

Proof. (1) As in the proof of the previous theorem we can write

$$\|B\|^2 = \|B(\mathcal{D},\mathcal{D})\|^2 + 2\left(\|B_{\phi\mathcal{D}^\perp}(\mathcal{D},\mathcal{D}^\perp)\|^2 + \|B_\nu(\mathcal{D},\mathcal{D}^\perp)\|^2\right) + \|B(\mathcal{D}^\perp,\mathcal{D}^\perp)\|^2.$$

We have already proved that $\|B_{\phi\mathcal{D}^\perp}(\mathcal{D},\mathcal{D}^\perp)\|^2 = p\left\{\|\nabla \ln f\|^2 + 1\right\}$. Hence we obtain the inequality. (We mention here that even if in the theorem used the manifold $\widetilde{M}$ was a Sasakian space form, the equality is still valid.)

(2) Assume now the equality sign holds identically. It follows

$$B(\mathcal{D},\mathcal{D}) = 0, \quad B(\mathcal{D}^\perp,\mathcal{D}^\perp) = 0, \quad B_\nu(\mathcal{D},\mathcal{D}^\perp) = 0. \tag{98}$$

Since N^T is totally geodesic in M, the first condition in (98) shows that N^T is totally geodesic in $\widetilde{M}$. Denote by σ_2 the second fundamental forms of $N^\perp$ in M. We have $g(\nabla_Z W, X) = g(\sigma_2(Z,W), X)$ for X tangent to N^T. On the other hand $g(\nabla_Z W, X) = -g(W, X(\ln f)Z) = -g(Z,W)X(\ln f)$. Next, one gets $\sigma_2(Z,W) = -g(Z,W)\nabla(\ln f)$ (because σ_2 is tangent to N^T). It follows that $N^\perp$ is totally umbilical in M. By using Gauss formula it follows that $N^\perp$ is also totally umbilical in $\widetilde{M}$.

Finally, since $B(\mathcal{D},\mathcal{D}) = 0$ and $B(\mathcal{D}^\perp,\mathcal{D}^\perp) = 0$ it follows that the mean curvature of M vanishes, so M is minimal in $\widetilde{M}$.

When $\widetilde{M} = \mathbf{R}^{2m+1}$ we get easily the result from the Theorem 3.3 (if the manifold N^T is compact then f is a constant).

(3) If the equality sign holds identically, the statement follows from (2). We must prove the converse, i.e. $N^\perp$ totally umbilical in $\widetilde{M}$ implies the equality sign.

We have from Lemma 3.2 that $\langle B(\mathcal{D},\mathcal{D}), \phi\mathcal{D}^\perp\rangle = 0$. So, since $T(M)^\perp = \phi\mathcal{D}^\perp$ and $B(\mathcal{D},\mathcal{D}) \subset T(M)^\perp$ it follows that $B(\mathcal{D},\mathcal{D}) = 0$.

If $N^\perp$ is totally umbilical in $\widetilde{M}$, then there exists a vector field $\widetilde{H}$, normal to $N^\perp$ (in $\widetilde{M}$) such that the second fundamental form $\widetilde{\sigma}_2$ of $N^\perp$ in $\widetilde{M}$ satisfies $\widetilde{\sigma}_2(Z,W) = g_{N^\perp}(Z,W)\widetilde{H}$. Since $\widetilde{\sigma}_2(Z,W) = \sigma_2(Z,W) + B(Z,W)$ and since $N^\perp$ is totally umbilical in $M = N^T \times_f N^\perp$ (and hence $\sigma_2(Z,W) = g_{N^\perp}(Z,W)H_2$ for some H_2 normal to $N^\perp$ in M) it follows that there exists a vector field N, normal to M (in $\widetilde{M}$ obviously) such that $B(Z,W) = g_{N^\perp}(Z,W)N$. Take Z,W in $\mathcal{D}^\perp$ unitary and orthogonal (in $N^\perp$) (we can do this since $p > 1$). Applying Lemma 2.1, statement 2, we deduce $\langle N, \phi W\rangle = \langle A_{\phi Z}W, Z\rangle = 0$ (since Z,W are orthogonal). But $N \in T(M)^\perp = \phi\mathcal{D}^\perp$. Taking $W = -\phi N$ we get $N = 0$, so $B(Z,W) = 0$ for all $Z,W \in \mathcal{D}^\perp$ and hence $B(\mathcal{D}^\perp,\mathcal{D}^\perp) = 0$. The third condition ($B(\mathcal{D},\mathcal{D}^\perp) \subset \phi\mathcal{D}^\perp$) which assures our conclusion is automatically satisfied.

