On amicable numbers.

To professor László Kalmár on his 50th birthday.

By P. ERDÖS in Jerusalem.

Denote by $\sigma(n)$ the sum of all divisors of n (n included). Two numbers a and b are called *amicable* if $\sigma(a)=\sigma(b)=a+b$. The smallest amicable numbers 220 and 284 where already known to PYTHAGORAS. The origin of the name amicable numbers is that if

$$\sigma_1(a)=\sum_{\substack{d\mid a\\ d<a}} d,$$

then a and b are amicable if and only if $\sigma_1(a)=b, \sigma_1(b)=a$. It is not yet known if there are infinitely many amicable numbers. It seems likely that this is true, in fact it can be conjectured that the number of amicable numbers less than n is greater than $n^{1-\varepsilon}$ for every $\varepsilon>0$ if $n>n_0(\varepsilon)$. Recently KANOLD[1]) gave an upper estimate for the number of amicable numbers. In fact he proved that the density of amicable numbers is less than 0·204, in other words he proved that the density of integers a for which there exists a b satisfying $\sigma(a)=\sigma(b)=a+b$ is less than 0·204.

In the present note we shall prove that the density of the amicable numbers is 0. By our method we could prove that the number of amicable numbers less than n is less than $c\cdot n/\log\log\log n$; very likely much more is true, the number of amicable numbers less than n is no doubt $o\left(\frac{n}{(\log n)^k}\right)$ for every k, but our method does not seem suitable for obtaining such a good result.

An old conjecture of CATALAN states that if we put

$$\sigma_1^{(n)}(a)=\sigma_1(\sigma_1^{(n-1)}(a)),$$

then for every a the sequence $\sigma_1^{(n)}(a)$ is bounded. In other words either $\sigma_1^{(n)}(a)=1$ for some n, or the sequence $\sigma_1^{(n)}(a)$ is periodic after omitting, if necessary, a finite number of terms in the beginning. It would not be hard

[1]) H. J. KANOLD, Über die asymptotische Dichte von gewissen Zahlenmengen. *Proceedings of the international congress of matematicians Amsterdam* 1954, p. 30.

to prove by our method that for every k the density of integers for which $\sigma_1^{(k)}(a) = a$ is 0.[2]) On the other hand questions like the above mentioned conjecture of CATALAN, or whether the density of integers a, for which there is an n with $\sigma_1^{(n)}(a) = 1$, exists seem inaccessible at present.

We shall need several Lemmas.

Lemma 1. *Let q_i be a sequence of primes satisfying $\sum_{i=1}^{\infty} \frac{1}{q_i} = \infty$. Denote by $v_q(n)$ the number of q's which divide n. Then the density of integers n with $v_q(n) < A$ is 0 for every A.*

Lemma 1 is a special case of a theorem of TURÁN[3]) which asserts in a weaker form that if $0 \leqq \psi(p) \leqq K$ for all primes p, $\sum_p \frac{\psi(p)}{p} = \infty$ and $\psi(n) = \sum_{p \mid n} \psi(p)$ where the summation is extended over all different prime factors of n, then for all but $o(N)$ values $\leqq N$ of n we have

$$\left| \psi(n) - \sum_{p \leqq N} \frac{\psi(p)}{p} \right| < \left(\sum_{p \leqq N} \frac{\psi(p)}{p} \right)^{\frac{3}{4}}. \tag{1}$$

In the present case we have only to take $\psi(p) = 1$ if p belongs to our q_i sequence, and $\psi(p) = 0$ for the other primes p.

Lemma 2. *Let A be any constant. Then the density of integers n for which*

$$\sigma(n) \not\equiv 0 \pmod{\left(\prod_{p \leqq A} p\right)^A}$$

is 0 for every A.

It suffices to show that the density of integers n for which there exists a $p \leqq A$ for which $\sigma(n) \not\equiv 0 \pmod{p^A}$ is 0. Let B be an arbitrary integer, $q_1, q_2, \ldots$ the primes satisfying $q \equiv -1 \pmod{p}$, $q_i > B$. It is well known that $\sum_{i=1}^{\infty} \frac{1}{q_i} = \infty$. Thus by Lemma 1 the density of the integers divisible by fewer than A of the q's is 0. Also the density of the integers divisible by the square of one of the q's is clearly at most $\sum_{i=1}^{\infty} \frac{1}{q_i^2} < \frac{1}{B}$. Now if n is divisible by at least A of the q's and by no q^2, then clearly $\sigma(n) \equiv 0 \pmod{p^A}$. Thus since B can be chosen arbitrarily large, the density of the integers with $\sigma(n) \not\equiv 0 \pmod{p^A}$ is 0.

---

[2]) It can be conjectured that for every k there exists infinitely many a's with $\sigma_1^{(k)}(a) = a$ and $\sigma_1^{(l)}(a) \neq a$ for $1 \leqq l < k$. This is a problem of O. MEISSNER. (See: DICKSON, History of the theory of numbers I, p. 49.) POULET observed that for $a = 12496$, $k = 5$ and for $a = 14316$, $k = 28$. (Ibid. p. 50.)

