

Semi-complements and complements in semi-modular lattices.

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1. Let L be a lattice with greatest and least elements¹⁾ and let a be any element of L . It is known that if L is modular, then the set of all complements of a is totally unordered (i. e., for any two complements x, y of a , $x \leq y$ implies $x = y$). It follows easily²⁾ that *if L is modular, then each complement of a is maximal in the (partly ordered) set of all semi-complements of a .*

In this paper we firstly show that the converse is also true, moreover not only for modular, but also for semi-modular lattices (Theorem 1). Next, this theorem gives a sufficient condition in order that a semi-complemented semi-modular lattice be also complemented (Corollary). Finally, using again Theorem 1, we prove a theorem concerning the structure of a special class of semi-complemented semi-modular lattices (Theorem 2).

2. Following R. CROISOT ([2], p. 85.), a lattice is said to be *semi-modular* if for any elements $a, b, c (\in L)$ which satisfy the inequalities

$$b \cap c < a < c < a \cup b,$$

there exists at least one element $t (\in L)$ such that

$$b \cap c < t \leq b$$

and

$$(a \cup t) \cap c = a.$$

Let L be a lattice which has a least element denoted by o . Then, by a *semi-complement* of $a (\in L)$ we mean ([5], p. 123.) an element $x (\in L)$ such

¹⁾ For the terminology see section 2.

²⁾ For, if x is any complement and $y (\geq x)$ is any semi-complement of a , then by $a \cup y \geq a \cup x = i$, (see the definitions in section 2), the element y satisfies (not only the equation $a \cap y = o$, but also) the equation $a \cup y = i$; that is, y is also a complement of a . But then, by the theorem cited above, it follows $x = y$ proving the maximality of x in the set of all semi-complements of a .

that $a \cap x = o$. If, moreover, $x \neq o$, then x is called a *proper semi-complement* of a . (Clearly, if x is a proper semi-complement of a , then $a \not\leq x$.) The set of all semi-complements of a will be denoted by $S(a)$.

The greatest element of a lattice L , if it exists, will be denoted by i . The set of all elements of L , differing from the (possibly existing) elements o, i , will be called the *interior* of L and denoted by $\mathfrak{I}(L)$.

A lattice L with least element o is said to be *semi-complemented* if every element of $\mathfrak{I}(L)$ has at least one proper semi-complement in L .³⁾

Let L be again a lattice with least element o . Then, by the *height* $h(a)$ of an element a of L we mean ([2], p. 10.) the maximum length of chains $(o =) x_0 < x_1 < \dots < x_n (= a)$ between o and a . When $h(a)$ is finite, a is called ([3], p. 243.) an *element of finite height*.

For all undefined terms and symbols the reader is referred to [1].

3. The main result of this paper is the following

Theorem 1. *Let L be any semi-complemented semi-modular lattice. If, for some $r (\in L)$, the set $S(r)$ of all semi-complements of r has a maximal element m , then L has a greatest element i and m is a complement of r .*

Corollary. *Let L be any semi-complemented semi-modular lattice. If, for each element r of the interior $\mathfrak{I}(L)$ of L , $S(r)$ contains at least one maximal element, then L has a greatest element i and it is complemented.*

PROOF. Since for the elements o, i , the assertion of the theorem is obvious, we need only consider the case that r is any element of $\mathfrak{I}(L)$. Clearly, it suffices to prove that if m is a proper semi-complement of r such that

$$(1) \quad r \cup m = d \quad \text{with } d \in \mathfrak{I}(L),$$

then there exists a semi-complement of r greater than m .

But, if for some elements r, m , condition (1) is satisfied, then d has a proper semi-complement x . Then we have

$$(2) \quad o < r \leq d, \quad o < m \leq d,$$

and, consequently,

$$(3) \quad x \cap m \leq x \cap d = o.$$

Clearly, the proof of the theorem may be accomplished by proving the following two assertions:

(i) *if z is an element of L such that*

$$(4) \quad o < z \leq x$$

³⁾ Since, obviously, $S(o) = L$, this definition is equivalent to that of [5], p. 123.

and

$$(5) \quad (z \cup m) \cap d = m,$$

then $z \cup m$ is a (proper) semi-complement of r and $z \cup m > m$;

(ii) there exists at least one element z having the properties assumed in (i).

