# On rings every subring of which is a multiple of the ring.

To the memory of my highly beloved teacher T. Szele.

By F. Szász in Debrecen.

In a previous paper we have determined all groups every cyclic subgroup of which is a power of the group [2]. We shall treat in this note a similar ring-theoretical problem.

We shall call an arbitrary ring $R$ *cyclic* if the additive group $R^+$ of $R$ is cyclic (see e. g. [1], pp. 326—327). The ring $I$ of rational integers is obviously cyclic.

An arbitrary ring $R$ is called *a ring with property P* if any subring $S$ of $R$ is a multiple $nR$ of the ring $R$, where $nR$ denotes the set of all elements $nr$ of $R$ (here $r \in R$ and $n \in I$). Then $nR$ is also an ideal in the ring $R$. It is clear that every homomorphic image $R'$ of a ring $R$ with property $P$ has likewise the property $P$.

We shall prove the following

**Theorem.** *An arbitrary ring $R$ is cyclic if and only if it has the property $P$.*

Proof. First of all we verify that a ring with property $P$ without divisors of zero is necessarily commutative. In fact, since every subring $S$ of $R$ is an ideal in $R$, for arbitrary elements $a \neq 0$ and $b$ of $R$ obviously $ab = a' \in \{a\}$ holds, where $\{M\}$ denotes the subring generated by the subset $M$ of $R$. By $aa' = a'a$ we obtain $a(ba - a') = (ab - a')a = 0$ and since $R$ has no divisors of zero, $a' = ab = ba$.

Now we distinguish two cases.

I. Let $R$ be a ring with property $P$ whose additive group is torsion free.

If $a \neq 0$ and $b \neq 0$ are elements of $R$ with $ab = 0$, then by $\{a\} = mR$ and $\{b\} = nR$ necessarily $\{0\} = \{a\} \cdot \{b\} = mRnR = mnR^2$. Therefore $R^2 = \{0\}$, consequently by the theorem of [2], $R$ is a zeroring over an infinite cyclic additive group.

Now we assume that $R$ has no divisors of zero. Then by our previous remark $R$ is commutative. For the element $a \neq 0$ of $R$ we have evidently $Ra \neq \{0\}$, therefore $Ra = nR$ for some natural number $n$. There exists there-

fore an element $b \neq 0$ of $R$ for which $ba = ab = na$. Consider the non-void subset $S$ of all elements $c$ of $R$ for which, with a certain rational integer $n_c$, an equation $ac = ca = n_c \cdot a$ holds. We can easily verify that $S$ is a subring of $R$, consequently $S = mR$, where by $0 \neq b \in S$ evidently $m \neq 0$. Therefore $Sa = (mR)a = mnR$. But then the mapping $r \to mnr$ is an isomorphism of the additive group $R^+$ of $R$ onto its subgroup $(mnR)^+$. On the other hand by the definition of $S$ the subring $S \cdot a = mn \cdot R$ is obviously cyclic, and on the basis of the preceding group-theoretical isomorphism the ring $R$ itself is cyclic.

II. Let $R$ be an arbitrary ring with property $P$ whose additive group $R^+$ is not torsion free. If $d$ is an element of finite order $n$ of the ring $R$, and $\{d\} = mR$, then $R^+$ is $k$-bounded, where $k \leqq m \cdot n$. The elements of $p$-power order of $R$ form a subring $R_p$, called the $p$-component of $R$, which is a direct summand of $R$, and has also property $P$. So we have to prove the theorem only for $p$-rings. We first observe that all subrings of $R_p$ are:

$$R_p,\ pR_p,\ p^2R_p, \ldots,\ p^iR_p, \ldots,\ p^{k-1}R_p \neq \{0\},\quad p^kR_p = \{0\}.$$

Since $p^iR_p/p^{i+1}R_p$ $(i = 0, 1, \ldots, k-1)$ has no proper left ideal, it is a prime field of characteristic $p$, or a zeroring over an additive group of order $p$. In both cases every $p^iR_p^+/p^{i+1}R_p^+$ is a group of order $p$, and so $R_p^+$ is a group of order $p^k$. But $p^{k-1}R_p^+ \neq 0$ shows that $R_p^+$ contains an element of order $p^k$, thus $R_p^+$ is cyclic.

REMARK. By the note [3] it is clear that an arbitrary ring $R$ is cyclic if and only if every non-trivial multiple $nR$ of $R$ is cyclic, where $nR$ is called trivial in the case $n = 1$, 0 or $-1$.

I am indebted to L. FUCHS and L. KOVÁCS for their valuable remarks.

## Bibliography.

[1] L. RÉDEI, Algebra I. *Budapest*, 1954.

[2] F. SZÁSZ, On groups every cyclic subgroup of which is a power of the group, *Acta Math. Acad. Sci. Hungar.* **6** (1956), 475—477.

[3] F. SZÁSZ, On groups every non-trivial power of which is cyclic, *Magyar Tud. Akad. Mat. Fiz. Oszt. Közl.* **5** (1955), 491—492.

*(Received September 11, 1955.)*