(4) If $p = 1$ we have $\dim(T_x(M))^\perp = 1$ for all $x \in M$; thus M it is a hypersurface in $\widetilde{M}$. Let μ the unit normal vector field of M (in $\widetilde{M}$). It follows that $Z = \phi\mu$ is tangent to M and unitary. Moreover we have $\mathcal{D}^\perp = \operatorname{span}[Z]$.

Suppose $B(\mathcal{D}^\perp,\mathcal{D}^\perp) = 0$; this means $B(Z,Z) = 0$. Thus we have $\langle A_\mu Z, Z\rangle$=0. It follows that $A_\mu Z \in \mathcal{D}$. Let $X \in \mathcal{D}$. We have

$$\langle A_\mu Z, X\rangle = \langle \widetilde{\nabla}_Z X, -\phi Z\rangle = \langle \widetilde{\nabla}_Z(\phi X) - \eta(X)Z, Z\rangle = (\phi X)(\ln f) - \eta(X).$$

Consider an adapted frame on $\mathcal{D} : \{e_i, \phi e_i, \xi\}$. We can write $A_\mu Z = \sum \alpha_i e_i + \sum \beta_i \phi e_i + \gamma\xi$ and so $\phi A_\mu Z = \sum \alpha_i \phi e_i + \sum(-\beta_i)e_i$. We have

$$\alpha_i = \langle \phi A_\mu Z, \phi e_i\rangle = \langle \nabla(\ln f), \phi e_i\rangle,\ \ -\beta_i = \langle \phi A_\mu Z, e_i\rangle = \langle \nabla(\ln f), e_i\rangle$$

It follows that $\phi A_\mu Z = \nabla \ln f$. Consequently $A_\mu Z = -\phi\nabla \ln f + \eta(A_\mu Z)\xi$. But $\eta(A_\mu Z) = \langle A_\mu Z, \xi\rangle = -\eta(\xi) = -1$.

Conversely one has that $A_\mu Z$ belongs to $\mathcal{D}$ and so $\langle A_\mu Z, Z\rangle = 0$ which is equivalent to $B(Z,Z) = 0$. □

For contact CR warped products in Sasakian space forms we have the following

Proposition 3.1. *Let $M = N^T \times_f N^\perp$ be a non-trivial (i.e. f non constant) complete, simply connected, contact CR warped product those second fundamental form satisfies $\|B\|^2 = 2p\left(\|\nabla \ln f\|^2 + 1\right)$ in a Sasakian space form $\widetilde{M}^{2m+1}(c)$. We have*

(1) N^T is a totally geodesic Sasakian submanifold of $\widetilde{M}^{2m+1}(c)$. Thus N^T is a Sasakian space form $N^{T^{2s+1}}(c)$.

(2) $N^\perp$ is a totally umbilical totally real submanifold of $\widetilde{M}^{2m+1}(c)$. Hence, $N^\perp$ is a real space form of constant sectional curvature. Denote it by ϵ.

(3) If $p > 1$, the function f satisfies

$$\|\nabla f\|^2 = \epsilon - \frac{c+3}{4}\, f^2. \tag{99}$$

Proof. (1) From the theorem above we have that N^T is totally geodesic submanifold in $\widetilde{M}^{2m+1}(c)$. By using Proposition 1.3, p. 49 from [49], it follows that N^T is of constant ϕ-sectional curvature c.

(2) Also from the above theorem we have that $N^\perp$ is totally umbilical submanifold in $\widetilde{M}^{2m+1}(c)$. Denoting by

$$H = -\nabla(\ln f) \tag{100}$$

we remark that the second fundamental form of $N^\perp$ in $\widetilde{M}$ can be written as $\widetilde{\sigma}_2(Z,W) = g(Z,W)\ H$.

As f is C^∞ on N^T and $g|_{N^T} \equiv g_{N^T}$ let us remark that $\|\nabla \ln f\|^2 \in C^\infty(N^T)$.

