

On a theorem of Shah.

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Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function; $\mu(r) = \mu(r, f)$ the maximum term for $|z| = r$ and $\nu(r) = \nu(r, f)$ its rank. It is known (see [1], pp. 80—81.) that if $f(z)$ be of order ρ , $0 \leq \rho \leq \infty$, then

$$(1) \quad \lim_{r \rightarrow \infty} \frac{\log \mu(r)}{\nu(r)} \leq \frac{1}{\rho} \leq \frac{1}{\lambda} \leq \overline{\lim}_{r \rightarrow \infty} \frac{\log \mu(r)}{\nu(r)}$$

where $\lambda = \lim_{r \rightarrow \infty} \frac{\log \log M(r)}{\log r}$.

In Theorems 2 and 3 of this note we give two different refinements of (1).

Theorem 1. Let $\Phi(x)$ be a real function, positive integrable L in any interval (Δ, r) where $\Delta > 0$;

$$(2) \quad \overline{\lim}_{x \rightarrow \infty} \frac{\log \Phi(x)}{\log x} = K.$$

Define

$$P(r) = \int_{\Delta}^r \frac{\Phi(x)}{x^{1+K}} dx; \quad Q(r) = r \int_r^{\infty} \frac{\Phi(x)}{x^{2+K}} dx;$$

then if $Q(r) \not\equiv 0$,

$$(3) \quad \overline{\lim}_{r \rightarrow \infty} \frac{P(r)}{Q(r)} \leq \frac{1}{K}.$$

PROOF. Assume that (3) is not true, so that for some positive $\alpha \left(< \frac{1}{K} \right)$ and X

$$(4) \quad P(x) \leq \alpha Q(x) \quad \text{for every } x > X.$$

If $K < \beta < K + \frac{1}{1+\alpha}$, we have $\int_X^\infty u^{-1-\beta} \Phi(u) du$ convergent, and so

$$\begin{aligned} \int_X^\infty t^{-\beta+K-1} Q(t) dt &= \int_X^\infty t^{-\beta+K} dt \int_t^\infty u^{-2-K} \Phi(u) du = \\ &= \int_X^\infty u^{-2-K} \Phi(u) du \int_X^u t^{-\beta+K} dt \leq \\ &\leq (1-\beta+K)^{-1} \int_X^\infty u^{-1-\beta} \Phi(u) du \end{aligned}$$

so that the left-hand integral is finite; and

$$\begin{aligned} \int_X^\infty t^{-\beta+K-1} Q(t) dt &\leq (1-\beta+K)^{-1} \int_X^\infty t^{-\beta+K} dP(t) \leq \\ &\leq (1-\beta+K)^{-1} \left[(t^{-\beta+K} P(t))_{t=\infty} + (\beta-K) \int_X^\infty t^{-\beta+K-1} P(t) dt \right]. \end{aligned}$$

But

$$\begin{aligned} t^{-\beta+K} P(t) &= t^{-\beta+K} \int_A^t \frac{\Phi(x)}{x^{1+K}} dx \\ &< t^{-\beta+K} \int_A^t \frac{x^{K+\frac{\beta-K}{2}}}{x^{1+K}} dx \quad \text{for } A > x_0 \\ &< t^{-\beta+K} \frac{t^{\frac{\beta-K}{2}} \cdot 2}{\beta-K}. \end{aligned}$$

Hence $(t^{-\beta+K} P(t))_{t=\infty} = 0$ and then by (4)

$$\begin{aligned} \int_X^\infty t^{-\beta+K-1} Q(t) dt &\leq (1-\beta+K)^{-1} (\beta-K) \int_X^\infty t^{-\beta+K-1} P(t) dt \leq \\ &\leq \alpha(1-\beta+K)^{-1} (\beta-K) \int_X^\infty t^{-\beta+K-1} Q(t) dt. \end{aligned}$$

Since $\alpha(\beta-K)/(1-\beta+K) < 1$ and $Q(t) \not\equiv 0$, this is a contradiction, and so (4) must fail for arbitrarily large values of X .

Theorem 2. *If $f(z)$ is an entire function of order zero, and not constant then*

$$(5) \quad \overline{\lim}_{r \rightarrow \infty} \frac{\int_{\Delta}^r t^{-1} L(t) \nu(t) dt}{r \int_r^{\infty} t^{-2} L(t) \nu(t) dt} = \infty$$

where $L(t)$ is any continuous nondecreasing function of t such that $\log L(t) = o(\log t)$.

This is, for $\rho = 0$, a strengthening of (1), which states that

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log \mu(r)}{\nu(r)} = \infty,$$

since, for a suitably large Δ

$$\int_{\Delta}^r t^{-1} L(t) \nu(t) dt \leq L(r) \int_{\Delta}^r t^{-1} \nu(t) dt \leq L(r) \log \mu(r)$$

and

$$r \int_r^{\infty} t^{-2} L(t) \nu(t) dt \geq r L(r) \nu(r) \int_r^{\infty} t^{-2} dt = L(r) \nu(r).$$

PROOF OF THEOREM 2. For an entire function of order zero

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log (L(r) \nu(r))}{\log r} = 0.$$

The theorem will therefore follow if we take $\Phi(x) = L(x) \nu(x)$ in Theorem 1.

Theorem 3. *If $f(z)$ be of lower order λ , $0 \leq \lambda \leq \infty$, then*

$$(6) \quad \frac{1}{\lambda} \leq \overline{\lim}_{r \rightarrow \infty} \frac{\log \mu(r, f)}{\nu(r, f^{(p)})}$$

where $\nu(r, f^{(p)})$ denotes the rank of the maximum term for $|z| = r$ of $f^{(p)}(z)$ (the p -th derivative of $f(z)$), where p is either an arbitrary constant integer or an integer valued function of r such that

$$p(\nu) = O(\nu / \log \nu), \quad \text{where } \nu = \nu(r, f^{(p)}).$$

This is a strengthening of (1) since (see [2])

$$(7) \quad \nu(r, f) \leq \frac{r \mu(r, f')}{\mu(r, f)} \leq \nu(r, f') \leq \dots$$

PROOF OF THEOREM 3. The lower order of $f(z)$ is the same as the lower order of $f^{(p)}(z)$. Hence from (1)

$$(8) \quad \frac{1}{\lambda} \leq \overline{\lim}_{r \rightarrow \infty} \frac{\log \mu(r, f^{(p)})}{\nu(r, f^{(p)})}.$$

But from (7) it follows that

$$\mu(r, f) \cong \frac{r^p \mu(r, f^{(p)})}{\nu(r, f') \dots \nu(r, f^{(p)})} \cong \frac{r^p \mu(r, f^{(p)})}{(\nu(r, f^{(p)}))^p}$$

$$\log \mu(r, f) \cong \log \mu(r, f^{(p)}) - p \log \nu(r, f^{(p)})$$

and the required result follows from (8).

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Bibliography.

- [1] S. M. SHAH, The maximum term of an entire series, *Math. Student* **19** (1942), 80—82.
- [2] Q. I. RAHMAN, On the derivatives of integral functions, *Math. Student* (to appear).

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