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On conformally flat special quasi Einstein manifolds

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Abstract. The object of the present paper is to study conformally flat special quasi Einstein manifold.

1. Introduction

The notion of quasi Einstein manifold was introduced by M. C. Chaki and R. K. Maity [1]. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a quasi Einstein manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$S(X,Y) = a\,g(X,Y) + b\,A(X)A(Y) \tag{1}$$

where a, b are scalars of which $b \neq 0$ and A is a non-zero 1-form such that

$$g(X,\rho) = A(X) \tag{2}$$

for all vector fields X; ρ being a unit vector field. If $b = 0$, then the manifold reduces to an Einstein manifold. In such a case a, b are called associated scalars, A is called the associated 1-form and ρ is called the generator of the manifold. An n-dimensional manifold of this kind is denoted by the symbol $(QE)_n$. Throughout this paper we assume that the associated scalar a is constant. We call such a quasi Einstein manifold a special quasi Einstein manifold and such a manifold is denoted by $S(QE)_n$.

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The object of the present paper is to study conformally flat special quasi Einstein manifold. In Section 2, it is shown that a conformally flat $S(QE)_n$ is of quasi-constant curvature [2] and such a manifold is a subprojective manifold in the sense of KAGAN [3]. Also it is shown that a conformally flat $S(QE)_n$ can be expressed as a warped product $I\ X_{e^q} M^*$ where M^* is an Einstein manifold. The notion of a special conformally flat manifold which generalizes the notion of a subprojective manifold [4] was introduced by CHEN and YANO [5]. In Section 4 of this paper it is shown that a conformally flat $S(QE)_n$ satisfying $a(n-2)-b>0$ is a special conformally flat manifold. Furthermore using a theorem of Chen's and Yano's paper referred to above, it is shown that a simply connected conformally flat $S(QE)_n$ $(n>3)$ satisfying $a(n-2)-b>0$ can be isometrically immersed in an Euclidean space E^{n+1} as a hypersurface.

2. Conformally flat $S(QE)_n$

In this section we assume that the manifold $S(QE)_n$ is conformally flat. Then $div\, C = 0$ where C denotes the Weyl's conformal curvature tensor and '*div*' denotes divergence.

Hence we have

$$\begin{aligned}(\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X)\\ = \frac{1}{2(n-1)}[g(Y,Z)\, dr(X) - g(X,Y)\, dr(Z)].\end{aligned} \tag{2.1}$$

Contracting (1) we get

$$r = an + b. \tag{2.2}$$

From (2.2) it follows that

$$dr(X) = db(X), \quad \text{since } a \text{ is a constant.} \tag{2.3}$$

(1) implies that

$$\begin{aligned}(\nabla_Z S)(X,Y) = db(Z)A(X)A(Y)\\ + b[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)].\end{aligned} \tag{2.4}$$

Substituting (2.4) in (2.1) and using (2.3) we get

$$\begin{aligned} &dr(X)A(Z)A(Y) + b[(\nabla_X A)(Z)A(Y) + A(Z)(\nabla_X A)(Y)] \\ &\quad - dr(z)A(Y)A(X) - b[(\nabla_Z A)(Y)A(X) + A(Y)(\nabla_Z A)(X)] \\ &= \frac{1}{2(n-1)}[g(Y,Z)dr(X) - g(X,Y)dr(Z)]. \end{aligned} \tag{2.5}$$

Puttint $Y = Z = e_i$ in the above expression where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, $1 \leq i \leq n$, we get

$$\frac{1}{2}dr(X) = dr(\rho)A(X) + b(\nabla_\rho A)(X) + b(\nabla_{e_i} A)(e_i)A(X). \tag{2.6}$$

Again putting $Y = Z = \rho$ in (2.5) yields

$$b(\nabla_\rho A)(X) = \frac{2n-3}{2(n-1)}[dr(X) - dr(\rho)A(X)]. \tag{2.7}$$

Substituting (2.7) in (2.6) we get

$$\frac{(n-2)}{2(n-1)}\, dr(X) + \frac{1}{2(n-1)}dr(\rho)A(X) + b(\nabla_{e_i} A)(e_i)A(X) = 0. \tag{2.8}$$

Now putting$X = \rho$ in (2.6) yields

$$b(\nabla_{e_i} A)(e_i) = -\frac{1}{2}dr(\rho). \tag{2.9}$$

From (2.8) and (2.9) it follows that

$$dr(X) = dr(\rho)A(X). \tag{2.10}$$

Putting $Y = \rho$ in (2.5) and using (2.10) we obtain

$$(\nabla_Z A)(X) - (\nabla_X A)(Z) = 0 \tag{2.11}$$

which implies that the 1-form A is closed. From (2.7) we get by virtue of (2.10)

$$(\nabla_\rho A)(Z) = 0, \quad \text{since } b \neq 0. \tag{2.12}$$

Now we consider the scalar function

$$f = \frac{1}{2(n-1)}\,\frac{dr(\rho)}{b}.$$

We have

$$\nabla_X f = \frac{1}{2(n-1)} \frac{dr(\rho)}{b^2} dr(X) + \frac{1}{2(n-1)b} d^2 r(\rho, X). \tag{2.13}$$

