

A note on entire and meromorphic functions.

By HARI SHANKAR in Moradabad.

1. Let $f(z)$ be an entire function and let

$$M(r, f) = M(r) = \max_{0 \leq \theta \leq 2\pi} |f(re^{i\theta})|,$$

$$\mathfrak{M}(r, f) = \mathfrak{M}(r) = \min_{0 \leq \theta \leq 2\pi} |f(re^{i\theta})|,$$

$n(r, a) = n(r)$ denote the number of zeros of $f(z) - a$ in $|z| \leq r$ each counted according to its multiplicity. Further let $|f(0)| = 1$, then M. L. CARTWRIGHT ([1], p. 14.) has proved that for $0 < r < R$

$$(1.1) \quad M(R) \geq \left(\frac{R}{r}\right)^{n(r)}$$

S. K. SINGH [3] proved that for $0 < r < R$

$$(1.2) \quad \frac{M(R)}{\mathfrak{M}(r)} \geq \left(\frac{R}{r}\right)^{n(r)},$$

$$(1.3) \quad \frac{\mathfrak{M}(R)}{M(r)} \geq \left(\frac{R}{r}\right)^{n(R)}.$$

Further he proved [4] that

$$(1.4) \quad \frac{M(R)}{\mathfrak{M}(r)} \geq \left(\frac{R}{r}\right)^{\frac{N(R)}{\log R}}, \quad (0 < r < R)$$

where

$$N(R) = \int_0^R \frac{n(t)}{t} dt.$$

The purpose of this note is to improve the results of SINGH and to establish a result (see § 2) on meromorphic functions. Let us indicate, first of

all, certain standard notations ([2], p. 6.) which we shall use here.

$$m\left(r, \frac{1}{f-a}\right) = m(r, a) = (2\pi)^{-1} \int_0^{2\pi} \log^+ |f(re^{i\theta}) - a|^{-1} d\theta, \quad (a \neq \infty),$$

$$m(r, f) = m(r, \infty) = (2\pi)^{-1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta.$$

$$N\left(r, \frac{1}{f-a}\right) = N(r, a) = \int_0^r \frac{n(t, a) - n(0, a)}{t} dt + n(0, a) \log r, \quad (a \neq \infty),$$

$$N(r, f) = N(r, \infty) = \int_0^r \frac{n(t, \infty) - n(0, \infty)}{t} dt + n(0, \infty) \log r,$$

$$T(r, f) = T(r) = m(r, f) + N(r, f).$$

As the expression $N(r, \infty)$ is identically zero for all entire functions, the NEVANLINNA characteristic function $T(r)$ for the function $f(z)$ reduces to

$$T(r) = m(r, f).$$

We prove here

Theorem. 1. Let $f(z)$ be an entire function and $|f(0)| = 1$, then for $0 < r < R$

$$(1.5) \quad \frac{\exp(T(R))}{\mathfrak{M}(r)} \cong \left(\frac{R}{r}\right)^{n(r)}$$

$$(1.6) \quad \frac{\mathfrak{M}(R)}{\exp(T(r))} \cong \left(\frac{R}{r}\right)^{n(R)}$$

$$(1.7) \quad \frac{\exp(T(R))}{\mathfrak{M}(r)} \cong \left(\frac{R}{r}\right)^{\frac{N(R)}{\log R}}$$

(exp (x) means e^x).

Since the inequality ([2], p. 24)

$$T(r) \leq \log^+ M(r, f) \leq \frac{R+r}{R-r} T(R), \quad (0 < r < R),$$

is satisfied by an entire function, (1.2), (1.3), and (1.4) follow immediately from (1.5), (1.6) and (1.7) respectively.

PROOF. By JENSEN's theorem we get

$$\begin{aligned} N(r, 0) &= \int_0^r \frac{n(t, 0)}{t} dt = (2\pi)^{-1} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta = \\ &= (2\pi)^{-1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta - (2\pi)^{-1} \int_0^{2\pi} \log^+ |f(re^{i\theta})|^{-1} d\theta = m(r, f) - m\left(r, \frac{1}{f}\right). \end{aligned}$$

Since $m\left(r, \frac{1}{f}\right) \geq 0$, we have

$$N(r) \leq m(r, f) = T(r).$$

Also

$$N(r) = \int_0^r \frac{n(t)}{t} dt = (2\pi)^{-1} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta \geq \log \mathfrak{M}(r).$$

Therefore

$$(1.8) \quad n(r) \log \frac{R}{r} \leq \int_r^R \frac{n(t)}{t} dt = N(R) - N(r) \leq T(R) - \log \mathfrak{M}(r).$$

Hence

$$\log \frac{\exp(T(R))}{\mathfrak{M}r} \geq \log \left(\frac{R}{r}\right)^{n(r)}$$

and (1.5) follows.

On the other hand

$$n(R) \log \frac{R}{r} \geq \int_r^R \frac{n(t)}{t} dt = N(R) - N(r) \geq \log \mathfrak{M}(R) - T(r).$$

Therefore

$$\log \frac{\mathfrak{M}(R)}{\exp(T(r))} \leq \log \left(\frac{R}{r}\right)^{n(R)}$$

and (1.6) follows.

Further, since $N(r)$ is an increasing convex function of $\log r$, and let O be the origin and $P: (\log R, N(R))$; $Q: (\log r, N(r))$ be two points on its graph, then the slope of OP is greater than the slope of OQ , consequently

$$\frac{N(R)}{\log R} \geq \frac{N(r)}{\log r}$$

or

$$\frac{N(R) - N(r)}{\log R - \log r} \cong \frac{N(R)}{\log R}.$$

Hence (1.8) gives

$$\frac{N(R)}{\log R} \leq \frac{T(R) - \log \mathfrak{M}(r)}{\log R - \log r}$$

from which (1.7) follows.

2. Let $w(z)$ be a non constant meromorphic function and (under the notations of § 1) let

$$T(r, w) = T(r) = m(r, \infty) + N(r, \infty)$$

be its NEVANLINNA's characteristic function.

We shall prove here

Theorem 2. If $0 < \alpha < 1$, then

$$\Phi(r) = T(\alpha r) - T(r)$$

is a non-increasing function of r .

PROOF. Let $0 < r_1 < r_2$, and consider two pair of points

$$(\log \alpha r_1, T(\alpha r_1)); (\log r_2, T(r_2))$$

$$(\log r_1, T(r_1)); (\log \alpha r_2, T(\alpha r_2))$$

lying on the graph of $T(r)$. Obviously the first pair of points contains the second pair. The mid-point of the corresponding chords have equal abscissa; for

$$\frac{\log \alpha r_1 + \log r_2}{2} = \frac{\log r_1 + \log \alpha r_2}{2}$$

and hence, as regards the ordinates we have due to the convexity of $T(r)$

$$\frac{T(\alpha r_1) + T(r_2)}{2} \cong \frac{T(r_1) + T(\alpha r_2)}{2}$$

and the result follows.

Bibliography.

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