The curvature tensor of Sasakian space form $\widetilde{M}$ is given by

$$\widetilde{R}_{VW}Z = \frac{c+3}{4}\left(\widetilde{g}(W,Z)V - \widetilde{g}(V,Z)W\right)$$

since $\eta|_{N^\perp}$ vanishes and $N^\perp$ is ϕ-anti-invariant (here V, W, Z are tangent to $N^\perp$). Now, taking into account that $\widetilde{g}(V,Z) = g(V,Z) = f^2 g_{N^\perp}(V,Z)$ for all V, Z tangent to $N^\perp$ we can write

$$\widetilde{R}_{VW}Z = \frac{c+3}{4}\ f^2\left(g_{N^\perp}(W,Z)V - g_{N^\perp}(V,Z)W\right). \tag{101}$$

On the other hand it can be easily proved

$$\widetilde{R}_{VW}Z = R^{N^\perp}_{VW}Z + f^2\left\{g_{N^\perp}(Z,W)\nabla_V H - g_{N^\perp}(V,Z)\nabla_W H\right\}$$

$$+ f^2 \{g_{N^\perp}(Z,W)B(V,H) - g_{N^\perp}(V,Z)B(W,H)\}.$$

But $V, W \in N^\perp$, $H \in N^T$ so $\nabla_V H = H(\ln f)V$ and $\nabla_V H = H(\ln f)V$. We also have $H(\ln f) = -\|H\|^2$. Hence,

$$\begin{aligned}\widetilde{R}_{VW}Z = R^{N^\perp}_{VW}Z - f^2\|H\|^2 \{g_{N^\perp}(Z,W)V - g_{N^\perp}(V,Z)W\} \\ + f^2 \{g_{N^\perp}(Z,W)B(V,H) - g_{N^\perp}(V,Z)B(W,H)\}.\end{aligned}$$

From (101) we have that $\widetilde{R}_{VW}Z$ is tangent to M so one obtains

$$\widetilde{R}_{VW}Z = R^{N^\perp}_{VW}Z - f^2\|\nabla(\ln f)\|^2 \{g_{N^\perp}(Z,W)V - g_{N^\perp}(V,Z)W\} \tag{102}$$

and

$$\begin{gathered} g_{N^\perp}(Z,W)B(V,H) = g_{N^\perp}(V,Z)B(W,H) \\ \text{for all } V, Z, W \text{ tangent to } N^\perp. \end{gathered} \tag{103}$$

The relations (101) and (102) yield to

$$R^{N^\perp}_{VW}Z = f^2 \left(\frac{c+3}{4} + \|\nabla(\ln f)\|^2\right) \{g_{N^\perp}(Z,W)V - g_{N^\perp}(V,Z)W\}.$$

The coefficient depends on the points of N^T so, it is a constant (with respect to $N^\perp$). It follows that $N^\perp$ is of constant sectional curvature. Denoting it by ϵ we have

$$\epsilon = f^2 \left(\frac{c+3}{4} + \|\nabla \ln f\|^2\right). \tag{104}$$

Since f is not constant (and so $\nabla \ln f \neq 0$) it follows that $\epsilon > f^2\ \frac{c+3}{4}$.

(3) The statement follows easily from (104). We know that $\nabla \ln f = \frac{1}{f}\nabla f$ so, $\epsilon = f^2\ \frac{c+3}{4} + \|\nabla f\|^2$. □

In the case that $\widetilde{M}^{2m+1} = \mathbf{R}^{2m+1}$ with the usual Sasakian structure, then $c = -3$ and thus $\epsilon = \|\nabla f\|^2$ which means that $N^\perp$ is a space form with positive curvature.

3.2. An example of contact CR-warped product in $\mathbf{R}^{2m+1}$ satisfying the "good" equality which does not satisfy $\|B\|^2 = 2p\left(\|\nabla(\ln f)\|^2+1\right)$.

Let $\mathbf{R}^{2s+1}$ be the Sasakian space form of ϕ sectional curvature -3 (cf. [35]). Let $S^p \subset \mathbf{R}^{p+1}$ be the unit sphere immersed in the Euclidian space $\mathbf{R}^{p+1}$. Let $\mathbf{R}^{2m+1}$ be also the Sasakian space form where $m = ph+s$ with h a positive integer, $1 < h \leq s$.