On the otherhand, (2.10) implies

$$d^2 r(Y, X) = d^2 r(\rho, Y) A(X) + dr(\rho)(\nabla_X A)(Y)$$

from which we get

$$d^2 r(\rho, Y) A(X) = d^2 r(\rho, X) A(Y). \tag{2.14}$$

Putting $X = \rho$ in (2.14) it follows that

$$d^2(\rho, Y) = d^2 r(\rho, \rho) A(Y) = hA(Y), \quad \text{where } h \text{ is a scalar function.}$$

Thus

$$\nabla_X f = \mu A(X) \tag{2.15}$$

where $\mu = \frac{1}{2(n-1)b}[h + \frac{dr(\rho)}{b} dr(\rho)]$, using (2.10).

Using (2.15) it is easy to show that $\omega(X) = \frac{1}{2(n-1)} \frac{dr(\rho)}{b} A(X) = fA(X)$ is closed. In fact,

$$d\omega(X, Y) = 0.$$

Using (2.10) and (2.11) in (2.7) we get

$$b[A(Z)(\nabla_X A)(Y) - A(X)(\nabla_Z A)(Y)]$$
$$= \frac{dr(\rho)}{2(n-1)}[g(Y, Z)A(X) - g(X, Y)A(Z)].$$

Now putting $Z = \rho$ in the above expression yields

$$(\nabla_X A)(Y) = \frac{1}{2(n-1)} \frac{dr(\rho)}{b}[A(X)A(Y) - g(X, Y)]. \tag{2.16}$$

Thus (2.16) can be written as follows:

$$(\nabla_X A)(Y) = -fg(X, Y) + \omega(X)A(Y) \tag{2.17}$$

where ω is closed. But this means that the vector field ρ corresponding to the 1-form A defined by $g(X, \rho) = A(X)$ is a proper concircular vector field ([4], [6]).

Hence we can state the following:

Theorem 1. *In a conformally flat $S(QE)_n$ $(n > 3)$, the vector field ρ defined by $g(X, \rho) = A(X)$ is a proper concircular vector field.*

It is known [3] that if a conformally flat manifold (M^n, g) $(n > 3)$ admits a proper concircular vector field, then the manifold is a subprojective manifold in the sense of Kagan. Since a conformally flat $S(QE)_n$ admits a proper concircular vector field, namely the vector field ρ, we can state as follows:

Theorem 2. *A conformally flat $S(QE)_n$ is a subprojective manifold in the sense of Kagan.*

K. Yano [7] proved that in order that a Riemannian space admits a concircular vector field, it is necessary and sufficient that there exists a coordinate system with respect to which the fundamental quadratic differential form may be written in the form

$$ds^2 = (dx^1)^2 + e^q g^*_{\alpha\beta} dx^\alpha dx^\beta$$

where $g^*_{\alpha\beta} = g^*_{\alpha\beta}(x^\gamma)$ are the functions of x^γ only $(\alpha, \beta, \gamma, \delta = 2, 3, \ldots, n)$ and $q = q(x^1) \neq$ constant is a function of x^1 only. Thus if a $S(QE)_n$ is conformally flat i.e., if it satisfies (2.1), it is a warped product $IX_{e^q}M^*$, where (M^*, g^*) is an $(n-1)$-dimensional Riemannian manifold. A. Gebarowski [8] proved that warped product $IX_{e^q}M^*$ satisfies (2.1) if and only if M^* is an Einstein manifold. Thus if $S(QE)_n$ satisfies (2.1), it must be a warped product $IX_{e^q}M^*$ where M^* is an Einstein manifold.

Thus we can state the following result:

Theorem 3. *A conformally flat $S(QE)_n$ $(n > 3)$ can be expressed as a warped product $IX_{e^q}M^*$ where M^* is an Einstein manifold.*

A conformally flat manifold (M^n, g) is said to be of quasi-constant curvature ([2]) if the curvature tensor $'R$ of type $(0, 4)$ is given by

$$\begin{aligned} 'R(X,Y,Z,W) = p[&g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &+ q[g(X,W)T(Y)T(Z) - g(X,Z)T(Y)T(W) \\ &+ g(Y,Z)T(X)T(W) - g(Y,W)T(X)T(Z)] \end{aligned} \tag{2.18}$$

where p and q are differentiable functions and the vector field corresponding to the 1-form T is a unit vector and $'R(X, Y, Z, W) = g(R(X, Y)Z, W)$.