[3]) P. TURÁN, Über einige Verallgemeinerungen eines Satzes von Hardy and Ramanujan. *J. London Math. Soc.* **11** (1936), 125—133.

Lemma 3. *Denote*

$$\sigma_A(n)=\sum_{\substack{d\mid n\\ d\leqq A}}\frac{n}{d}.$$

Then to every ε *and* η_i *there exists an* A_0, *so that for* $A>A_0$ *the number of integers* $n\leqq x$ *for which* $\sigma(n)-\sigma_A(n)>\eta_i n$ *is less than* εx.

We evidently have

$$\sum_{n=1}^{x}(\sigma(n)-\sigma_A(n))=\sum_{n=1}^{x}\sum_{\substack{d\mid n\\ d\leqq A}}\frac{n}{d}=\sum_{d_1>A}\sum_{d_2\leqq x/d_1}d_2<\sum_{d>A}\frac{x^2}{d^2}<\frac{x^2}{A}. \tag{2}$$

If Lemma 3 would not be true we would have $\sigma(n)-\sigma_A(n)\geqq\eta_i n$ for at least εx integers d not exceeding x. Thus

$$\sum_{n=1}^{x}(\sigma(n)-\sigma_A(n))>\eta_i\sum_{d\leqq\varepsilon x}d>\frac{\eta_i\varepsilon^2x^2}{4}. \tag{3}$$

Thus for $A>\frac{4}{\eta_i\varepsilon^2}$ (3) contradicts (2) proving so Lemma 3.

Theorem. *The density of amicable numbers is* 0.

Denote by (a_i, b_i), $a_i<b_i$, $i=1,2,\ldots$ the sequence of pairs of amicable numbers. It clearly will be sufficient to show that the sequence a_i, $i=1,2,\ldots$ has density 0. We split the sequence a_i into two classes. Let $A=A(\varepsilon)$ be sufficiently large. In the first class are the a's for which there exists a $p\leqq A$ with $\sigma(a_i)\not\equiv 0 \pmod{p^A}$. It follows from Lemma 2 that the density of the a's of the first class is 0.

For the a's of the second class $\sigma(a_i)\equiv 0 \pmod{p^A}$ for every $p\leqq A$. Thus clearly if $d\leqq A$ and $d\mid a_i$, then $\sigma(a_i)\equiv 0 \pmod d$. Hence $\sigma(a_i)-a_i=b_i\equiv 0 \pmod d$. It follows from Lemma 3 that except for at most εx of the a's not exceeding x we have

$$\frac{\sigma_A(a_i)}{a_i}\geqq\frac{\sigma(a_i)}{a_i}-\eta_i. \tag{4}$$

Since ε can be chosen arbitrarily small it suffices to consider the a's of the second class which satisfy (4). We have from (4) and from the fact that $d\leqq A$, $d\mid a_i$ implies $d\mid b_i$

$$\frac{\sigma(b_i)}{b_i}\geqq\frac{\sigma_A(a_i)}{a_i}\geqq\frac{\sigma(a_i)}{a_i}-\eta_i. \tag{5}$$

Now we have $\sigma(a_i)=\sigma(b_i)=a_i+b_i$. Thus from (5)

$$\eta_i\geqq\frac{\sigma(a_i)}{a_i}-\frac{\sigma(b_i)}{b_i}=\frac{b_i}{a_i}-\frac{a_i}{b_i}$$

or

$$1<\frac{b_i}{a_i}<1+\eta_i. \tag{6}$$

Hence from (6) and $\sigma(a_i) = a_i + b_i$ we have

$$2 < \frac{\sigma(a_i)}{a_i} < 2 + \eta_i. \tag{7}$$

Now it is well know[4]) that the density of the integers satisfying $\sigma(n)/n \leqq c$ exists and is a continuous function of c. Thus it follows from (7) that the density of the a's of the second class satisfying (4) is for sufficiently small η_i less than ε, and since ε can be chosen arbitrarily small, the proof of our theorem is completed.

REMARK. Lemma 2 together with Lemma 3 implies that for every ε the density of integers a, b for which

$$\frac{\sigma(b)}{b} < \frac{\sigma(a)}{a} - \varepsilon, \quad b = \sigma(a) - a$$

is 0 for every ε. By a more complicated argument we can show that the density of integers a, b for which

$$\frac{\sigma(b)}{b} > \frac{\sigma(a)}{a} + \varepsilon$$

is also 0 for every ε. Thus except for a sequence of density 0

$$\frac{\sigma(b)}{b} = \frac{\sigma(a)}{a} + o(1).$$

(Received January 25, 1955.)

[4]) H, DAVENPORT, Über Numeri Abundantes. *Sitzungsberichte der Preußischen Akademie der Wiss.* **26/29** (1933), 830—837. — Also: P. ERDŐS, On the density of the abundant numbers. *J. London Math. Soc.* **9** (1934), 278—282. — P. ERDŐS, On the density of some sequences of numbers. *J. London Math. Soc.* **10** (1935), 120—125.