Consider the map $r : \mathbf{R}^{2s+1} \times S^p \longrightarrow \mathbf{R}^{2m+1}$ defined by

$$r(x_1, y_1, \ldots, x_s, y_s, z, w^0, w^1, \ldots, w^p) = (w^0x_1, w^0y_1, \ldots, w^px_1, w^py_1, \ldots$$
$$\ldots, w^0x_h, w^0y_h, \ldots, w^px_h, w^py_h, x_{h+1}, y_{h+1}, \ldots, x_s, y_s, z)$$

where $(w^0)^2+(w^1)^2+\ldots+(w^p)^2 = 1$. On $\mathbf{R}^{2m+1}$ we consider the (local) coordinates

$$\{X_j^\alpha, Y_j^\alpha, X_a, Y_a, Z\}, \quad \alpha = 0, \ldots, p,\ j = 1, \ldots, h,\ a = h+1, \ldots, s.$$

With this notation the equations of the map r are given by

$$r : \begin{cases} X_i^\alpha = w^\alpha x_i, & Y_i^\alpha = w^\alpha y_i, \\ X_a = x_a, & Y_a = y_a,\ Z = z. \end{cases}$$

Proposition 3.2. *We have*

(1) *r is an isometric immersion between the warped product $\mathbf{R}^{2s+1} \times_f S^p$ and $\mathbf{R}^{2m+1}$. The warped function is $f = \frac{1}{2}\sqrt{\sum_{i=1}^h (x_i^2+y_i^2)}$.*

(2) *$\mathbf{R}^{2s+1}$ is a $\widetilde{\phi}$ invariant submanifold in $\mathbf{R}^{2m+1}$, i.e. $\widetilde{\phi}(r_*T(\mathbf{R}^{2s+1})) \subset r_*T(\mathbf{R}^{2s+1})$ (we put $\widetilde{\ }$ for structures on $\mathbf{R}^{2m+1}$).*

(3) *S^p is a $\widetilde{\phi}$ anti-invariant submanifold in $\mathbf{R}^{2m+1}$, i.e. $\widetilde{\phi}(r_*T(S^p)) \subset (r_*T(S^p))^\perp$.*

Proof. (1) An ordinary exercise shows that r is an immersion and $\widetilde{g}(r_*X, r_*Y) \circ r = g(X,Y)$ for X, Y tangent to $\mathbf{R}^{2s+1}$. A vector field $Z = Z^\alpha \frac{\partial}{\partial w^\alpha}$ is tangent to S^p if and only if $\sum_\alpha w^\alpha Z^\alpha = 0$. Doing the computations one gets that $\widetilde{g}(r_*Z, r_*W) = \frac{1}{4}\sum_{i=1}^h (x_i^2+y_i^2)\left(\sum_{\alpha=0}^p Z^\alpha W^\alpha\right) = f^2 g_{\mathbf{R}^{p+1}}(Z,W)$ where Z, W are vector fields tangent to the sphere S^p. Then, it is easy to prove that $\widetilde{g}(r_*X, r_*Z)$ vanishes for all X tangent to $\mathbf{R}^{2s+1}$ and Z tangent to S^p. Thus we have the statement.

(2) Let us remark that $\widetilde{\phi}(r_*X) = r_*(\phi X)$ which means that $\phi = \widetilde{\phi}|_{\mathbf{R}^{2s+1}}$.

(3) Let Z be a tangent vector field on $S^p \subset \mathbf{R}^{p+1}$ given by $Z = \sum Z^\alpha \frac{\partial}{\partial w^\alpha}$ with the tangency condition $\sum w^\alpha Z^\alpha = 0$. We have $\widetilde{\phi}(r_*Z) = \sum_{i,\alpha} Z^\alpha \big(x_i \frac{\partial}{\partial Y_i^\alpha} - y_i \frac{\partial}{\partial X_i^\alpha}\big)$. Making the computations we obtain that $\widetilde{g}(r_*X, \widetilde{\phi}(r_*Z))$ and $\widetilde{g}(r_*W, \widetilde{\phi}(r_*Z))$ vanish (X is tangent to $\mathbf{R}^{2s+1}$ and W is tangent to the sphere). This means that $\widetilde{\phi}(r_*Z)$ is normal to $r(\mathbf{R}^{2s+1} \times_f S^p)$ and hence S^p is $\widetilde{\phi}$-anti-invariant submanifold in $\mathbf{R}^{2m+1}$. □