Since the manifold is conformally flat, the curvature tensor is given by

$$'R(X,Y,Z,W) = \frac{1}{(n-2)}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + S(X,W)g(Y,Z) - S(Y,W)g(X,Z)] + \frac{r}{(n-1)(n-2)}[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)]. \tag{2.19}$$

Hence by virtue of (1) we can express (2.19) as follows:

$$\begin{aligned} 'R(X,Y,Z,W) &= \left(\frac{2a}{n-2} + \frac{r}{(n-1)(n-2)}\right) \\ &\times \big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big] \\ &+ \frac{b}{n-2}\big[g(X,W)A(Y)A(Z) - g(X,Z)A(Y)A(W) \\ &+ g(Y,Z)A(X)A(W) - g(Y,W)A(X)A(Z)\big] \\ &= p\big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big] \\ &+ q\big[g(X,W)A(Y)A(Z) - g(X,Z)A(Y)A(Z) + g(Y,Z)A(X)A(W) \\ &- g(Y,W)A(X)A(Z)\big] \end{aligned}$$

where $p = \frac{2a}{n-2} + \frac{r}{(n-1)(n-2)}$ and $q = \frac{b}{n-2}$.

Hence we can state the following:

Theorem 4. *A conformally flat $S(QE)_n$ $(n > 3)$ is a manifold of quasi-constant curvature.*

3. Special conformally flat $S(QE)_n$ $(n > 3)$

The notion of a special conformally flat manifold which generalizes the notion of subprojective manifold was introduced by Chen and Yano [5]. According to them a conformally flat manifold is said to be a special conformally flat manifold if the tensor H of type $(0,2)$ defined by

$$H(X,Y) = -\frac{1}{(n-2)}S(X,Y) + \frac{r}{2(n-1)(n-2)}g(X,Y) \tag{3.1}$$

is expressible in the form

$$H(X,Y) = -\frac{\alpha^2}{2} g(X,Y) + \beta(X\alpha)(Y\alpha) \tag{3.2}$$

where α and β are two scalars such that α is positive. In virtue of (1) we can express (3.1) as

$$\begin{aligned} H(X,Y) = &\left[-\frac{a}{n-2} + \frac{r}{2(n-1)(n-2)} \right] g(X,Y) \\ &- \frac{b}{n-2} A(X)A(Y). \end{aligned} \tag{3.3}$$

We now put

$$\alpha^2 = \frac{2a}{n-2} - \frac{r}{(n-1)(n-2)} = \frac{a(n-2)-b}{(n-1)(n-2)}. \tag{3.4}$$

Then

$$2\alpha(X\alpha) = -\frac{dr(\rho)}{(n-1)(n-2)} A(X), \quad \text{using (2.10).} \tag{3.5}$$

Hence (3.3) can be expressed as

$$H(X,Y) = -\frac{\alpha^2}{2} g(X,Y) + \beta A(X)A(Y) \tag{3.6}$$

where $\beta = \frac{4b(r-2an+2a)(n-1)}{\lambda^2}$, $\lambda = dr(\rho)$.

Suppose that $a(n-2) - b > 0$, then α is not zero. Hence from (3.4) it follows that α may be taken as positive. From (3.6) we conclude that the manifold under consideration is a special conformally flat manifold.

It is known from a theorem of Chen's and Yano's paper [5] that every simply connected special conformally flat manifold can be isometrically immersed in a Euclidean space E^{n+1} as a hypersurface.

We can therefore state the following:

Theorem 5. *Every simply connected conformally flat $S(QE)_n$ ($n>3$) satisfying $a(n-2) - b > 0$ can be isometrically immersed in a Euclidean space E^{n+1} as a hypersurface.*

References

[1] M. C. Chaki and R. K. Maity, On quasi Einstein manifold, *Publ. Math. Debrecen* **57** (2000), 297–306.

[2] B. Y. Chen and K. Yano, Hypersurfaces of a conformally flat space, *Tensor N.S.* **26** (1972), 318–322.

[3] T. Adati, On subprojective spaces, III, *Tohoku Math. J.* **3** (1951), 343–358.

[4] J. A. Schouten, Ricci-Calculus, *Springer*, *Berlin*, 1954.

[5] B. Y. Chen and K. Yano, Special conformally flat spaces and canal hypersurfaces, *Tohoku Math. J.* **25** (1973), 177–184.

[6] K. Yano, Concircular geometry, I, *Proc. Imp. Acad. Tokyo* **16** (1940), 195–200.

[7] K. Yano, On the torseforming direction in Riemannian spaces, *Proc. Imp. Acad. Tokyo* **20** (1944), 340–345.

[8] A. Gebarowski, Nearly conformally symmetric warped product manifolds, *Bulletin of the Institute of Mathematics Academia Sinica* **20:4** (1992), 359–371.

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