Assertion (i) may be proved by direct calculation. Indeed, m being a semi-complement of r , by (2) and (5) we get

$$r \cap (z \cup m) = r \cap r \cap (z \cup m) \leq r \cap (d \cap (z \cup m)) = r \cap m = o;$$

further, by (4) and (3),

$$z \cap m \leq x \cap m = o < z,$$

which implies $z \cup m > m$.

In order to prove (ii), we consider the element

$$(6) \quad v = (x \cup m) \cap d.$$

Then, by (6) and (2),

$$v = (x \cup m) \cap d \geq m \cap m = m;$$

that is, $v \geq m$.

If $v = m$, then by (6) we have $(x \cup m) \cap d = m$. Further, by definition, $o < x (\leq x)$. Thus the conditions (4), (5) for $z = x$ are satisfied.

It remains to consider the case $v > m$. We then show that the elements x, m, v satisfy the inequalities

$$(7) \quad (o =) x \cap v < m < v < x \cup m.$$

Firstly, by (6), (3) and (2),

$$x \cap v = x \cap (x \cup m) \cap d = x \cap d = o < m.$$

Next, $m < v$ by assumption. Finally, (6) implies immediately that $v \leq x \cup m$ and, again by (6), $v = x \cup m$ would imply

$$x \leq x \cup m = v = (x \cup m) \cap d \leq d,$$

a contradiction to the definition of x . Thus the inequalities in (7) are verified.

L being semi-modular, it follows that there exists an element t such that

$$(8) \quad (o =) x \cap v < t \leq x$$

and

$$(9) \quad m = (m \cup t) \cap v.$$

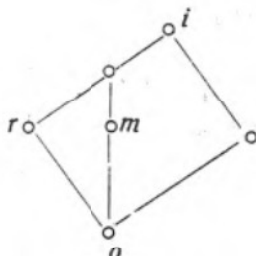
It follows by (8) that $t \cup m \leq x \cup m$. This implies, by (9) and (6),

$$(10) \quad m = (t \cup m) \cap v = (t \cup m) \cap (x \cup m) \cap d = (t \cup m) \cap d.$$

From (8) and (10) we see that now the conditions (4), (5) for $z = t$ are satisfied. Hence also assertion (ii) is proved.

By the theorem, the corollary is obvious.

We remark that if L fails to be semi-modular, then Theorem 1 is in general false. For example, the lattice given by the diagram



is semi-complemented and m is a maximal semi-complement of r ; however, m is not a complement of r .

Further, it is easy to see that Theorems 2, 3 in the paper [4] of the author are special cases of our present corollary.

We prove also

Theorem 2. *Let L be any semi-complemented semi-modular lattice of infinite length in which every element of $\mathfrak{J}(L)$ is of finite height. Then, for each element r of $\mathfrak{J}(L)$ and for each integer $K (\geq 0)$, there exists a semi-complement x of r whose height is equal to K .*

PROOF. Earlier ([3], Theorem 2) we have essentially shown that, under the assumptions of the present theorem, no element of $\mathfrak{J}(L)$ has complements, even if the greatest element i exists in L . Hence, by Theorem 1, $S(r)$ ($r \in \mathfrak{J}(L)$) contains no maximal element. It follows that there exists an infinite ascending chain

$$o < m_1 < m_2 < \dots < m_k < \dots \quad (m_k \in S(r); k = 1, 2, \dots).$$

Clearly, $h(m_k) \geq K$. If $h(m_k) = K$, then our theorem is proved. If, however, $h(m_k) > K$, then let $\mathcal{K}$ denote the (uniquely defined) index such that

$$(11) \quad h(m_{\mathcal{K}}) \leq K < h(m_{\mathcal{K}+1}).$$

Since $m_{\mathcal{K}+1}$ is of finite height, there exists a chain

$$(m_{\mathcal{K}} =) \bar{m}_0 < \bar{m}_1 < \dots < \bar{m}_l (= m_{\mathcal{K}+1})$$

between $m_{\mathcal{K}}$ and $m_{\mathcal{K}+1}$ which is maximal in the sense that $h(\bar{m}_k) = h(\bar{m}_{k-1}) + 1$ for all k ($1 \leq k \leq l$). It follows from (11) that $h(\bar{m}_{k_0}) = K$ for some k_0 ($0 \leq k_0 \leq l-1$). Moreover, by $r \cap \bar{m}_{k_0} \leq r \cap m_{\mathcal{K}+1} = o$, the element $\bar{m}_{k_0}$ is a semi-complement of r . This completes the proof of the theorem.

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