Proposition 3.3. *The second fundamental form of the warped product $\mathbf{R}^{2s+1} \times_f S^p$ in $\mathbf{R}^{2m+1}$ satisfies*

$$\|B\|^2 = 2p \left\{ \|\nabla \ln f\|^2 - \Delta \ln f + 1 \right\}.$$

Proof. On $\mathbf{R}^{2m+1}$ we will consider the vector fields $A_i^\alpha = 2\left(\frac{\partial}{\partial X_i^\alpha} + Y_i^\alpha \frac{\partial}{\partial Z}\right)$, $B_i^\alpha = 2\frac{\partial}{\partial Y_i^\alpha}$ for $\alpha = 1, \ldots, p$ and $i = 1, \ldots, s$ and similarly A_a, B_a for $a = h+1, \ldots, m$. Denote by $\widetilde{\xi} = 2\frac{\partial}{\partial Z}$. We have

$$r_*A_i = \sum_\alpha w^\alpha A_i^\alpha, \quad r_*B_i = \sum_\alpha w^\alpha B_i^\alpha, \quad r_*A_a = A_a, \quad r_*B_a = B_a, \quad r_*\xi = \widetilde{\xi}$$

(we denote with the same letters the vector fields A_a and B_a on $\mathbf{R}^{2s+1}$ and $\mathbf{R}^{2m+1}$ respectively). Let Z be a vector field tangent to the sphere S^p. We have

$$\nabla_Z \frac{\partial}{\partial x_i} = \frac{x_i}{4f^2} Z, \quad \nabla_Z \frac{\partial}{\partial y_i} = \frac{y_i}{4f^2} Z, \quad \nabla_Z \frac{\partial}{\partial x_a} = 0,$$

$$\nabla_Z \frac{\partial}{\partial y_a} = 0, \quad \nabla_Z \frac{\partial}{\partial z} = 0.$$

Since $r_*Z = \frac{1}{2}\sum_{\alpha,i} Z^\alpha \left(x_i A_i^\alpha + y_i B_i^\alpha\right)$ we obtain by using the Gauss formula $\widetilde{\nabla}_{r_*Z} r_*A_i = \sum_\alpha Z^a A_i^\alpha$ and $\widetilde{\nabla}_{r_*Z} r_*B_i = \sum_\alpha Z^a B_i^\alpha$. Hence

$$\begin{cases} B(Z, A_i) = \sum_{\alpha,j} Z^\alpha \left[\left(\delta_{ij} - \frac{x_i x_j}{4f^2}\right) A_j^\alpha - \frac{x_i x_j}{4f^2} B_j^\alpha\right] \\ B(Z, B_i) = \sum_{\alpha,j} Z^\alpha \left[-\frac{x_i x_j}{4f^2} A_j^\alpha + \left(\delta_{ij} - \frac{x_i x_j}{4f^2}\right) B_j^\alpha\right]. \end{cases}$$

Let us take Z unitary (on the product manifold). We get

$$\|B(Z, A_i)\|^2 = \frac{1}{f^2}\left(1 - \frac{x_i^2}{4f^2}\right), \quad \|B(Z, B_i)\|^2 = \frac{1}{f^2}\left(1 - \frac{y_i^2}{4f^2}\right)$$

$$\|B(Z,\xi)\|^2 = 1, \quad \|B(Z,A_a)\|^2 = \|B(Z,B_a)\|^2 = 0.$$

So, $\|B(\mathcal{D},\mathcal{D}^\perp)\|^2 = \frac{p}{2f^2}\,(2h-1)+p$. But $B(\mathcal{D},\mathcal{D}) = 0$ and $B(\mathcal{D}^\perp,\mathcal{D}^\perp) = 0$. It follows that $\|B\|^2 = 2p\left(\frac{2h-1}{f^2}+1\right)$. Note that $\nabla \ln f = \frac{1}{2f^2}\sum_{i=1}^{h}\left(x_iA_i+y_iB_i\right)$ and thus $\|\nabla \ln f\|^2 = \frac{1}{f^2}$. Making the usual computations we obtain $\Delta \ln f = \frac{2}{f^2}\,(1-h)$. This ends the proof. □

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